# EDISP (Z-transform )
# (English) Digital Signal Processing
# Z-Transform lecture

February 21, 2011

# $z$-transform

$\mathcal{Z}$ – a generalization of DTFT, similar to $\mathcal{L}$ as a generalization of CTFT

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$\longrightarrow$ DTFT is equal to $X(z)$ at unit circle $z = e^{j\theta}$

**Convergence:** same as for DTFT of $x[n] \cdot r^{-n}$ (substitute $z = r \cdot e^{j\theta}$)

$$\sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right| < \infty$$

example: $u[n]$ is not absolutely summable; $u(n) \cdot r^{-n}$ can be, if $|r^{-1}| < 1$

$\longrightarrow$ $\mathcal{Z}(u[n])$ is convergent for $r > 1$.

# Properties:

- Linearity *(each student will recite the formula at 4 a.m.)*,
- shift $x(n-n_0) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} \cdot X(z)$,
- multiplication $z_0^n \cdot x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z/z_0)$,
- transform differentiation $nx(n) \overset{\mathcal{Z}}{\longleftrightarrow} -zdX(z)/dz$,
- conjugation $x^*(n) \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*)$,
- time reversal $x(-n) \overset{\mathcal{Z}}{\longleftrightarrow} X(1/z)$,
- initial value $x(0) = \lim_{z\to\infty} X(z)$ if $x(n) = 0$ for $n < 0$ *(hint: limit of each term . . . )*
- multiplication $x_1(n) \cdot x_2(n) \overset{\mathcal{Z}}{\longleftrightarrow} 1/(2\pi j) \oint_C X_1(v) X_2(z/v) v^{-1} \, dv$ **complex!** convolution

What happens to ROC with above transformations?

# Examples

- $x(n) = a^n u(n)$ (causal)

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

- $x(n) = -a^n u(-n-1)$ (non-causal)

$$X(z) = -\sum_{n=-\infty}^{-1} (az^{-1})^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| < |a|$$

- $x(n) = \begin{cases} a^n & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$ (finite)

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-(az^{-1})} = \frac{1}{z^{(N-1)}} \frac{z^N - a^N}{z-a}$$

# Inverse $z$ - transform

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} \, dz$$

- Power series expansion (e.g for finite series) $\longrightarrow$ "a series of deltas"
- Partial fraction expansion: $X(z)$ a rational function with $M$ zeros and $N$ distinct poles,

$$X(z) = \sum_{r=0}^{M-N} B_r \cdot z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}, \quad A_k = (1 - d_k z^{-1}) \cdot X(z)\big|_{z=d_k}$$

Are we looking for a

- causal (or maybe only right-sided)
- anticausal (...left-sided)
- noncausal (...two-sided?)

solution?

Remark: *these terms are overloaded – understand them as a short for "signal that may be an imp. response of a causal filter" etc.*

# Z-transform pairs

Remember about ROC!

In the table only causal prototypes are shown.

| | | |
|---|---|---|
| $\delta(n)$ | — | $1$ |
| $u(n)$ | — | $\frac{1}{1-z^{-1}}$ |
| $a^n u(n)$ | — | $\frac{1}{1-az^{-1}}$ |
| $n \cdot a^n u(n)$ | — | $\frac{az^{-1}}{(1-az^{-1})^2}$ |

# Z-transform of a convolution

$$y[n] = x[n] * h[n] \longrightarrow y(n) = \sum_{k=-\infty}^{+\infty} x(k) \cdot h(n-k)$$

$$Y(z) = \sum_{n=-\infty}^{+\infty} \left[ \sum_{k=-\infty}^{+\infty} x(k) \cdot h(n-k) \right] z^{-n} =$$

$$= \sum_{k=-\infty}^{+\infty} \left[ x(k) \sum_{n=-\infty}^{+\infty} h(n-k) z^{-n} \right] =$$

(we substitute $m = n - k$ so $n = k - m$)

$$= \sum_{k=-\infty}^{+\infty} \left[ x(k) \sum_{m=-\infty}^{+\infty} h(m) z^{-k-m} \right] =$$

$$= \sum_{k=-\infty}^{+\infty} x(k) z^{-k} \sum_{m=-\infty}^{+\infty} h(m) z^{-m}$$

$$Y(z) = X(z) \cdot H(z)$$

And this is the main application of $z$-transform.

# $Z$-transform and difference equations (1)

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$a_0 = 1$ traditionally
Simpler case of $N = 0$ (FIR, no recursion)

$$\begin{aligned}
y(n) &= \sum_{k=0}^{M} b_k x(n-k) \\
Y(z) &= \sum_{k=0}^{M} b_k \mathcal{Z}[x(n-k)] = \\
&= \sum_{k=0}^{M} b_k X(z) z^{-k} = \\
&= X(z) \cdot \sum_{k=0}^{M} b_k z^{-k} = \\
&= X(z) \cdot H(z)
\end{aligned}$$

# $Z$-transform and difference equations (2)

Now the general case:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$Y(z) = X(z) \cdot \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Recall that the transform is linear, and shift is represented by $z-k$ operator.