EDISP (FFT)
(English) Digital Signal Processing
FFT lecture

November 5, 2007

Fast DFT algorithms $\longrightarrow$ FFT

- Direct computation with pre-computed $W_N = e^{-j2\pi/N}$ *(twiddle factors)* :

$$X\left(e^{j\theta_k}\right) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$\longrightarrow$ complexity: N^2

- Goertzel algorithm: $X(k) = y_k(N)$, where

$$y_k(n) = \sum_{r=0}^{N-1} x(r) W_N^{-k(n-r)}$$

$\longrightarrow$ filtering: $y_k(n) = x(n) + y_k(n-1) \cdot W_n^{-k}$

- Decimation in time **FFT** (first stage):

$$X(k) = \sum_{n \text{even}} x(n) W_N^{nk} + \sum_{n \text{odd}} x(n) W_N^{nk} =$$

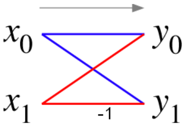
$$= \sum_{r=0}^{N/2-1} x(2r)(W_{N/2})^{rk} + W_N^k \sum_{r=0}^{N/2-1} x(2r+1)(W_{N/2})^{rk}$$

radix-2 FFT

$$X(k) = \sum_{n \text{even}} x(n) W_N^{nk} + \sum_{n \text{odd}} x(n) W_N^{nk} =$$

$$\sum_{r=0}^{N/2-1} x(2r)(W_{N/2})^{rk} + W_N^k \sum_{r=0}^{N/2-1} x(2r+1)(W_{N/2})^{rk}$$

- If $N = 2^L$ ... We can continue with this trick - decimating each half into sub-halves, each sub-half into sub-sub ... L times
- for $k > N/2$, $W_N^k = -W_N^{k-N/2}$
- DFT with size 1 is rather trivial

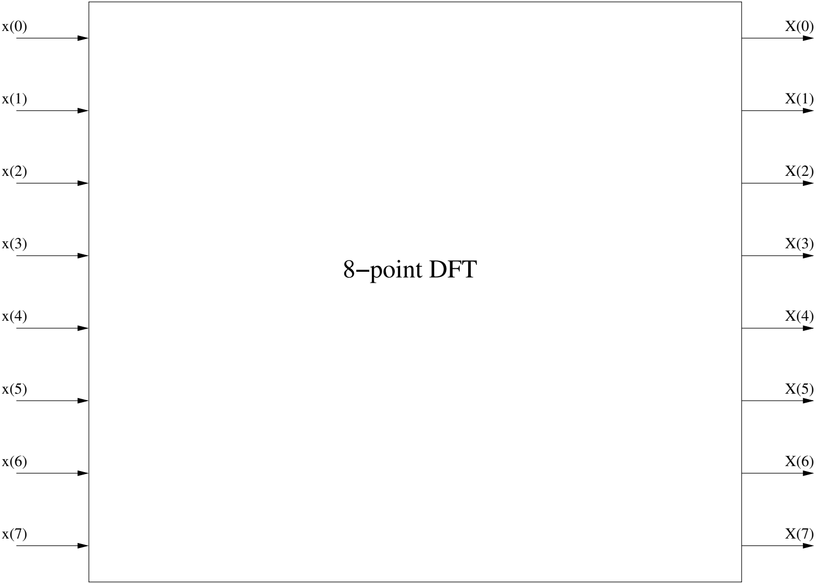


Effect: We have L layers of $N/2$ butterflies. Each butterfly is one multiplication, one addition, one subtraction. In the result, we have $O(N \log_2 N)$ operations

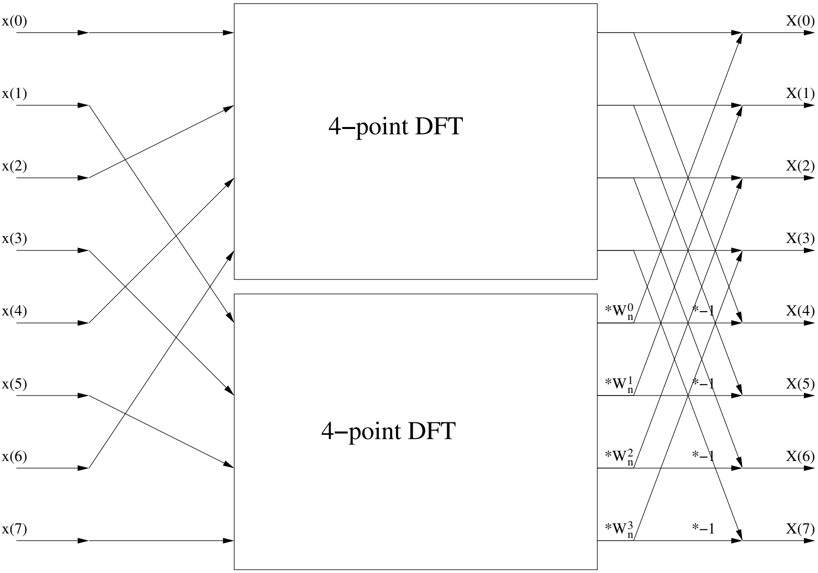
FFT inventors

James W. Cooley and John W. Tukey, "An algorithm for the machine calculation of complex Fourier series," Math. Comput. 19, pp. 297-301 (1965).

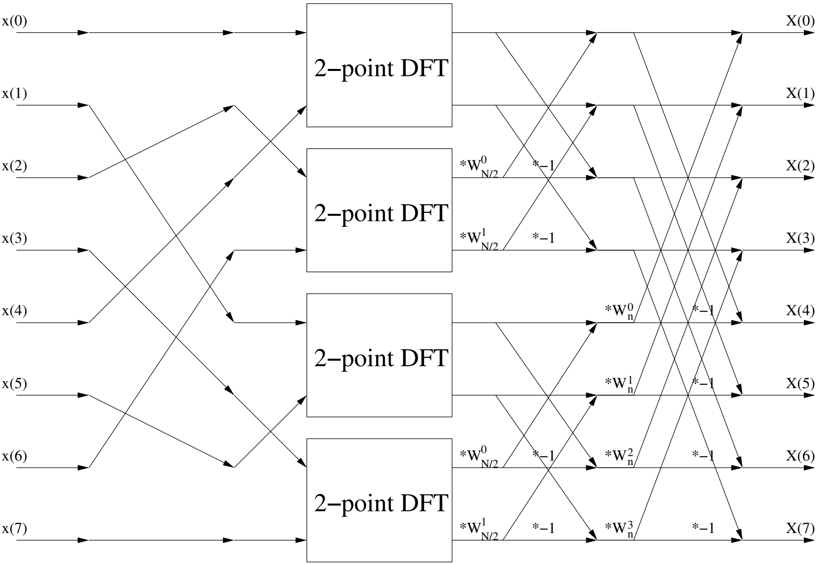
8-point radix-2 FFT



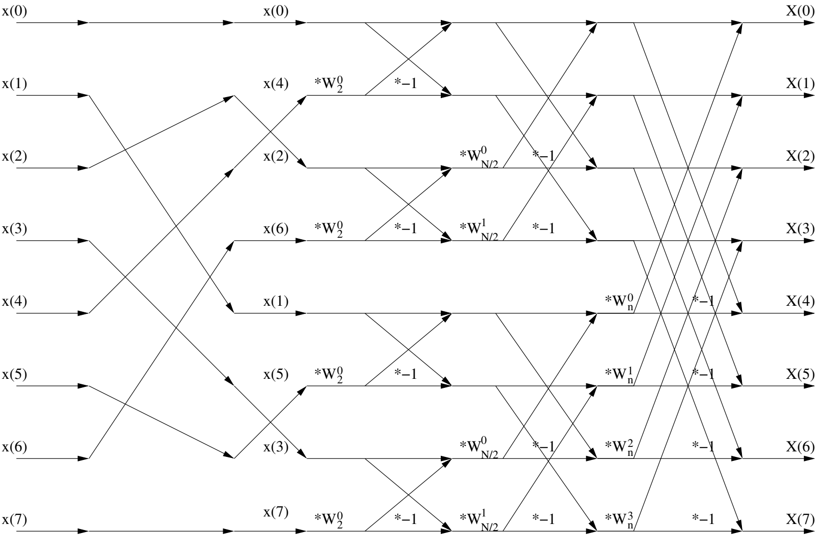
8-point radix-2 FFT



8-point radix-2 FFT

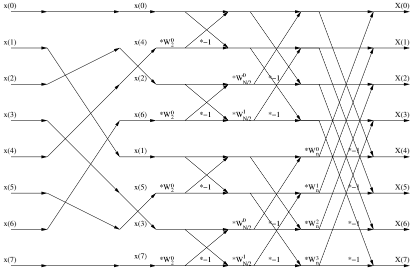


8-point radix-2 FFT



Indexing for FFT

How to describe the sequence of numbers: 0, 4, 2, 6, 1, 5, 3, 7?



- $0 = 000_2$ is at position 000_2
- $4 = 100_2$ is at position 001_2
- $2 = 010_2$ is at position 010_2
- $6 = 110_2$ is at position 011_2
- $1 = 001_2$ is at position 110_2
- $5 = 101_2$ is at position 101_2
- $3 = 011_2$ is at position 110_2
- $7 = 111_2$ is at position 111_2

$\longrightarrow$ bit-reversal does the job!

Processors designed for FFT do have the bit-reversal mode of indexing. (And they do a butterfly in one or two cycles)

Decimation in frequency FFT

- We split the definition formula for k even $(= 2r)$ or odd $(= 2r + 1)$
- We note that $W_N^{2nr} = W_{N/2}^{nr}$ or $W_N^{n(2r+1)} = W_N^n \cdot W_{N/2}^{nr}$
- Further, for $n > N/2$ $W_N^n = -W_N^{n-N/2}$
- and so on - please sketch the DIF FFT diagram by yourselves

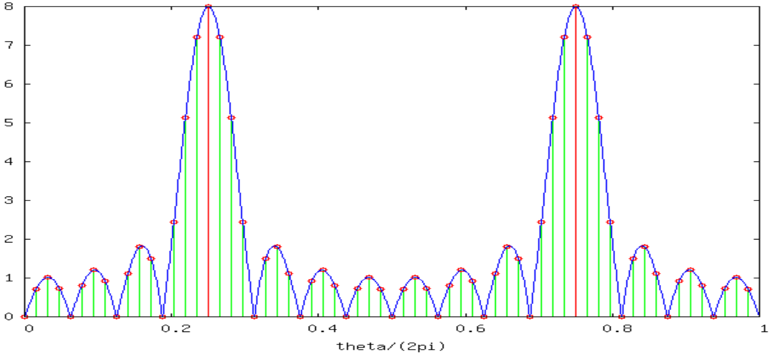
$\longrightarrow$ here, we need to re-index the frequencies...

Specials

- Non-radix2 FFT - slower than radix2, but still faster than direct
- Chirp-z transform - one use of it is to calculate FT for θ's not equal to $2\pi/N$
- Non-uniform FFT ...
- FFTW - the Fastest FFT in the West - a free library, used by many free and commercial products (Frigo & Johnson from MIT)

DFT resolution

- N-point DFT $\longrightarrow$ frequency sampled at $\theta_k = \frac{2\pi k}{N}$, so the resolution is f_s/N
- If we want more, we use $N_1 > N$ filling with zeros (zero-padding)
- but IDFT will give N_1-periodic signal
- and the spectrum will have *sidelobes*



More at the next lecture. . .

Summary

- DTFT - spectrum of a discrete-time signal (defined for a limited-energy signal or a limited mean power signal in a different manner) *periodic, continuous or discrete function of* θ
- DFT - samples of DTFT of a limited duration signal (or a segment....) *periodic, discrete* $X(k)$
- FFT - a trick (method[s]) to compute DFT efficiently