EDISP (LTIlect)
(English) Digital Signal Processing
DT systems, LTI

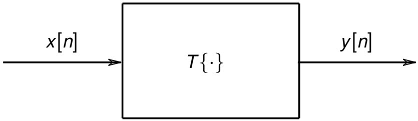
November 26, 2015

DT systems

A DT system: an operator mapping an input sequence $x[n]$ into an output sequence $y[n]$.

$$y[n] = T\{x[n]\}$$

$\longrightarrow$ A rule (formula) for computing $y(n)$ from $x(n)$



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Implementations:

- PC program
- matlab m-file
- custom VLSI or FPGA
- programmable digital signal processor

Examples:

$$y(n) = 3 \cdot x(n)$$

$$y(n) = \frac{x(n) + x(n-1)}{2}$$

$$y(n) = \frac{1}{M} \sum_{k=0}^{M-1} x(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)$$

$$y(n) = x(n)^2$$

Linear & time-invariant DT systems

Linearity property

$$T\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\} = \alpha_1 T\{x_1[n]\} + \alpha_2 T\{x_2[n]\}$$

in other words:

if

$$x_1[n] \longrightarrow y_1[n]$$
$$x_2[n] \longrightarrow y_2[n]$$

then

$$\alpha x_1[n] \longrightarrow \alpha y_1[n] \quad \text{(scaling, homogeneity)}$$
$$x_1[n] + x_2[n] \longrightarrow y_1[n] + y_2[n] \quad \text{(additivity)}$$

Time invariance (shift invariance)

If

$$T\{x[n]\} = y[n]$$

then

$$\forall n_0, \quad T\{x[n-n_0]\} = y[n-n_0]$$

Shift does not modify result $\leftrightarrow$ System properties do not change

Linear systems - examples

- $y(n) = 3 \cdot x(n)$ – is linear; it is also *memoryless*
- $y(n) = \frac{x(n)+x(n-1)}{2}$ (not memoryless):

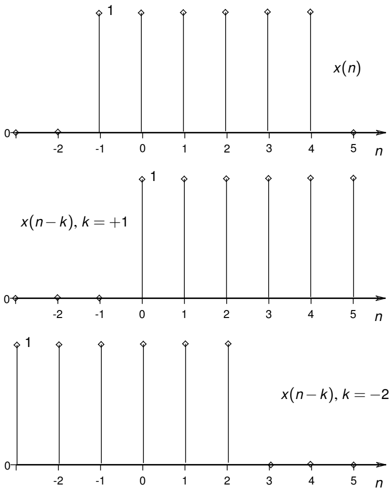
$$\begin{aligned} T\{\alpha_1 x_1(n) + \alpha_2 x_2(n)\} &= \frac{[\alpha_1 x_1(n)+\alpha_2 x_2(n)]+[\alpha_1 x_1(n-1)+\alpha_2 x_2(n-1)]}{2} = \\ &= \frac{\alpha_1 x_1(n)+\alpha_1 x_1(n-1)}{2} + \frac{\alpha_2 x_2(n)+\alpha_2 x_2(n-1)}{2} = \\ &= \alpha_1 \frac{x_1(n)+x_1(n-1)}{2} + \alpha_2 \frac{x_2(n)+x_2(n-1)}{2} = \\ &= \alpha_1 y_1(n) + \alpha_2 y_2(n) \ \text{ cnd} \end{aligned}$$

---

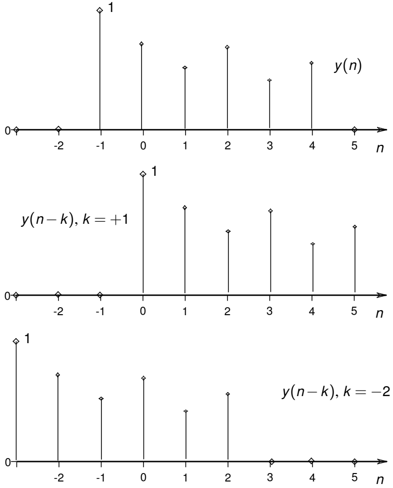
not L) $y(n) = (x(n))^2$ because
$T\{x_1(n) + x_2(n)\} = (x_1(n) + x_2(n))^2 = (x_1(n))^2 + (x_2(n))^2 + [2 \cdot x_1(n) x_2(n)]$

shift example

Input signals $x[n-k]$.

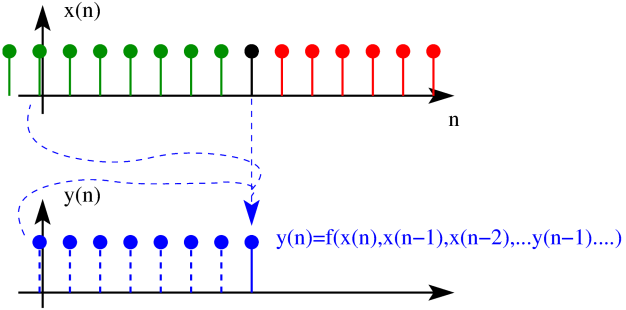


Responses $T\{x[n-k]\}$ of TI system $T\{.\}$



Other properties: **causality**, stability

causality



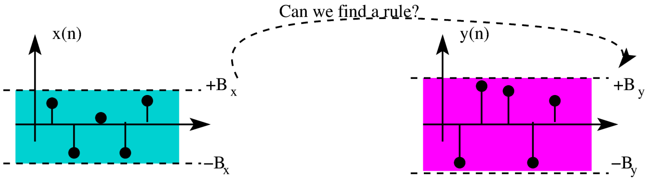
$\longrightarrow y(n_0)$ depends only on $x(n),\ n \leq n_0$ (*important in real-time implementations, unimportant for off-line processing)*

Other properties: causality, **stability**

stability

$\longrightarrow$ bounded input causes bounded output [BIBO]

bounded $\longrightarrow \exists B_x : \quad \forall n \ \ |x(n)| \leq B_x < \infty$



Decimator (compressor)

$$y(n) = x(Mn)$$

$\longrightarrow$ L, but not TI (*prove it!*)

1-st order difference

forward: $y(n) = x(n+1) - x(n) \longrightarrow$ noncausal
backward: $y(n) = x(n) - x(n-1) \longrightarrow$ causal

Accumulator

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

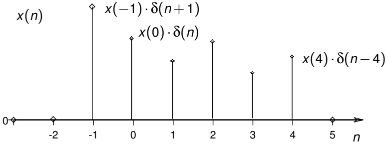
$\longrightarrow$ unstable; (*hint: feed it with* $u[n]$)

LTI systems: impulse response

$h[n] = T\{\delta[n]\} \longrightarrow$ impulse response of $T\{.\}$
$h[n]$ characterizes completely system $T\{.\}$ – we may compute its response for any input $x[n]$.

- Decompose $x[n]$ into weighted sum of impulses $\delta[n-k]$

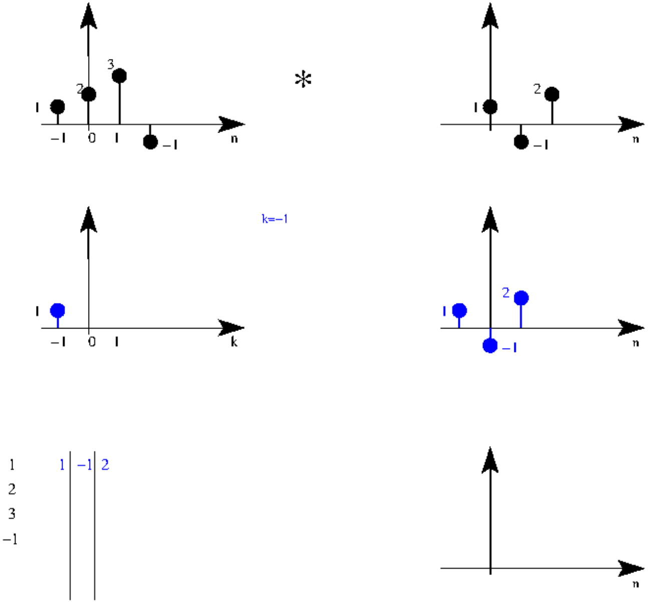
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



- Superpose responses (use LTI properties)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$\longrightarrow$ this is a **convolution sum**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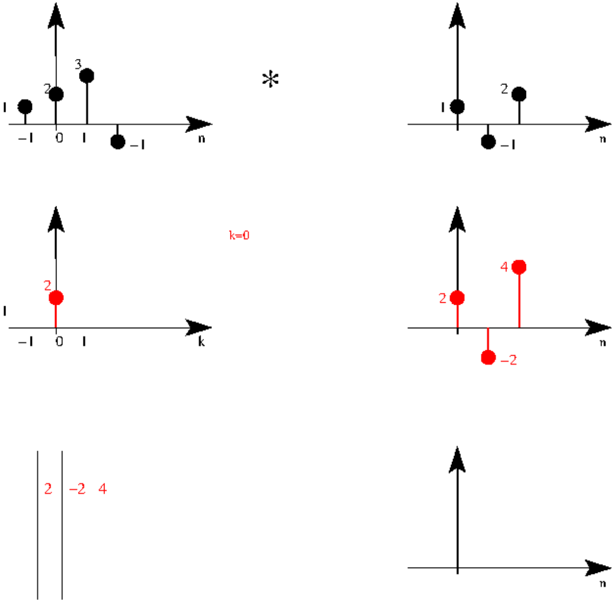
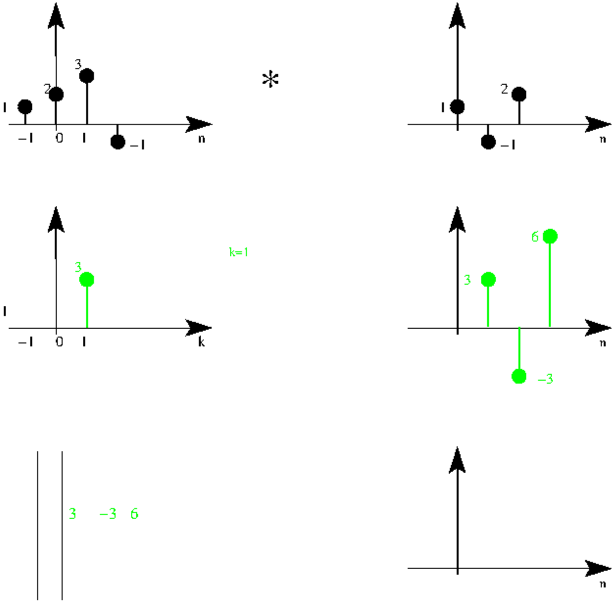
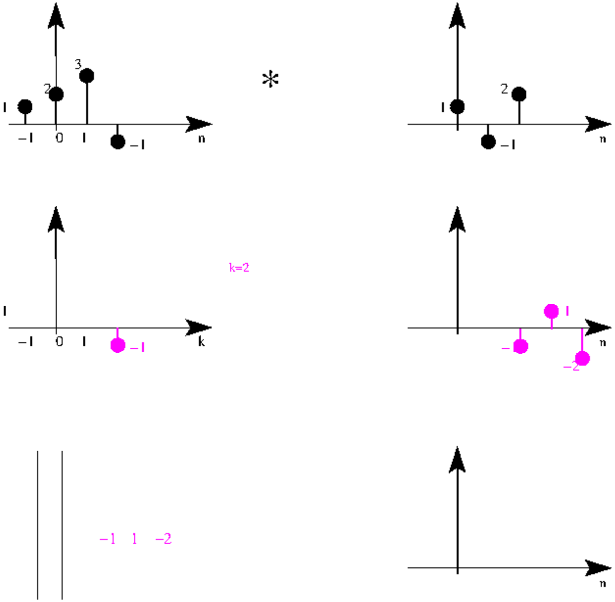
Convolution example



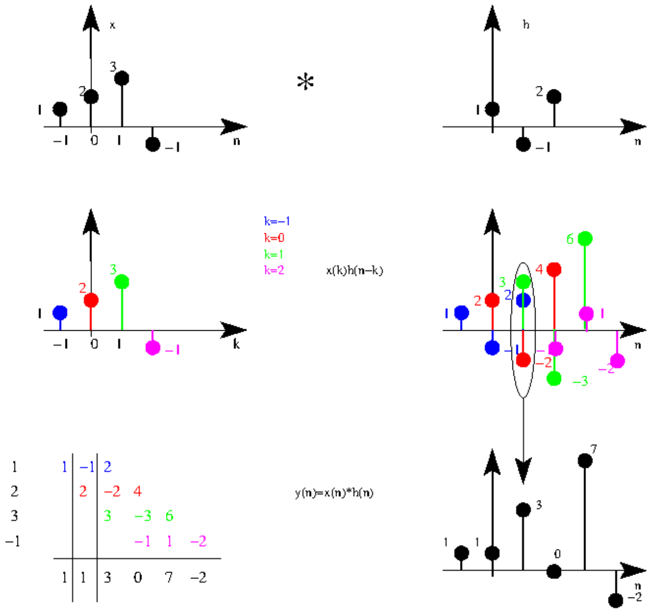
Convolution example



Convolution example



Convolution example



Convolution properties

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

we denote as

$$y[n] = x[n] * h[n]$$

Properties of "$*$"

"$*$" is commutative: $x[n] * h[n] = h[n] * x[n]$

"$*$" distributes over addition $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * (h_2[n])$

$\longrightarrow$ Convolution of a signal $x[n]$ with given fixed $h[n]$ is linear!

System and $h[n]$

- causality $\Leftrightarrow h[n] = 0, \quad n < 0$. A hint: $y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$
- stability $\Leftrightarrow S = \sum_{k=-\infty}^{\infty} |h(k)| < \infty$

Linear difference equations

... describe an important class of LTI systems.

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k), \quad a_0 = 1 \text{ (traditionally)}$$

or

$$\begin{aligned} y(n) \quad = \quad & -a_1 \cdot y(n-1) - a_2 \cdot y(n-2) - \ldots - a_n \cdot y(n-N) + \\ +b_0 \cdot x(n) \quad & +b_1 \cdot x(n-1) + b_2 \cdot x(n-2) + \ldots + b_n \cdot x(n-M) \end{aligned}$$

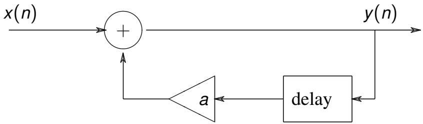
Note: if, for a given input $x_p[n]$, an output sequence $y_p[n]$ satisfies given difference equation,

$$y[n] = y_p[n] + y_h[n]$$

will also satisfy the equation, if $y_h[n]$ is a solution to $\sum_{k=0}^{N} a_k y(n-k) = 0$ (homogenous equation).

Difference equation – example

An equation: $y(n) = a \cdot y(n-1) + x(n)$
with input
$x(n) = 0, \; n < 0$
$x(n) \neq 0, \; n > 0.$



$$
\begin{aligned}
y(0) &= a \cdot \mathbf{y(-1)} + x(0) \\
y(1) &= a \cdot y(0) + x(1) \\
y(2) &= a \cdot y(1) + x(2) \\
&\dots
\end{aligned}
$$

Initial condition: $y(-1) = \alpha$
Let $x[n] = \delta[n]$

$$
\begin{aligned}
y(0) &= a \cdot \alpha + 1 \\
y(1) &= a(a \cdot \alpha + 1) = a^2\alpha + a \\
y(2) &= a^3\alpha + a^2 \\
&\dots \\
y(n) &= a^{n+1}\alpha + a^n
\end{aligned}
$$

Difference equation – impulse response (example continued

$y(n) = a \cdot y(n-1) + x(n)$
Initial condition: $y(-1) = \alpha$
$x[n] = \delta[n]$
Solution: $y(n) = a^{n+1}\alpha + a^n$

Find a homogenous part!

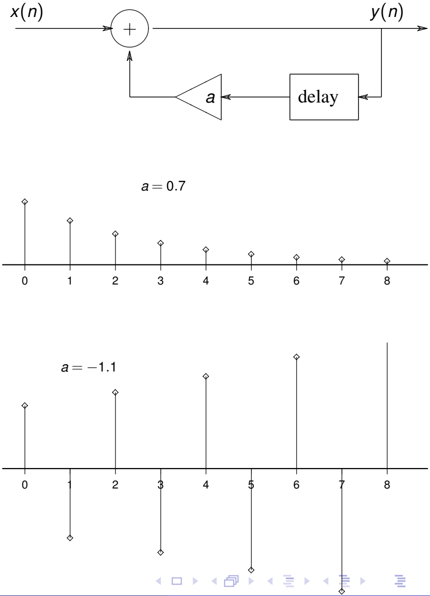
Stability:

$1 < a$: $a^n \to \infty$

$0 < a < 1$: $a^n \to 0$

$-1 < a < 0$: $a^n \to 0$

$a < -1$: $a^n \to ???$



$\mathcal{Z}$-transform – what and why

- DTFT – a transform based on periodic decomposition (basis: $e^{jn\theta}$ sequences)
- responses of most LTI systems – made of short sequences (FIR) and decaying complex exponentials $r^n e^{jn\theta}$ (IIR)
- $r^n e^{jn\theta} = (re^{j\theta})^n = z^n$ can be a good basis

$\mathcal{Z}$-transform is a tool for analyzing transient signals, such as an impulse response of a system.

$\mathcal{Z}$-transform definition

$\mathcal{Z}$ – a generalization of DTFT, similar to $\mathcal{L}$ as a generalization of CTFT

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$\longrightarrow$ DTFT is equal to $X(z)$ at unit circle $z = e^{j\theta}$

Convergence: same as for DTFT of $x[n] \cdot r^{-n}$ (substitute $z = r \cdot e^{j\theta}$)

$$\sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right| < \infty$$

example: $u[n]$ is not absolutely summable; $u(n) \cdot r^{-n}$ can be, if $|r^{-1}| < 1$

$\longrightarrow$ $\mathcal{Z}(u[n])$ is convergent for $r > 1$.

Properties:

- Linearity *(each student will recite the formula at 4 a.m.)*,
- shift $x(n-n_0) \stackrel{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} \cdot X(z)$,
- multiplication $z_0^n \cdot x(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z/z_0)$,
- transform differentiation $nx(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} -zdX(z)/dz$,
- conjugation $x^*(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} X^*(z^*)$,
- time reversal $x(-n) \stackrel{\mathcal{Z}}{\longleftrightarrow} X(1/z)$,
- initial value $x(0) = \lim_{z\to\infty} X(z)$ if $x(n) = 0$ for $n < 0$ *(hint: limit of each term . . .)*
- multiplication $x_1(n) \cdot x_2(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} 1/(2\pi j) \oint_C X_1(v) X_2(z/v) v^{-1} \, dv$ **complex! convolution**

What happens to ROC with above transformations?

Examples

- $x(n) = a^n u(n)$ (causal)

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

- $x(n) = -a^n u(-n-1)$ (non-causal)

$$X(z) = -\sum_{n=-\infty}^{-1} (az^{-1})^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| < |a|$$

- $x(n) = \begin{cases} a^n & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$ (finite)

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1 - (az^{-1})^N}{1 - (az^{-1})} = \frac{1}{z^{(N-1)}} \frac{z^N - a^N}{z - a}$$

$\mathcal{Z}$-transform pairs

Remember about ROC (Region of Convergence):

- causal signal (non-zero only when $n \geq 0$) – outside unit circle
- anticausal signal (non-zero only when $n \leq 0$) – inside unit circle

In the table only causal prototypes are shown.

$\delta(n)$	—	1
$u(n)$	—	$\frac{1}{1-z^{-1}}$
$a^n u(n)$	—	$\frac{1}{1-az^{-1}}$
$n \cdot a^n u(n)$	—	$\frac{az^{-1}}{(1-az^{-1})^2}$

Inverse $\mathcal{Z}$ - transform

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz$$

- Power series expansion (e.g for finite series) $\longrightarrow$ "a series of deltas"
- Partial fraction expansion: $X(z)$ a rational function with M zeros and N distinct poles $\longrightarrow$ $X(z) = num(z)/den(z) = quot(z) + rem(z)/den(z)$

$$X(z) = \sum_{r=0}^{M-N} B_r \cdot z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}, \quad A_k = (1 - d_k z^{-1}) \cdot X(z)\Big|_{z=d_k}$$

Are we looking for a

- causal (or maybe only right-sided)
- anticausal (... left-sided)
- noncausal (... two-sided?)

solution?

Remark: *these terms are overloaded – understand them as a short for "signal that may be an imp. response of a causal filter" etc.*

$H(z)$ to $h(n)$ (or how to find $\mathcal{Z}^{-1}$)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{2} b_k z^{-k}}{\sum_{m=0}^{2} a_m z^{-m}} = A\frac{\prod_{k=0}^{2}(1 - c_k z^{-1})}{\prod_{m=0}^{2}(1 - d_m z^{-1})}$$

Zeros at $z = c_k \longrightarrow (1 - c_k z^{-1}) = \frac{z - c_k}{z - 0}$ (plus pole at $z = 0$).

Poles at $z = d_m \longrightarrow \frac{1}{(1 - d_m z^{-1})} = \frac{z - 0}{z - d_m}$ (plus a zero at $z = 0$).

Knowing poles $\longrightarrow$ decompose into a polynomial + partial fractions:

$$H(z) = \sum_{r=0}^{M-N} B_r \cdot z^{-r} + \sum_{m=1}^{N} \frac{A_m}{1 - d_m z^{-1}}$$

$$h(n) = \sum_{r=0}^{M-N} B_r \cdot \delta(n - r) + \sum_{m=1}^{N} A_m u(n)(d_m)^n$$

So, two conjugate poles (at d_m and d_m^*) give a decaying cosine!

Z-transform of a convolution

$$y[n] = x[n] * h[n] \longrightarrow y(n) = \sum_{k=-\infty}^{+\infty} x(k) \cdot h(n-k)$$

$$Y(z) = \sum_{n=-\infty}^{+\infty} \left[\sum_{k=-\infty}^{+\infty} x(k) \cdot h(n-k) \right] z^{-n} =$$

$$= \sum_{k=-\infty}^{+\infty} \left[x(k) \sum_{n=-\infty}^{+\infty} h(n-k) z^{-n} \right] =$$

(we substitute $m = n - k$ so $n = k - m$)

$$= \sum_{k=-\infty}^{+\infty} \left[x(k) \sum_{m=-\infty}^{+\infty} h(m) z^{-k-m} \right] =$$

$$= \sum_{k=-\infty}^{+\infty} x(k) z^{-k} \sum_{m=-\infty}^{+\infty} h(m) z^{-m}$$

$$Y(z) = X(z) \cdot H(z)$$

And this is the main application of z-transform.

Z-transform and difference equations (1)

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$a_0 = 1$ traditionally
Simpler case of $N = 0$ (FIR, no recursion)

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

$$\begin{aligned} Y(z) &= \sum_{k=0}^{M} b_k \mathcal{Z}[x(n-k)] = \\ &= \sum_{k=0}^{M} b_k X(z) z^{-k} = \\ &= X(z) \cdot \sum_{k=0}^{M} b_k z^{-k} = \\ &= X(z) \cdot H(z) \end{aligned}$$

Z-transform and difference equations (2)

Now the general case:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

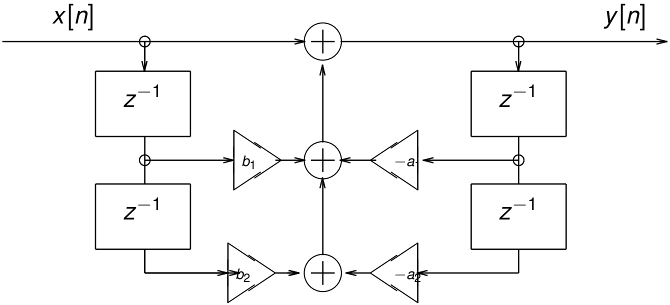
$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$Y(z) = X(z) \cdot \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Recall that the transform is linear, and time shift by k is represented by z^{-k} operator.

System and its difference equation



$$y(n) = x(n) + b_1 x(n-1) + b_2 x(n-2) - a_1 y(n-1) - a_2 y(n-2)$$

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = x(n) + b_1 x(n-1) + b_2 x(n-2)$$

$$\sum_{m=0}^{2} a_m y(n-m) = \sum_{k=0}^{2} b_k x(n-k)$$

Difference equation and $H(z)$

z^{-1} - time shift (delay) operator

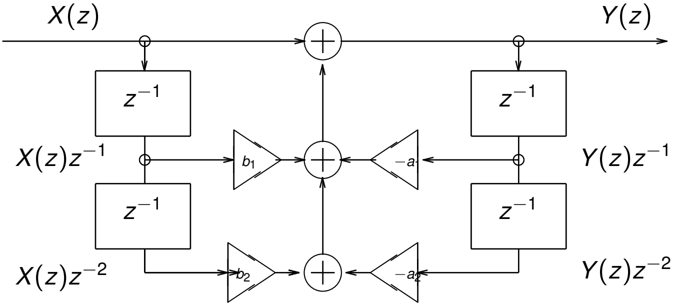
$$\sum_{m=0}^{2} a_m y(n-m) = \sum_{k=0}^{2} b_k x(n-k)$$

$$\sum_{m=0}^{2} a_m Y(z) z^{-m} = \sum_{k=0}^{2} b_k X(z) z^{-k}$$

$$Y(z) \sum_{m=0}^{2} a_m z^{-m} = X(z) \sum_{k=0}^{2} b_k z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{2} b_k z^{-k}}{\sum_{m=0}^{2} a_m z^{-m}}$$

System and its $H(z)$



$$Y(z) = X(z) + b_1 X(z)z^{-1} + b_2 X(z)z^{-2} - a_1 Y(z)z^{-1} - a_2 Y($$

$$Y(z) + a_1 Y(z)z^{-1} + a_2 Y(z)z^{-2} = X(z) + b_1 X(z)z^{-1} + b_2 X(z)z^{-2}$$

$$\sum_{m=0}^{2} a_m Y(z)z^{-m} = \sum_{k=0}^{2} b_k X(z)z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{2} b_k z^{-k}}{\sum_{m=0}^{2} a_m z^{-m}}$$

System defined by $H(z)$ + complex sinusoid

$$x(n) = e^{jn\theta} \longrightarrow \boxed{\text{h(n)}} \longrightarrow y(n) = ?$$

$$y(n) = \sum_k h(k) \cdot e^{j(n-k)\theta} =$$
$$= \sum_k h(k) \cdot e^{j(-k)\theta} \cdot e^{jn\theta} =$$
$$= e^{jn\theta} \sum_k h(k) \cdot e^{j(-k)\theta} =$$
$$= e^{jn\theta} H(e^{j\theta})$$
$$H(e^{j\theta}) = A(\theta) e^{j\phi(\theta)}$$

$A(\theta)$ - magnitude, $\phi(\theta)$ - phase of $H(e^{j\theta})$
If $x(n)$ is periodic - we can decompose it into harmonics (linearity).

As you can see, when we deal with periodic signals, we use $H(e^{j\theta}) = H(z)|_{z=e^{j\theta}}$

System defined by $H(z)$ + sine/cosine signal

We say $H(z)$ meaning “transfer function”, but we immediatly substitute $z = e^{j\theta}$....

$$x(n) = e^{jn\theta} \longrightarrow \boxed{\text{h(n)}} \longrightarrow y(n) = e^{jn\theta} H(e^{j\theta})$$

so if $x(n) = cos(n\theta) = 1/2 \cdot (e^{jn\theta} + e^{-jn\theta})$
then $y(n) = 1/2 \cdot (H(e^{j\theta})e^{jn\theta} + H(e^{-j\theta})e^{-jn\theta})$

$$H(e^{j\theta}) = A(\theta)e^{j\phi(\theta)}$$

and $y(n) = A(\theta) \cdot 1/2 \cdot (e^{jn\theta} \cdot e^{j\phi(\theta)} + e^{-jn\theta} \cdot e^{-j\phi(\theta)})$
(for a real h(n) $\phi(\theta)$ is odd: $\phi(-\theta) = -\phi(\theta)$)
and $y(n) = A(\theta) \cdot 1/2 \cdot (e^{j(n\theta+\phi(\theta))} + e^{-j(n\theta+\phi(\theta))})$

$$y(n) = A(\theta) \cdot cos(n\theta + \phi(\theta))$$

Repeat the same with $sin()$ $\longrightarrow$ at home.

Example: $x(n) = 3 + 5sin(0.1\pi n)$ $\longrightarrow$ a DC component and a 0.1π sinusoidal signal.
So $y(n) = A(0) \cdot 3 + A(0.1\pi) \cdot 5sin(0.1\pi n + \phi(0.1\pi))$.
Note: With periodic signals use Fourier and *NOT* Z-transform !!!