EDISP (NWL2)
(English) Digital Signal Processing
Transform, FT, DFT

October 13, 2011

Transform concept

We want to analyze the signal $\longrightarrow$ represent it as "built of" some buliding blocks (well known signals), possibly scaled

$$x[n] = \sum_k A_k \phi_k[n]$$

- The number k of "blocks" $\phi_k[n]$may be finite, infinite, or even a continuum (then $\sum \longrightarrow \int$)
- Scaling coefficients A_k are usually real or complex numbers
- ϕ_k are complex harmonics $e^{j\theta_k n}$ or cosines or *wavelets* ...
- If the representation (*expansion*) is unique for a class of functions, the set $\phi_k[n]$ is called a *basis* for this class.
- The above representation is an "*Inverse ...* transform". The "...transform" (the forward one) is the way to calculate A_k coefficients from the given signal $x[n]$.
- The forward transform is mathematically a *cast* onto the basis ϕ_k, and it is calculated with *inner product*, *scalar product* of a signal with a *dual basis* $\tilde{\phi}_k$ functions $A_k = \langle x[n], \tilde{\phi}_k[n] \rangle$ (for an orthogonal transform, $\tilde{\phi}_k = \phi_k$)

In a Fourier transform, we take the basis representing different frequencies.

Fourier spectrum of a limited energy signal

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty \quad ,(x[n] \in \ell^2)$$

$X(e^{j\theta})$ – a continuous, periodic function.

Fourier spectrum definition:

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})\, e^{jn\theta}\, d\theta$$

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\theta}$$

$\longrightarrow$ inverse transform

Linearity: $ax[n] + by[n] \overset{\mathcal{F}}{\longleftrightarrow} aX(e^{j\theta}) + bY(e^{j\theta})$

Time shift: $x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-jn_0\theta} X(e^{j\theta})$,

Frequency shift: $e^{-jn\theta_0} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\theta-\theta_0)})$

Convolution: $x[n] * y[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\theta}) \cdot Y(e^{j\theta})$,

Modulation: $x[n] \cdot y[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\phi}) \cdot Y(e^{j\theta-\phi}) d\phi$

(Parseval's): $E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\theta})|^2 d\theta$

Example

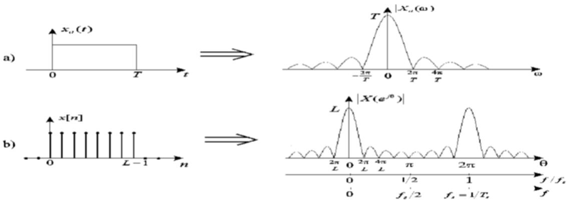
We sample $x_a(t)$ with $T_s = T/L$

$$x_a(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t < T \\ 0 & \text{for} \quad \text{other } t \end{cases} \qquad x[n] = \begin{cases} 1 & \text{for} \quad n = 0, 1, \ldots, L-1 \\ 0 & \text{for} \quad \text{other } n \end{cases}$$

$$X_a(\omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\omega t}\, dt \qquad X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\theta}$$

$$X_a(\omega) = T\, \frac{\sin(\omega T/2)}{\omega T/2}\, e^{-j\omega T/2} \qquad X(e^{j\theta}) = e^{-j(L-1)\theta/2}\, \frac{\sin(L\theta/2)}{\sin(\theta/2)}$$

(hint: $\left(\sum_{n=0}^{N-1} q^n = (1-q^N)/(1-q)\right)$)



Periodic (limited mean power) signal FT

The signal is periodic with period N $\longrightarrow$ no component that is nonperiodic or periodic with different period.
Conclusion: only N-periodic components (this includes N/k: N/2, N/3, etc.)$\longrightarrow e^{j2\pi nk/N}$

$$\frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2 < \infty \quad ,$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}$$

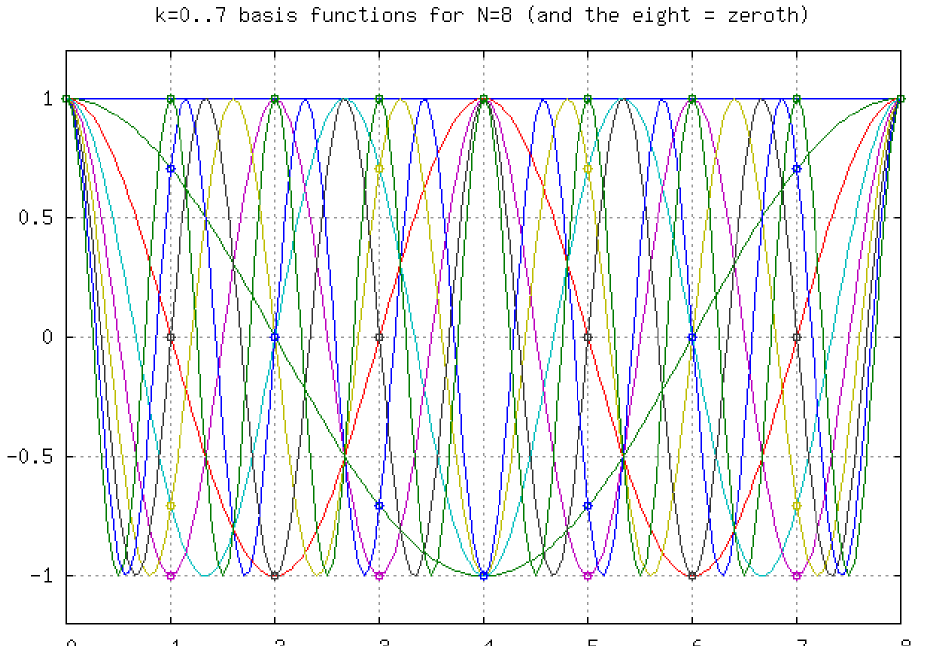
Fourier spectrum definition:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad -\infty < k < \infty$$

$\longrightarrow$ inverse transform

We represent $x[n]$ as a sum of N complex discrete harmonics with angular frequencies $\theta_k = \frac{2\pi}{N}\cdot k\ ,\quad k = 0,\ 1,\ \ldots,\ N-1$

8 basis functions for N=8 (real part only)

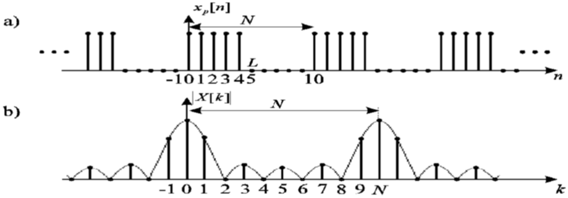


Example

$x_p[n]$ with period $N = 10$ has $L = 5$ nonzero samples
$(n = 0, 1, \ldots L-1)$

$$X(k) = \sum_{n=0}^{N-1} x_p(n) \mathrm{e}^{-\mathrm{j}\, 2\pi kn/N} = \sum_{n=0}^{L-1} \mathrm{e}^{-\mathrm{j}\, 2\pi kn/N} = \mathrm{e}^{-\mathrm{j}(L-1)\pi k/N} \frac{\sin(L\pi k/N)}{\sin(\pi k/N)}, \quad k$$

The amplitude spectrum $|X[k]| = \left| \frac{\sin(L\theta_{k/2})}{\sin(\theta_{k/2})} \right|, \quad \theta_k = 2\pi k/N$ is shown



Discrete Fourier Transform

- A signal $x[n]$ defined for $-\infty < n < \infty$
- Its spectrum $X(e^{j\theta})$ defined for continuous $0 \leq \theta < 2\pi$
- Life is short ...

$\longrightarrow$ Let us take a fragment of $x[n]$: $x_0[n]$, $n = 0, 1, \ldots, N-1$

$$x_0[n] = x[n]g[n], \text{ where } g[n] = \begin{cases} 1 & \text{for} \quad n = 0, 1, \ldots, N-1 \\ 0 & \text{for} \quad \text{others } n \end{cases}$$

$g[n]$ – *window (gate?)* function (here: a *rectangular window*) ($w[n]$ we reserve for *white* noise)

$\longrightarrow$ We take only N values of $\theta_k = \frac{2\pi}{N}k, \quad k = 0, 1, \ldots, N-1$

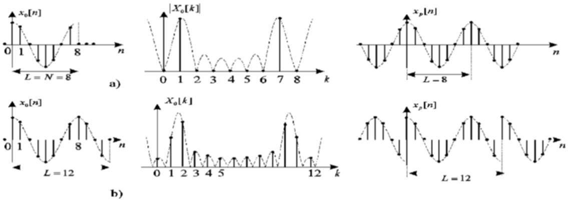
$$X_0\left(e^{j\theta_k}\right) = \sum_{n=0}^{N-1} x_0(n)\, e^{-jn\theta_k} = \sum_{n=0}^{N-1} x_0(n) e^{-j2\pi nk/N}$$

DFT properties

Orthogonality – (see next slide)

Periodicity As we sample the spectrum, the reconstructed signal is periodic with period *N*. If we compute IDFT for $-\infty < n < \infty \ldots$

- A non-periodic signal was reconstructed as periodic
- A periodic signal was reconstructed as *N*-periodic



DFT as an orthogonal transform

An orthogonal transform (e.g. DFT) is a decomposition of a function (signal) on a set of orthogonal basis functions $\phi_k[n]$.

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} A(k)\cdot\phi_k[n]$$

Because of $\phi_k[n]$ orthogonality, $A(k)$ are easy to calculate:

$$A(k) = \sum_{n=0}^{N-1} x(n)\cdot\phi_k^*(n)$$

Basis sequences (transform kernel) have to be orthogonal:

$$\frac{1}{N}\sum_{k=0}^{N-1} \phi_k(n)\cdot\phi_m^*(n) = \begin{cases} 1 & m = k \\ 0 & \textit{otherwise} \end{cases}$$ Scalar product is zero = orthogonal!

DFT basis functions $\phi_k(n) = \mathrm{e}^{-\mathrm{j}n\theta_k} = \mathrm{e}^{-\mathrm{j}2\pi nk/N}$ are orthogonal – we chose θ_k so it be!

Inverse DFT

Let's take forward DFT definition as a linear equation set, with $x_0[n]$ as unknowns. When we multiply both sides by $\mathrm{e}^{\mathrm{j}2\pi rk/N}$, $r = 0, 1, \ldots, N-1$ and sum for $k = 0, 1, \ldots, N-1$

$$\sum_{k=0}^{N-1} X_0(k)\,\mathrm{e}^{\mathrm{j}\,2\pi rk/N} = \sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_0(n)\mathrm{e}^{-\mathrm{j}2\pi nk/N}\right]\mathrm{e}^{\mathrm{j}2\pi rk/N} =$$

$$= \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} x_0(n)\,\mathrm{e}^{\mathrm{j}2\pi k(r-n)/N} = \sum_{n=0}^{N-1} x_0(n)\sum_{k=0}^{N-1}\mathrm{e}^{\mathrm{j}2\pi k(r-n)/N}$$

$$\sum_{k=0}^{N-1}\mathrm{e}^{\mathrm{j}2\pi k(r-n)/N} = \begin{cases} N, & r = n \\ 0, & r \neq n \end{cases} \Rightarrow \sum_{k=0}^{N-1} X_0(k)\mathrm{e}^{\mathrm{j}2\pi rk/N} = N\; x_0(r), \quad r = 0, 1$$

$$x_0(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_0(k)\mathrm{e}^{\mathrm{j}2\pi nk/N}, \quad n = 0, 1, \ldots, N-1$$

Forward and Inverse DFT - transformation matrix

$$
\begin{aligned}
\mathcal{F}(x) &= F \cdot x \\
F^{-1} \cdot \mathcal{F}(x) &= F^{-1} \cdot F \cdot x \\
F^{-1} \cdot \mathcal{F}(x) &= x
\end{aligned}
$$

algebraic trivia:
How to construct F matrix? $F_{kn} = e^{-j2\pi nk/N}$
What is F^{-1}? (not-so-trivial, but see IDFT slide)
Note nice properties of F matrix...