Fast DFT algorithms (FFT)

- Direct computation:

$$X\left(e^{j\theta_k}\right) \;=\; \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

where $W_N = e^{-j2\pi/N}$, $\longrightarrow$ complexity: N^2

- Goertzel algorithm: $X(k) = y_k(N)$, where

$$y_k(n) = \sum_{r=0}^{N-1} x(r) W_N^{-k(n-r)}$$

$\longrightarrow$ filtering: $y_k(n) = x(n) + y_k(n-1) \cdot W_n^{-k}$

- Decimation in time FFT:

$$X(k) = \sum_{n\text{even}} x(n) W_N^{nk} + \sum_{n\text{odd}} x(n) W_N^{nk} = \sum_{r=0}^{N/2-1} x(2r)(W_{N/2})^{rk} + W_N^k \sum_{r=0}^{N/2-1} x(2r+1)(W_{N/2})^{rk}$$

z-transform

$\mathcal{Z}$ – a generalization of DTFT, similar to $\mathcal{L}$ as a generalization of CTFT

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$\longrightarrow$ DTFT is equal to $X(z)$ at umit circle $z = e^{j\theta}$

Convergence: same as for DTFT of $x[n] \cdot r^{-n}$ (substitute $z = r \cdot e^{j\theta}$)

$$\sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right| < \infty$$

example: $u[n]$ is not absolutely summable; $u(n) \cdot r^{-n}$ can be, if $|r^{-1}| < 1$ $\longrightarrow \mathcal{Z}(u[n])$ is convergent for $r > 1$.

Properties:

- Linearity,
- shift $x(n-n_0) \stackrel{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} \cdot X(z)$,
- multiplication $z_0^n \cdot x(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z/z_0)$,
- transform differentiation $nx(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} -zdX(z)/dz$,
- conjugation $x^*(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} X^*(z^*)$,
- time reversal $x(-n) \stackrel{\mathcal{Z}}{\longleftrightarrow} X(1/z)$,
- initial value $x(0) = \lim_{z\to\infty} X(z)$ if $x(n) = 0$ for $n < 0$ *(hint: limit of each term . . .)*
- multiplication $x_1(n) \cdot x_2(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} 1/(2\pi j) \oint_C X_1(v) X_2(z/v) v^{-1} dv$ **complex!** convolution

Examples

- $x(n) = a^n u(n)$ (causal)

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

- $x(n) = -a^n u(-n-1)$ (non-causal)

$$X(z) = -\sum_{n=-\infty}^{-1} (az^{-1})^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| < |a|$$

- $x(n) = \begin{cases} a^n & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$ (finite)

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-(az^{-1})} = \frac{1}{z^N - 1} \frac{z^N - a^N}{z-a}$$

Inverse z - transform

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} \, dz$$

- Partial fraction expansion: $X(z)$ a rational function with M zeros and N poles,

$$X(z) = \sum_{r=0}^{M-N} B_r \cdot z^{-r} \; + \; \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}, \quad A_k = (1 - d_k z^{-1}) \cdot X(z)\big|_{z=d_k}$$

- Power series expansion (e.g for finite series)