Course outline & schedule

Lecture [203]	Week			Lab [022]
DT signals & systems review	3.10.01	1		
Frequency domain methods	8.10.01	2	11.10.01	Signals, systems, frequency
DTFT properties, implementations	15.10.01	3	18.10.01	Signals, systems, frequency
DTFT - applications	22.10.01	4	25.10.01	Spectral analysis (determ.)
z-transform, filters	29.10.01	5		
Test I. Instantaneous spectrum	5.11.01	6	8.11.01	Spectral analysis (determ.)
Stochastic signals	12.11.01	7	15.11.01	Instantaneous spectrum
Autocorrelation & PSD estimation	19.11.01	8	22.11.01	Instantaneous spectrum
Parametric and nonparametric modeling	26.11.01	9	29.11.01	Analysis of stochastic signals
IIR filter design	3.12.01	10	6.12.01	Analysis of stochastic signals
FIR filter design	10.12.01	11	13.12.01	Filter design
Test II. Signal processors	17.12.01	12	20.12.01	Filter design
		13		
		14	3.01.02	Signal processors
2D signal processing	7.01.02	15	7.01.02	Signal processors
Sampling, decimation, interpolation	14.01.02	16	17.01.02	Image processing
Advanced techniques overview	21.01.02	17	24.01.02	Image processing

The lab is 4 hours, every second week (interlaced) in two groups of max. 11 students

DT signal frequency concept

Continuous time cosine:		Discrete time cosine:
$x_a(t) = \cos \omega t$	$\omega \in \mathrm{R}$	$x(n) = \cos \omega n t_s$
$\omega = 2\pi f$		$x(n) = \cos 2\pi f \, n \frac{1}{f_s}$
		$x(n) = \cos \theta n$
$T = \frac{2\pi}{f}$	$\leftarrow$ period ? $\rightarrow$	$N_0 = \frac{2\pi}{\theta}$
$x(t) = x(t + 2kT)$		$x(n) = x(n + kN)$
		$x(n+N)$ defined only if $N \in \mathrm{I}$
Always	$\leftarrow$ periodic $\rightarrow$	only if $N_0 = N/M$ (!!)

Normalized angular frequency θ: interval of 2π may be assumed as $[0, 2\pi)$ or $[-\pi, \pi)$.

$$\cos n(\theta + k \cdot 2\pi) = \cos(n\theta + n \cdot k \cdot 2\pi) = \cos n\theta$$

Fourier spectrum of a limited energy signal

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty \quad ,$$

$X(e^{j\theta})$ – a continuous, periodic function.

Fourier spectrum definition:

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\theta}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})\, e^{jn\theta} d\theta$$

$\longrightarrow$ inverse transform

Linearity: $ax[n] + by[n] \overset{\mathcal{F}}{\longleftrightarrow} aX(e^{j\theta}) + bY(e^{j\theta})$

Time shift: $x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-jn_0\theta} X(e^{j\theta})$,

Frequency shift: $e^{-jn\theta_0} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j(\theta-\theta_0)}\right)$

Convolution: $x[n] * y[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\theta}) \cdot Y(e^{j\theta})$,

Modulation: $x[n] \cdot y[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\phi}) \cdot Y(e^{j\theta-\phi}) d\phi$

(Parseval's): $E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |X(e^{j\theta})|^2 d\theta$

Example

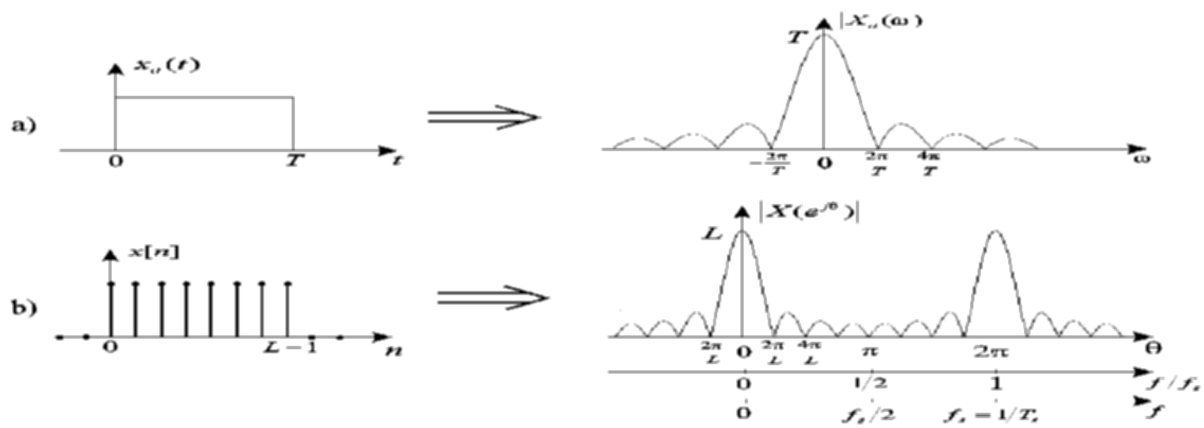
We sample $x_a(t)$ with $T_s = T(L-1)$

$$x_a(t) = \begin{cases} 1 & \text{for} \quad 0 \le t \le T \\ 0 & \text{for} \quad \text{other } t \end{cases} \qquad x[n] = \begin{cases} 1 & \text{for} \quad n = 0, 1, \dots, L-1 \\ 0 & \text{for} \quad \text{other } n \end{cases}$$

$$X_a(\omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\omega t} \, dt \qquad X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\theta}$$

$$X_a(\omega) = T \, \frac{\sin(\omega T/2)}{\omega T/2} \, e^{-j\omega T/2} \qquad X(e^{j\theta}) = e^{-j(L-1)\theta/2} \, \frac{\sin(L\theta/2)}{\sin(\theta/2)}$$

(hint: $\left(\sum_{n=0}^{N-1} q^n = (1-q^N)/(1-q)\right)$)



Periodic (limited mean power) signal FT

$$\frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2 < \infty \quad ,$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}$$

Fourier spectrum definition:

$\longrightarrow$ inverse transform

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad -\infty < k < \infty$$

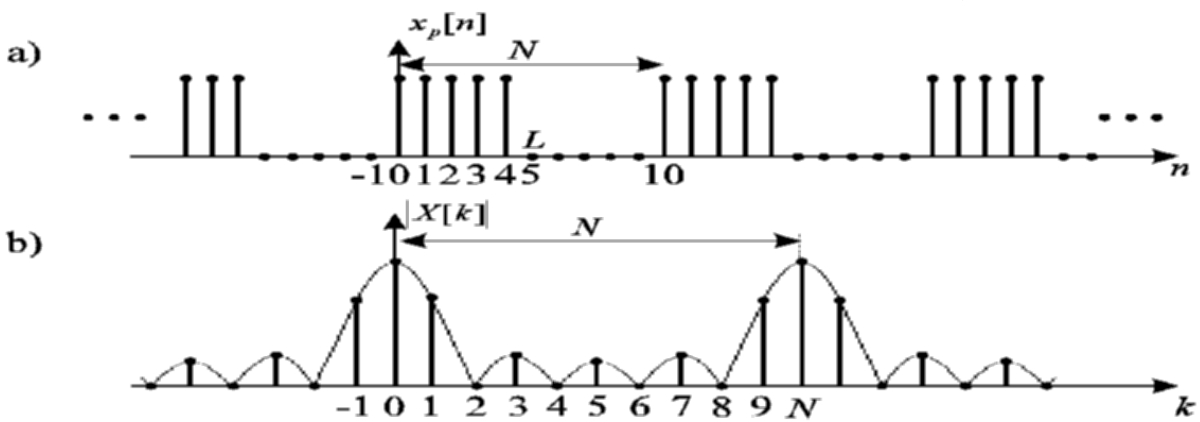
We represent $x[n]$ as a sum of N complex discrete sinusoids with angular frequencies $\theta_k = \frac{2\pi}{N}\cdot k\,, \quad k = 0,\, 1,\, \ldots,\, N-1$

Example

$x_p[n]$ with period $N = 10$ has $L = 5$ nonzero samples $(n = 0, 1, \ldots L-1)$

$$X(k) = \sum_{n=0}^{N-1} x_p(n) e^{-j\,2\pi kn/N} = \sum_{n=0}^{L-1} e^{-j\,2\pi kn/N} = e^{-j(L-1)\pi k/N} \frac{\sin(L\pi k/N)}{\sin(\pi k/N)}, \qquad k = 0, 1, \ldots$$

The amplitude spectrum $|X[k]| = \left| \frac{\sin(L\theta_{k/2})}{\sin(\theta_{k/2})} \right|$, $\theta_k = 2\pi k/N$ is shown



Discrete Fourier Transform

- A signal $x[n]$ defined for $-\infty < n > \infty$
- Its spectrum $X(e^{j\theta})$ defined for continuous $0 \leq \theta < 2\pi$
- Life is short . . .

$\longrightarrow$ Let us take a section of $x[n]$: $x_0[n],\, n = 0,\, 1,\, \ldots,\, N-1$

$$x_0[n] = x[n]g[n], \text{ where } g[n] = \begin{cases} 1 & \text{for} \quad n = 0,\, 1,\, \ldots,\, N-1 \\ 0 & \text{for} \quad \text{others } n \end{cases}$$

$g[n]$ – *window (gate?)* function (here: a *rectangular window*) ($w[n]$ we reserve for *white* noise)

$\longrightarrow$ We take only N values of $\theta_k = \frac{2\pi}{N}k, \quad k = 0,\, 1,\, \ldots,\, N-1$

$$X_0\left(e^{j\theta_k}\right) = \sum_{n=0}^{N-1} x_0(n)\, e^{-jn\theta_k} = \sum_{n=0}^{N-1} x_0(n) e^{-j2\pi nk/N}$$

Inverse DFT

Let's take forward DFT definition as a linear equation set, with $x_0[n]$ as unknowns. When we multiply both sides by $\mathrm{e}^{\mathrm{j}2\pi rk/N}$, $r = 0, 1, \ldots, N-1$ and sum for $k = 0, 1, \ldots, N-1$

$$\sum_{k=0}^{N-1} X_0(k)\,\mathrm{e}^{\mathrm{j}\,2\pi rk/N} = \sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_0(n)\mathrm{e}^{-\mathrm{j}\,2\pi nk/N}\right]\mathrm{e}^{\mathrm{j}\,2\pi rk/N} =$$

$$= \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} x_0(n)\,\mathrm{e}^{\mathrm{j}\,2\pi k(r-n)/N} = \sum_{n=0}^{N-1} x_0(n)\sum_{k=0}^{N-1}\mathrm{e}^{\mathrm{j}\,2\pi k(r-n)/N}$$

$$\sum_{k=0}^{N-1}\mathrm{e}^{\mathrm{j}\,2\pi k(r-n)/N} = \begin{cases} N, & r = n \\ 0, & r \neq n \end{cases} \Rightarrow \sum_{k=0}^{N-1} X_0(k)\mathrm{e}^{\mathrm{j}\,2\pi rk/N} = N\;x_0(r), \quad r = 0, 1, \ldots, N-1$$

$$x_0(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_0(k)\mathrm{e}^{\mathrm{j}\,2\pi nk/N}, \quad n = 0, 1, \ldots, N-1$$

DFT properties

Orthogonality – a decomposition of a function (signal) on a set of orthogonal basis functions. We've seen that our basis functions are orthogonal (with $\sum_{k=0}^{N-1} x[n] \cdot y[n]$ as a scalar product), thus the decomposition is correct. This is a good reason to chose θ_k this way.

Periodicity As we sample the spectrum, the reconstructed signal is periodic with period N.

- A non-periodic signal was reconstructed as periodic
- A periodic signal was reconstructed as N-periodic