EDISP (Z-transform)
(English) Digital Signal Processing
Z-Transform lecture

November 19, 2007

z-transform

$\mathcal{Z}$ – a generalization of DTFT, similar to $\mathcal{L}$ as a generalization of CTFT

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$\longrightarrow$ DTFT is equal to $X(z)$ at unit circle $z = e^{j\theta}$
Convergence: same as for DTFT of $x[n] \cdot r^{-n}$ (substitute $z = r \cdot e^{j\theta}$)

$$\sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right| < \infty$$

example: $u[n]$ is not absolutely summable; $u(n) \cdot r^{-n}$ can be, if $|r^{-1}| < 1$
$\longrightarrow \mathcal{Z}(u[n])$ is convergent for $r > 1$.

Properties:

- Linearity,
- shift $x(n-n_0) \xleftrightarrow{\mathcal{Z}} z^{-n_0} \cdot X(z)$,
- multiplication $z_0^n \cdot x(n) \xleftrightarrow{\mathcal{Z}} X(z/z_0)$,
- transform differentiation $nx(n) \xleftrightarrow{\mathcal{Z}} -zdX(z)/dz$,
- conjugation $x^*(n) \xleftrightarrow{\mathcal{Z}} X^*(z^*)$,
- time reversal $x(-n) \xleftrightarrow{\mathcal{Z}} X(1/z)$,
- initial value $x(0) = \lim_{z\to\infty} X(z)$ if $x(n) = 0$ for $n < 0$ *(hint: limit of each term . . .)*
- multiplication $x_1(n) \cdot x_2(n) \xleftrightarrow{\mathcal{Z}} 1/(2\pi j) \oint_C X_1(v) X_2(z/v) v^{-1} \, dv$ **complex!** convolution

Examples

- $x(n) = a^n u(n)$ (causal)

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

- $x(n) = -a^n u(-n-1)$ (non-causal)

$$X(z) = -\sum_{n=-\infty}^{-1} (az^{-1})^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| < |a|$$

- $x(n) = \begin{cases} a^n & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$ (finite)

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1 - (az^{-1})^N}{1 - (az^{-1})} = \frac{1}{z^N - 1} \frac{z^N - a^N}{z - a}$$

Inverse z - transform

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} \, dz$$

- Partial fraction expansion: $X(z)$ a rational function with M zeros and N poles,

$$X(z) = \sum_{r=0}^{M-N} B_r \cdot z^{-r} \; + \; \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}, \quad A_k = (1 - d_k z^{-1}) \cdot X(z)\big|_{z=d_k}$$

- Power series expansion (e.g for finite series)

Z-transform of a convolution

$$y[n] = x[n] * h[n] \longrightarrow y(n) = \sum_{k=-\infty}^{+\infty} x(k) \cdot h(n-k)$$

$$Y(z) = \sum_{n=-\infty}^{+\infty} \left[\sum_{k=-\infty}^{+\infty} x(k) \cdot h(n-k) \right] z^{-n} =$$

$$= \sum_{k=-\infty}^{+\infty} \left[x(k) \sum_{n=-\infty}^{+\infty} h(n-k) z^{-n} \right] =$$

(we substitute $m = n - k$ so $n = k - m$)

$$= \sum_{k=-\infty}^{+\infty} \left[x(k) \sum_{m=-\infty}^{+\infty} h(m) z^{-k-m} \right] =$$

$$= \sum_{k=-\infty}^{+\infty} x(k) z^{-k} \sum_{m=-\infty}^{+\infty} h(m) z^{-m}$$

$$Y(z) = X(z) \cdot H(z)$$

And this is the main application of z-transform.

Z-transform and difference equations (1)

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$a_0 = 1$ traditionally
Simpler case of $N = 0$ (FIR, no recursion)

$$\begin{aligned} y(n) &= \sum_{k=0}^{M} b_k x(n-k) \\ Y(z) &= \sum_{k=0}^{M} b_k \mathcal{Z}[x(n-k)] = \\ &= \sum_{k=0}^{M} b_k X(z) z^{-k} = \\ &= X(z) \cdot \sum_{k=0}^{M} b_k z^{-k} = \\ &= X(z) \cdot H(z) \end{aligned}$$

Z-transform and difference equations (2)

Now the general case:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$Y(z) = X(z) \cdot \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Recall that the transform is linear, and shift is represented by $z-k$ operator.