Homework1 – LTI systems, FT of DT signals

1. Determine whether the system has the following properties: stability, causality, linearity, time-invariance, memorylessness. Present your reasoning!

$$T(x[n]) = a \cdot x[n]$$
$$T(x[n]) = z[n] \cdot x[n], \text{ where } z[n] = (-1)^n \text{ is a fixed sequence of coefficients}$$
$$T(x[n]) = x[n - n_0]$$
$$T(x[n]) = ax[n] + b$$
$$T(x[n]) = ax[n] + bx[n-3]$$
$$T(x[n]) = y[n],\ y(n) = ax(n) + b \cdot n$$

2. For the systems from previous item that are LTI, calculate impulse responses and unit responses. Does it make sense to analyze impulse response of a system that is not LTI? (Why?)

3. An LTI system is described by its impulse response $h[n]$. For input $x[n]$ it gives output $y[n]$.
   (a) $h[n] = u(n) - u(n-N)$, $x[n] = u(n) - u(n-M)$; find $y[n]$
   (b) $h[n]$ is nonzero from $n = 0$ to $N-1$, $x[n]$ is nonzero from $n = 0$ to $M-1$; where may $y[n]$ be nonzero?
   (c) $h[n]$ is nonzero only for $N_0 \leq n \leq N_1$, $[x[n]$ is nonzero only for $N_2 \leq n \leq N_3$. Find N_4 and N_5 which fulfill $[y[n]$ is nonzero only for $N_4 \leq n \leq N_5$. Express N_4 and N_5 in terms of N_0, N_1, N_2, N_3.

4. A DT signal $x[n]$ was created by sampling a 6 kHz sine wave with 10 μs sampling period. Find the normalized frequency, normalized angular frequency, period of $x[n]$.

5. An LTI system has an impulse response $h[n]$. How can you calculate the step response $k[n]$ from $h[n]$? ($k[n]$ is the response of the system when $u[n]$ is at the input).

6. Let $x[n]$ be a finite length sequence of length N. Let us define two sequences of length $N_2 = 2N$:

$$x_1(n) = \begin{cases} x(n) & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{a zero-padded version of } x[n]$$
$$x_2(n) = \begin{cases} x(n) & 0 \leq n \leq N-1 \\ -x(n-N) & N \leq n \leq 2N-1 \end{cases} \qquad \text{a combination of } x_1(n) \text{ and a sequence which can be computed from } x_1$$

   X_1, X_2, X_3 denote DFT's of respective x's.
   - How to compute $X[k]$ from $X_1[k]$ (note that we compute transform of short one from long one; the other way it is much harder!)?
   - How to compute $X_2[k]$ from $X_1[k]$? *(hint: DFT is linear)*

   Hint: Start from sketching an example of x_1, x_2, x_3.

7. Let $x[n]$ be a periodic sequence with period N_1. Thus $x[n]$ is also periodic for period $N_2 = 2N_1$. We may compute $X_1[k]$ – N-point DFT of $x[n]$ and $X_2[k]$ – $2N$-point DFT of $x[n]$.
   - express X_2 in terms of X_1
     Hint: it is easy with even samples $X_2(2m)$, *harder with odd ones* $X_2(2m+1)$, $m \in \mathbb{Z}$ *(set of integers)*
   - invent an example with $N_1 = 4$ and calculate X_1 and X_2 by hand.

   If it was too easy, try with $N_3 = 3N$; *when bored, try also* $N_4 = 4N$; *start from guessing pairs of samples which are identical (or just scaled).*

8. *For hardcore math crackers only*
   $x[n]$ – real, finite length sequence. $\mathcal{F}$ denotes Fourier transform of an L_2 signal

$$X(e^{j\omega}) = \mathcal{F}(x[n])$$
$$X[k] = \text{DFT}(x[n])$$
$$\Im\{X[k]\} = 0$$

   Prove or reject: $\Im\{X(e^{j\omega})\} = 0$ where : $\Im$ denotes imaginary part operator.
   Hint: imagine two cases:
   (a) $x[n]$ nonzero from 0 to L
   (b) $x[n]$ nonzero from $-L/2$ to $L/2$ and symmetric around 0 (and DFT definition modified appropriately)