

Test 2 (2010/11) **version A** – inst. spectrum, z-transform, filters  
 Please mark your name and test version on all your answer pages

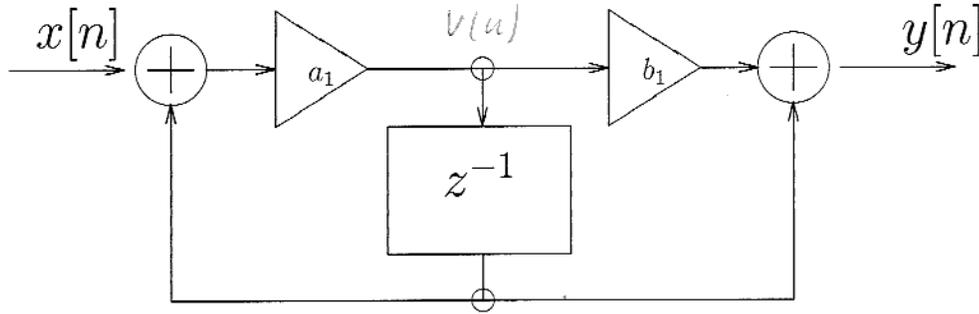
1. (3 p.) The instantaneous spectrum  $X(e^{j\theta}, n)$  of a signal

$$x(n) = \delta(n + 4) + \delta(n - 4)$$

is computed using rectangular window  $g(k)$  of length  $K = 9$ .

- sketch  $|X(e^{j\theta}, n)|$  for all  $\theta$  and calculate  $X(e^{j\theta}, n)$  at  $\theta = 0$  and  $\theta = \pi$  for:
  - (a)  $n = -4$ .
  - (b)  $n = 0$ .
  - (c)  $n = +4$ .

2. (5 p.) Analyze a filter described with the following graph:

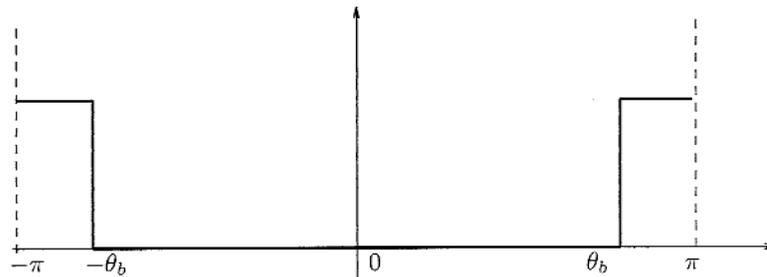


Assume  $a_1 = 0.9$   $b_1 = 1$ . *Hint: introduce additional signal  $v[n]$  at the input of the delay block; it will be cancelled when calculating  $Y/X$ .*

- (a) Find  $H(z)$ ,  $h(n)$ . Check if the filter is stable
  - (b) Sketch approximate  $A(\theta)$
  - (c) Calculate response  $y(n)$  for  $x(n) = \delta(n - 1) - \delta(n + 1)$
  - (d) Calculate response  $y(n)$  for  $x(n) = \cos(n\pi) + \sin(n\pi/2)$
3. (2 p.) Calculate the z-transform and determine ROC (region of convergence) for the series:
- (a)  $\delta[n+1]$
  - (b)  $\delta[n-1] - \delta[n+1]$
  - (c)  $u[n-1] \cdot (-1)^n$

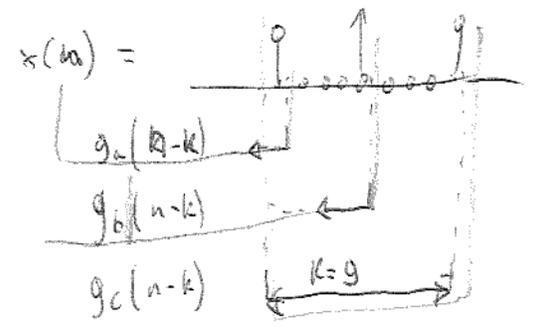
(optional - extra points) Sketch the Fourier transforms for above signals, if they exist.

4. (2 p.) A causal FIR filter of the order 8 was designed from windowed Inverse Fourier Transform of the zero-phase ideal filter frequency response (see figure). A rectangular window was used. Ideal filter cutoff was at  $\theta_b = (3/4)\pi$ .
- (a) Sketch the phase characteristics  $\phi(\theta)$  of the resulting filter.
  - (b) Find the approximate width of the transition band in the amplitude characteristics.
  - (c) Sketch the impulse response.



A

1.



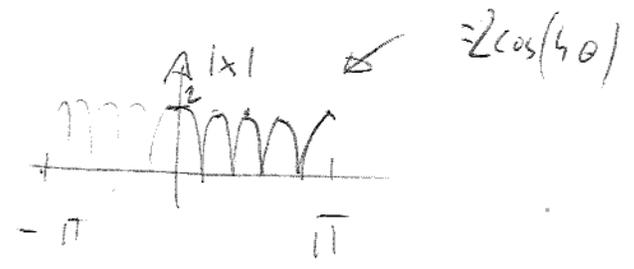
↑  
three cases of  
window positioning,

a)  $x \cdot g = \delta(n+4) \rightarrow X(\theta, n) = e^{i4\theta}$

$|X| = 1$   
 $X(0, -4) = 1; X(\pi, -4) = 1$

b) = a)

c)  $x \cdot g = x(n) \rightarrow X(\theta, n) = e^{i4\theta} + e^{-i4\theta}$



2.  $V(z) = \begin{cases} z^{-1} V(z) + X(z) \\ \dots \end{cases} \rightarrow X(z) = \frac{1}{a_1} V(z) - z^{-1} X(z) = \left( \frac{1}{a_1} - z^{-1} \right) V(z)$   
 $Y(z) = b_1 V(z) + z^{-1} V(z)$

$H(z) = \frac{Y(z)}{X(z)} = \frac{b_1 V(z) + z^{-1} V(z)}{\left( \frac{1}{a_1} - z^{-1} \right) V(z)} = a_1 \frac{b_1 + z^{-1}}{1 - a_1 z^{-1}}$

~~$H(z) = \frac{b_1}{a_1} \frac{1}{1 - a_1 z^{-1}} + z^{-1} \frac{1}{1 - a_1 z^{-1}}$~~

~~$H(z) = \frac{a_1 b_1 + a_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{\left( \frac{a_1}{2} - \frac{1}{2} \right) (1 - a_1 z^{-1}) + a_1 b_1 + \left( \frac{b_1}{2} - \frac{1}{2} \right)}{1 - a_1 z^{-1}}$~~

~~$= \frac{(a_1 z^{-1} - 1) + a_1 b_1 + 1}{1 - a_1 z^{-1}} = -1 + \frac{1 + a_1 b_1}{1 - a_1 z^{-1}}$~~

this is a decomposition  
- see in the appendix

$h(n) = -\delta(n) + u(n) \cdot a_1^n \cdot (1 + a_1 b_1)$

Z cont.

b) stable  $\because |a_1| < 1$  Yes

$$c) A(\theta) \quad A(\theta) = a_1 \cdot \frac{b_1 + z^{-1}}{1 - a_1 z^{-1}} \Bigg|_{z=e^{j\theta}} = 0.999 \cdot \frac{1 + e^{-j\theta}}{1 - 0.999 e^{-j\theta}}$$

pole at  $\theta = 0$

$$\theta = 0 \rightarrow 0.999 \cdot \frac{1+1}{1-0.999} \approx \frac{1.8}{0.1} = 18 = A(\theta)$$

zero at  $\theta = \pi$

$$\theta = \pi \rightarrow A(\pi) = 0$$

$$\theta = \frac{\pi}{2} \rightarrow \left| \frac{1+j}{1-0.999j} \right| \approx \frac{\sqrt{2}}{\sqrt{1.81}} \approx \sqrt{0.9} = 1.05 = A\left(\frac{\pi}{2}\right)$$

$\varphi\left(\frac{\pi}{2}\right) \approx 0.48\pi$



d)  $h(n) = -\delta(n) + u(n) \cdot a_1^n \cdot (1 - a_1 b_1) =$

$$= -\delta(n) + \underbrace{(1 - 0.9)}_{= 0.1} \cdot u(n) \cdot 0.9^n$$

$$y(n) = -\delta(n-1) + 0.1 \cdot u(n-1) \cdot 0.9^{(n-1)} + \delta(n+1) + 0.1 \cdot u(n+1) \cdot 0.9^{(n+1)}$$

e)  $x(n) = \cos n\pi + \sin n\pi/2 \rightarrow$  periodic signal

$$y(n) = A(\pi) \cdot \cos(n\pi - \varphi(\pi)) + A\left(\frac{\pi}{2}\right) \cdot \sin\left(n \cdot \frac{\pi}{2} - \varphi\left(\frac{\pi}{2}\right)\right)$$

$\approx 1.05$   $= 0.48\pi$

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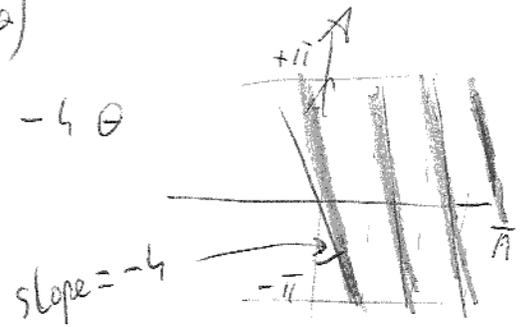
4. order = 8, length = 9, shift to <sup>make</sup> causal  $n_0 = 4$ .

~~$\phi(\theta) = \dots$~~

$$H(\theta) = e^{-jn_0\theta} \cdot A(\theta)$$

a)

$$\phi(\theta) = -n_0\theta = -4\theta$$



b)  $h(n) = h_{ideal}(n) \cdot \Pi_9(n)$

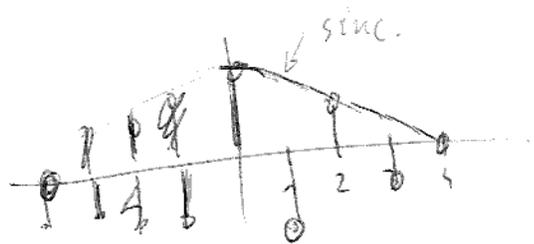
↪ rect imp with length of  $L=9$

$$H(\theta) = H_{ideal}(\theta) * \frac{\sin L\theta/2}{\sin \theta/2} \quad \rightarrow \text{main lobe } \frac{4\pi}{L} = \frac{4\pi}{9}$$

c)  $h_{ideal}(n) = (-1)^n \cdot h_{LP,ideal}(n)$

$$\rightarrow \frac{\pi}{\theta_b} \cdot \frac{\sin n\theta_{bLP}}{\pi n}$$

$$\theta_{bLP} = \pi - \theta_b = \frac{\pi}{4}$$



Appendix - decompose a rational function

$$A \cdot (1 - a_1 z^{-1}) + B = a_1 b_1 + a_1 z^{-1}$$

$$A + B = a_1 b_1$$

$$- ( -A a_1 + B = a_1 )$$

$$2A a_1 = a_1 b_1 - a_1$$

$$-A a_1 = a_1$$

$$2A a_1 = a_1 (b_1 - 1)$$

$$A = -1$$

$$A = \frac{b_1 - 1}{2}$$

$$B = a_1 b_1 + 1$$

$$B = a_1 b_1 - \frac{a_1}{2} - \frac{1}{2}$$



$$\frac{a_1 B + a_1 z^{-1}}{1 - a_1 z^{-1}} =$$

$$\frac{-1 + 1}{2}$$

$$\frac{(a_1 z^{-1} - 1) + a_1 b_1 + 1}{1 - a_1 z^{-1}}$$

$$\left( \frac{b_1}{2} - \frac{1}{2} \right)$$

$$\left( \frac{b_1}{2} - \frac{1}{2} \right) (1 - a_1 z^{-1}) + a_1 b_1 - \left( \frac{b_1}{2} - \frac{1}{2} \right) =$$

$$-a_1 \left( \frac{b_1}{2} - \frac{1}{2} \right) a_1 z^{-1} + a_1 b_1 =$$

$$= \frac{a_1 b_1}{2} \cdot a_1 z^{-1} - \frac{1}{2} a_1 z^{-1} + a_1 b_1$$

