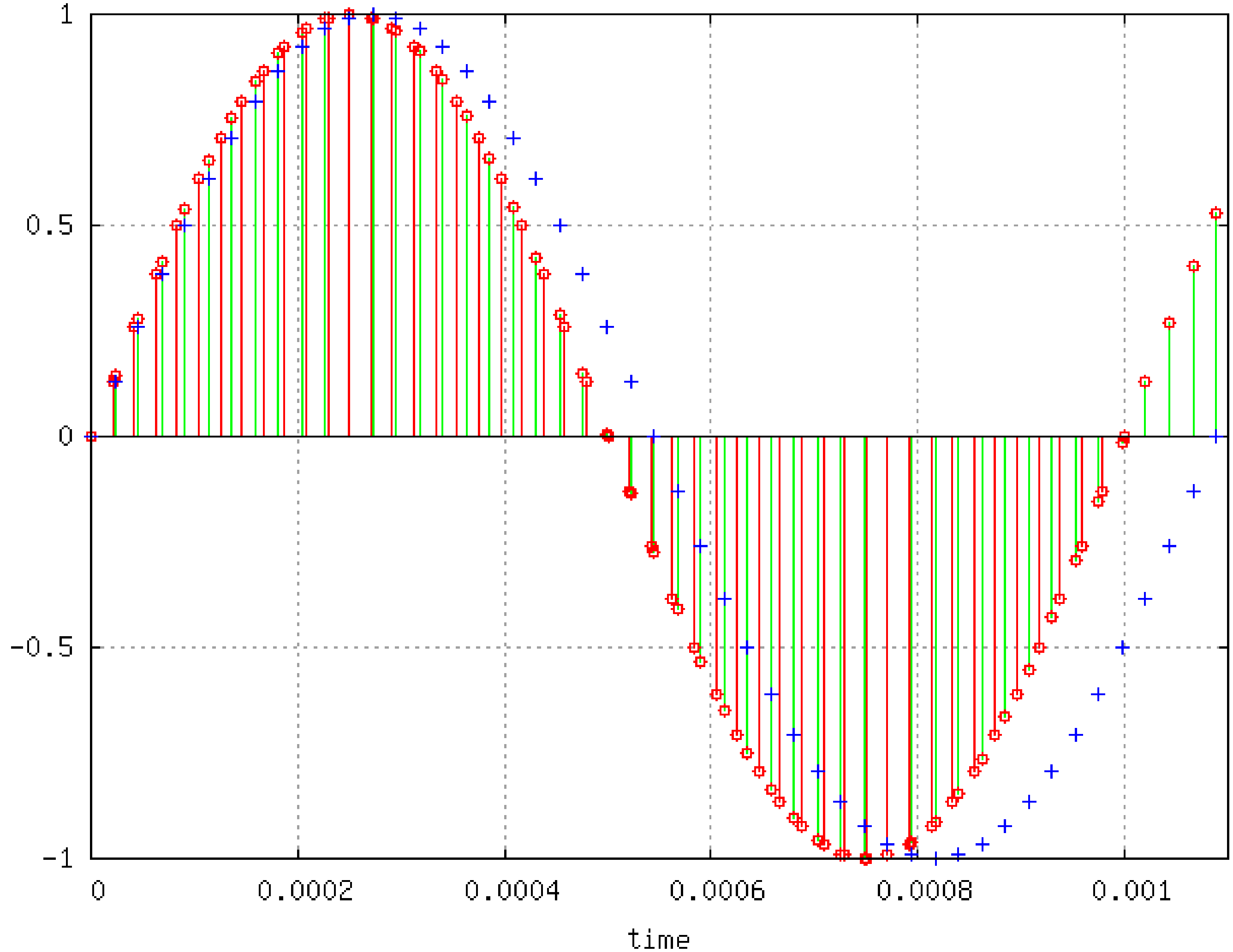
Frequency in a DT signal

	CD audio system	DAT audio system
Sampling:	44100 Hz	48000 Hz
Nyquist:	22050 Hz	24000 Hz
t_s	22.676μs	20.833μs
1kHz: samples per period	44.1	48
1kHz: moved from CD to DAT	1kHz	48/44.1=1.0884 kHz

We need a good definition of frequency!


1
0.5
0
-0.5
-1
0
0.0002
0.0004
0.0006
0.0008
0.001
time

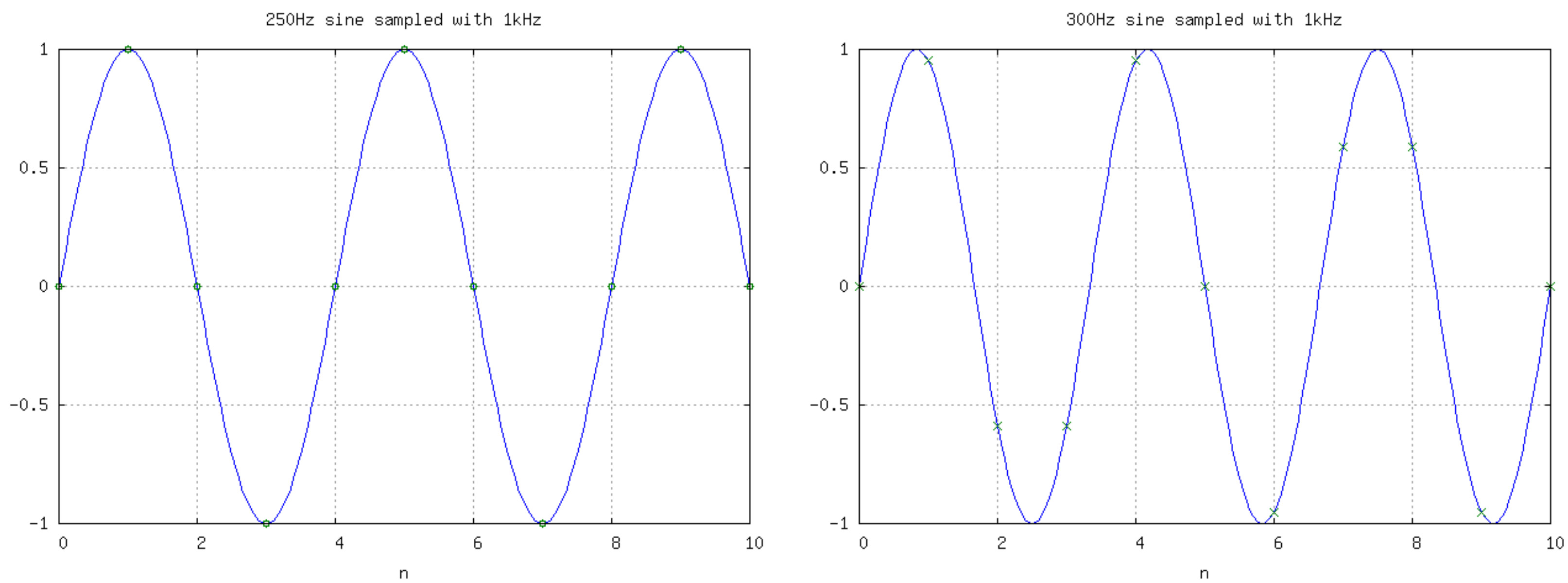
DT signal frequency concept

Continuous time cosine:		Discrete time cosine:	
$x_a(t) = \cos \omega t$	$\omega \in \mathrm{R}$	$x(n) = \cos \omega n t_s$	$t = n t_s$
$\omega = 2\pi f$		$x(n) = \cos 2\pi f \, n \frac{1}{f_s}$	$f_n = \frac{f}{f_s}$
		$x(n) = \cos \theta n$	$\theta = 2\pi \frac{f}{f_s}$
$T = \frac{1}{f} = \frac{2\pi}{\omega}$	← period ? →	$N_0 = \frac{1}{f_n} = \frac{2\pi}{\theta}$	
$x(t) = x(t + kT)$		$x(n) = x(n + kN)$	
		$x(n+N)$ defined only if $N \in \mathrm{I}$	
Always	← periodic →	only if $N_0 = N/M$ (!!)	

Normalized angular frequency θ: interval of 2π may be assumed as $[0, 2\pi)$ or $[-\pi, \pi)$.

$$\cos n(\theta + k \cdot 2\pi) = \cos(n\theta + n \cdot k \cdot 2\pi) = \cos n\theta$$

Periodicity example



Transform concept

We want to analyze the signal $\longrightarrow$ represent it as "built of" some buliding blocks (well known signals), possibly scaled

$$x[n] = \sum_k A_k \phi_k[n]$$

- The number k of "blocks" $\phi_k[n]$may be finite, infinite, or even a continuum (then $\sum \longrightarrow \int$)
- Scaling coefficients A_k are usually real or complex numbers
- ϕ_k are complex harmonics $e^{j\theta_k n}$ or cosines or *wavelets* ...
- If the representation (*expansion*) is unique for a class of functions, the set $\phi_k[n]$ is called a *basis* for this class.
- The above representation is an "*Inverse ...* transform". The "...transform" (the forward one) is the way to calculate A_k coefficients from the given signal $x[n]$.
- The forward transform is mathematically a *cast* onto the basis ϕ_k, and it is calculated with *inner product*, *scalar product* of a signal with a *dual basis* $\tilde{\phi}_k$ functions $A_k = \langle x[n], \tilde{\phi}_k[n] \rangle$ (for an orthogonal transform, $\tilde{\phi}_k = \phi_k$)

In a Fourier transform, we take the basis representing different frequencies.

Fourier spectrum of a limited energy signal

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty \quad ,$$

$X(e^{j\theta})$ – a continuous, periodic function.

Fourier spectrum definition:

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\theta}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})\, e^{jn\theta} d\theta$$

$\longrightarrow$ inverse transform

Linearity: $ax[n] + by[n] \stackrel{\mathcal{F}}{\longleftrightarrow} aX(e^{j\theta}) + bY(e^{j\theta})$

Time shift: $x[n-n_0] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-jn_0\theta} X(e^{j\theta})$,

Frequency shift: $e^{-jn\theta_0} x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j(\theta-\theta_0)})$

Convolution: $x[n] * y[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\theta}) \cdot Y(e^{j\theta})$,

Modulation: $x[n] \cdot y[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\phi}) \cdot Y(e^{j\theta-\phi}) d\phi$

(Parseval's): $E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\theta})|^2 d\theta$

Example

We sample $x_a(t)$ with $T_s = T/L$

$$x_a(t) = \begin{cases} 1 & \text{for} \quad 0 \le t < T \\ 0 & \text{for} \quad \text{other } t \end{cases}$$

$$X_a(\omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\omega t}\, dt$$

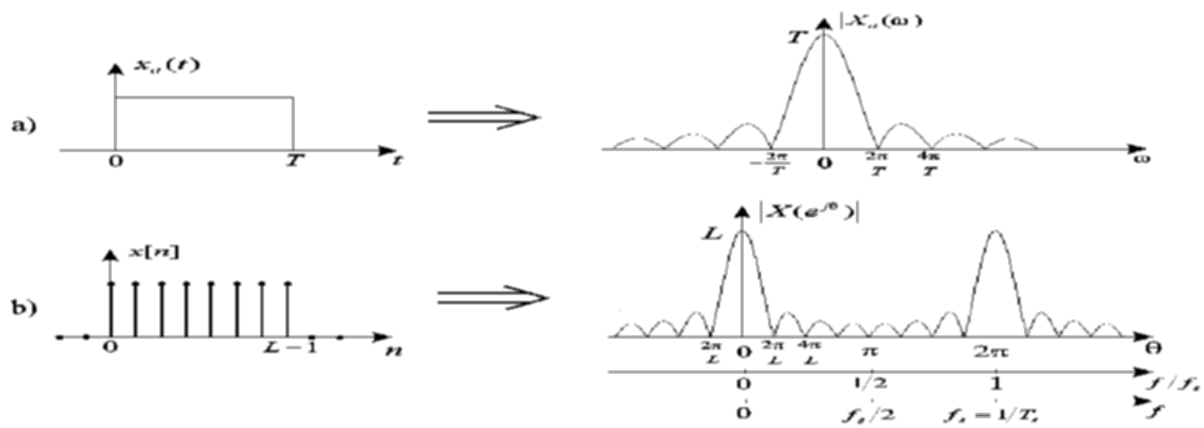
$$X_a(\omega) = T \frac{\sin(\omega T/2)}{\omega T/2} e^{-j\omega T/2}$$

$$x[n] = \begin{cases} 1 & \text{for} \quad n = 0, 1, \ldots, L-1 \\ 0 & \text{for} \quad \text{other } n \end{cases}$$

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\theta}$$

$$X(e^{j\theta}) = e^{-j(L-1)\theta/2} \frac{\sin(L\theta/2)}{\sin(\theta/2)}$$

(hint: $\left(\Sigma_{n=0}^{N-1} q^n = (1 - q^N)/(1-q)\right)$)



Periodic (limited mean power) signal FT

$$\frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2 < \infty \quad ,$$

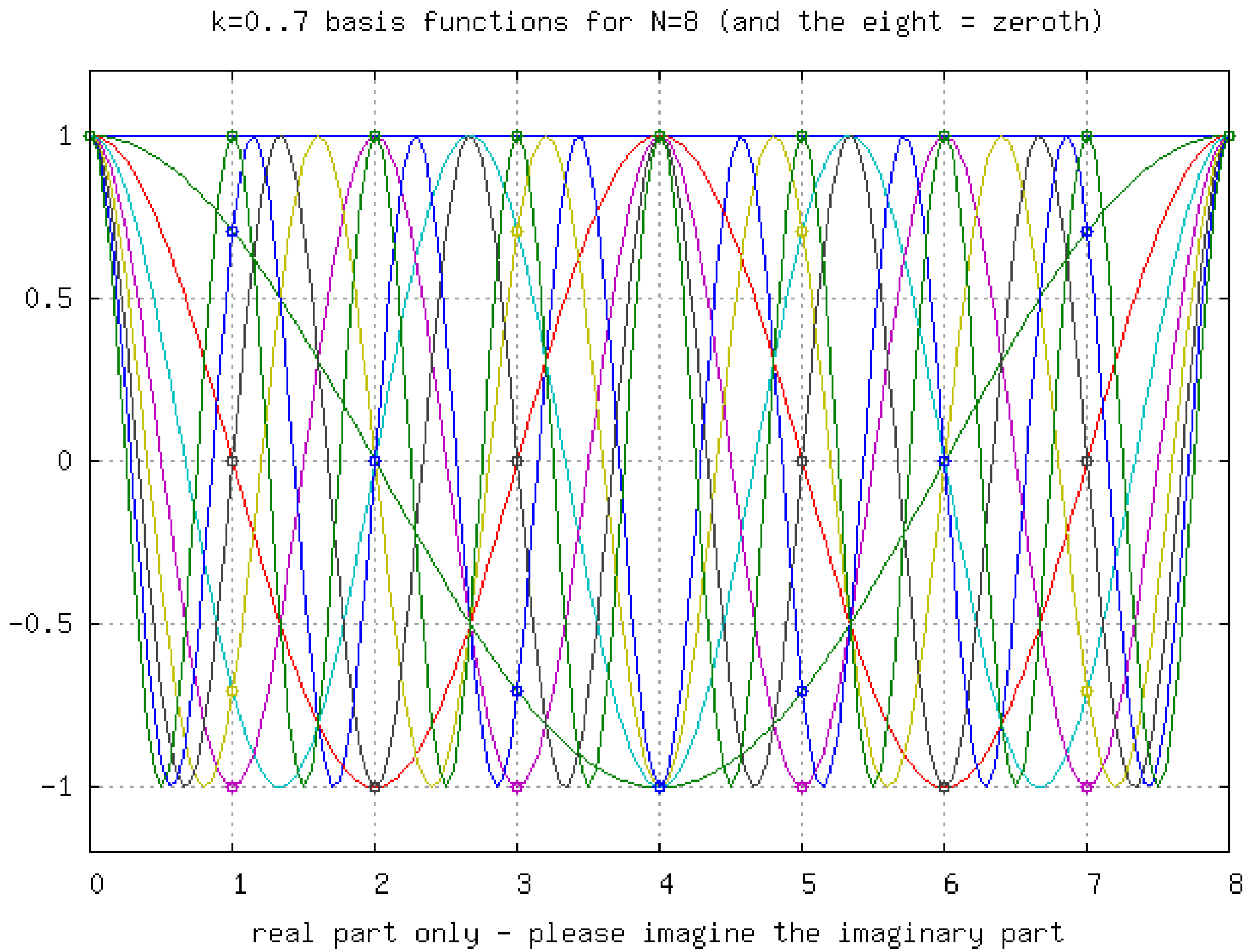
$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}$$

Fourier spectrum definition:

$\longrightarrow$ inverse transform

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad -\infty < k < \infty$$

We represent $x[n]$ as a sum of N complex discrete harmonics with angular frequencies $\theta_k = \frac{2\pi}{N} \cdot k \,, \quad k = 0,\, 1,\, \ldots,\, N-1$

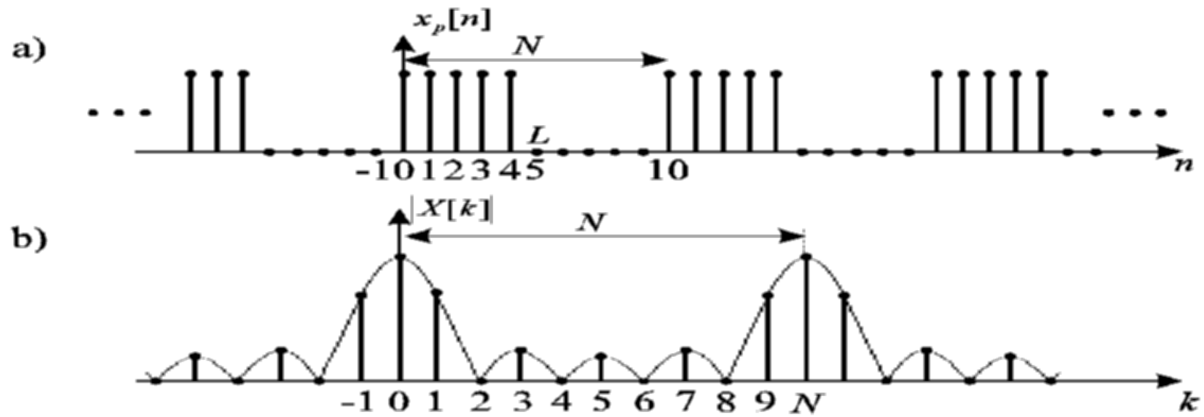

k=0..7 basis functions for N=8 (and the eight = zeroth)
1
0.5
0
-0.5
-1
0
1
2
3
4
5
6
7
8
real part only - please imagine the imaginary part

Example

$x_p[n]$ with period $N = 10$ has $L = 5$ nonzero samples $(n = 0, 1, \ldots L-1)$

$$X(k) = \sum_{n=0}^{N-1} x_p(n) e^{-j\,2\pi kn/N} = \sum_{n=0}^{L-1} e^{-j\,2\pi kn/N} = e^{-j(L-1)\pi k/N} \frac{\sin(L\pi k/N)}{\sin(\pi k/N)}, \quad k = 0, 1, \ldots$$

The amplitude spectrum $|X[k]| = \left|\frac{\sin(L\theta_{k/2})}{\sin(\theta_{k/2})}\right|$, $\theta_k = 2\pi k/N$ is shown



Discrete Fourier Transform

- A signal $x[n]$ defined for $-\infty < n < \infty$
- Its spectrum $X(e^{j\theta})$ defined for continuous $0 \leq \theta < 2\pi$
- Life is short . . .

$\longrightarrow$ Let us take a fragment of $x[n]$: $x_0[n]$, $n = 0, 1, \ldots, N-1$

$$x_0[n] = x[n]g[n], \text{ where } g[n] = \begin{cases} 1 & \text{for} \quad n = 0, 1, \ldots, N-1 \\ 0 & \text{for} \quad \text{others } n \end{cases}$$

$g[n]$ – *window (gate?)* function (here: a *rectangular window*) ($w[n]$ we reserve for *white* noise)

$\longrightarrow$ We take only N values of $\theta_k = \frac{2\pi}{N}k$, $\quad k = 0, 1, \ldots, N-1$

$$X_0\left(\mathrm{e}^{\mathrm{j}\theta_k}\right) = \sum_{n=0}^{N-1} x_0(n)\ \mathrm{e}^{-\mathrm{j}n\theta_k} = \sum_{n=0}^{N-1} x_0(n)\mathrm{e}^{-\mathrm{j}2\pi nk/N}$$

Inverse DFT

Let's take forward DFT definition as a linear equation set, with $x_0[n]$ as unknowns. When we multiply both sides by $\mathrm{e}^{\mathrm{j}2\pi rk/N}$, $r = 0, 1, \ldots, N-1$ and sum for $k = 0, 1, \ldots, N-1$

$$\sum_{k=0}^{N-1} X_0(k)\,\mathrm{e}^{\mathrm{j}\,2\pi rk/N} = \sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_0(n)\mathrm{e}^{-\mathrm{j}\,2\pi nk/N}\right]\mathrm{e}^{\mathrm{j}\,2\pi rk/N} =$$

$$= \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} x_0(n)\,\mathrm{e}^{\mathrm{j}\,2\pi k(r-n)/N} = \sum_{n=0}^{N-1} x_0(n)\sum_{k=0}^{N-1}\mathrm{e}^{\mathrm{j}\,2\pi k(r-n)/N}$$

$$\sum_{k=0}^{N-1}\mathrm{e}^{\mathrm{j}\,2\pi k(r-n)/N} = \begin{cases} N, & r = n \\ 0, & r \neq n \end{cases} \Rightarrow \sum_{k=0}^{N-1} X_0(k)\mathrm{e}^{\mathrm{j}\,2\pi rk/N} = N\ x_0(r), \quad r = 0, 1, \ldots, N-1$$

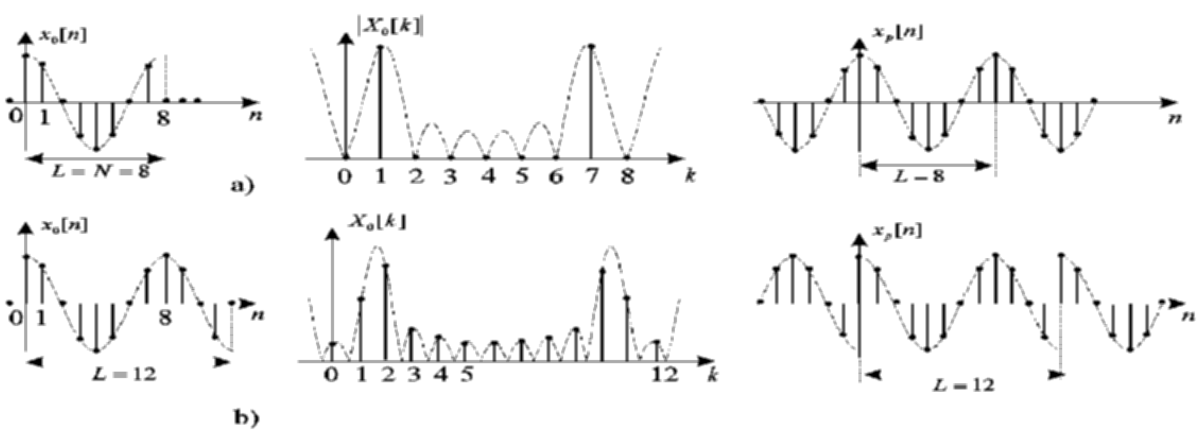
$$x_0(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_0(k)\mathrm{e}^{\mathrm{j}\,2\pi nk/N}, \quad n = 0, 1, \ldots, N-1$$

DFT properties

Orthogonality – (see next slide)

Periodicity As we sample the spectrum, the reconstructed signal is periodic with period N. If we compute IDFT for $-\infty < n < \infty \ldots$

- A non-periodic signal was reconstructed as periodic
- A periodic signal was reconstructed as N-periodic



DFT as an orthogonal transform

An orthogonal transform (e.g. DFT) is a decomposition of a function (signal) on a set of orthogonal basis functions $\phi_k[n]$.

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} A(k)\cdot\phi_k[n]$$

Because of $\phi_k[n]$ orthogonality, $A(k)$ are easy to calculate:

$$A(k) = \sum_{n=0}^{N-1} x(n)\cdot\phi_k^*(n)$$

Basis sequences (transform kernel) have to be orthogonal:

$$\frac{1}{N}\sum_{k=0}^{N-1}\phi_k(n)\cdot\phi_m^*(n) = \begin{cases} 1 & m=k \\ 0 & otherwise \end{cases}$$ Scalar product is zero = orthogonal!

DFT basis functions $\phi_k(n) = \mathrm{e}^{-\mathrm{j}n\theta_k} = \mathrm{e}^{-\mathrm{j}2\pi nk/N}$ are orthogonal – we chose θ_k so it be!