EDISP (NWL2)
(English) Digital Signal Processing
Transform, FT, DFT

March 1, 2017

Transform concept

We want to analyze the signal $\longrightarrow$ represent it as "built of" some building blocks (well known signals), possibly scaled

$$x[n] = \sum_k A_k \phi_k[n]$$

$\longrightarrow$ Linear combination of $\phi_k[n]$ functions

- "forward transform" $\longrightarrow$ For a given $x[n]$ we want to find coefficients A_k
- "inverse transform" $\longrightarrow$ We know A_k, we reconstruct $x[n]$
- The number k of "blocks" $\phi_k[n]$ may be finite, infinite, or even a continuum (then $\sum \longrightarrow \int$)
- Scaling coefficients A_k are usually real or complex numbers
- ϕ_k are complex harmonics $e^{j\theta_k n}$ or cosines or *wavelets* ...

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$\longrightarrow$ Linear combination of $\phi_k[n]$ functions

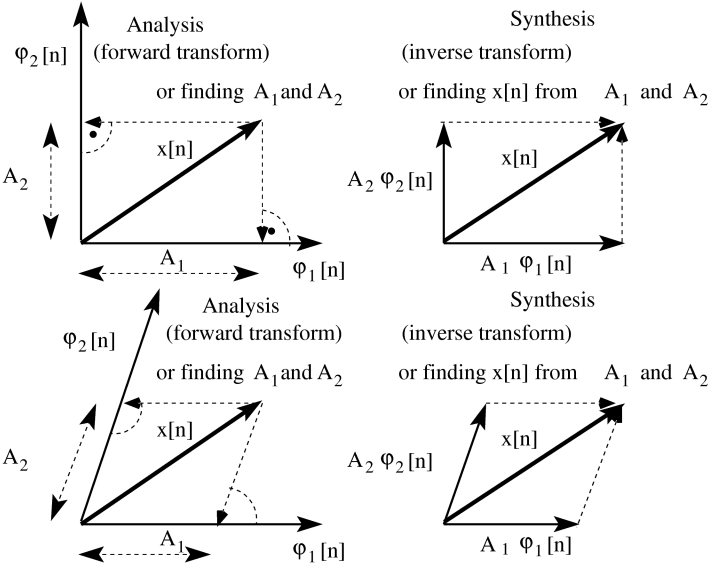
- If the representation by A_k(*expansion*) is unique for a class of functions, the set $\phi_k[n]$ is called a *basis* for this class.
- The forward transform is mathematically a *cast (projection)* onto the basis ϕ_k, and it is calculated with *inner product*, (a.k.a *scalar product*, *dot product*) of a signal with a *dual basis* $\tilde{\phi}_k$ functions

$$A_k = \frac{\langle x[n],\, \tilde{\phi}_k[n] \rangle}{\langle \tilde{\phi}_k[n],\, \tilde{\phi}_k[n] \rangle}$$

  - for an orthogonal transform, $\tilde{\phi}_k = \phi_k$,
  - only sometimes bases are normalized so the denominator $\langle \tilde{\phi}_k[n],\, \tilde{\phi}_k[n] \rangle = 1$

In a Fourier transform, we take the basis representing different frequencies.

Side remark: orthogonal and non-orthogonal basis



This is why we like the orthogonal case ...

Fourier spectrum (Fourier transform – FT) of a limited energy signal

... as we talk about DT signals, it is a DTFT

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty \quad ,(x[n] \in \ell^2)$$

$X(e^{j\theta})$ – a continuous, periodic function.

Fourier spectrum definition:

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})\, e^{jn\theta}\, d\theta$$

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\theta}$$

$\longrightarrow$ inverse transform

Linearity: $ax[n] + by[n] \overset{\mathcal{F}}{\longleftrightarrow} aX(e^{j\theta}) + bY(e^{j\theta})$

Time shift: $x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-jn_0\theta} X(e^{j\theta})$,

Frequency shift: $e^{jn\theta_0} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\theta-\theta_0)})$

Convolution: $x[n] * y[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\theta}) \cdot Y(e^{j\theta})$,

Modulation: $x[n] \cdot y[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\phi}) \cdot Y(e^{j\theta-\phi}) d\phi$

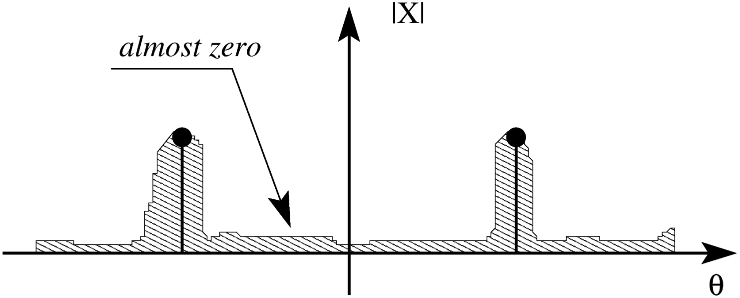
(Parseval's): $E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\theta})|^2 d\theta$

A simple example

A very long piece of sinusoid: $\sin(\theta_x n)$ for $n \in 0, 1, \ldots, N-1$ (or $1/2je^{+jn\theta_x} - 1/2je^{-jn\theta_x}$)
Scalar product of $1/2je^{+jn\theta_x}$ with $e^{-jn\theta}$

- when $\theta = \theta_x$: $\sum_{n=0}^{N-1} 1/2je^{0} = jN/2$
- when $\theta \neq \theta_x$: $\sum_{n=0}^{N-1} 1/2je^{jn\theta_x - \theta} = \ldots$ $jN/2$ times mean value of a complex sine $\longrightarrow$ almost zero

Similar for $-\theta_x$, and we get two strong components in spectrum at $\pm\theta_x$ (plus "almost zero" around – for thorough analysis see next slide)



A full-fledged example

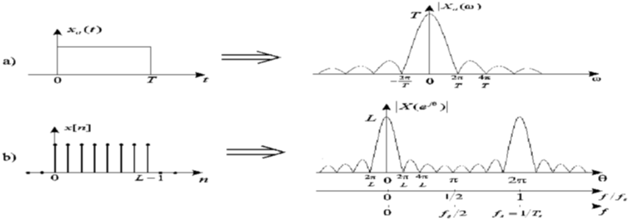
We sample $x_a(t) = \mathrm{rect}(\frac{t-0.5}{T})$ with $T_s = T/L$

$$x_a(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t < T \\ 0 & \text{for} \quad \text{other } t \end{cases} \qquad x[n] = \begin{cases} 1 & \text{for} \quad n = 0, 1, \ldots, L-1 \\ 0 & \text{for} \quad \text{other } n \end{cases}$$

$$X_a(\omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\omega t} \, dt \qquad X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\theta}$$

$$X_a(\omega) = T \, \frac{\sin(\omega T/2)}{\omega T/2} \, e^{-j\omega T/2} \qquad X(e^{j\theta}) = e^{-j(L-1)\theta/2} \, \frac{\sin(L\theta/2)}{\sin(\theta/2)}$$

(hint: $\left(\sum_{n=0}^{N-1} q^n = (1 - q^N)/(1-q) \right)$)



At home: *Repeat calculations for* $x_a = \mathrm{rect}(\frac{t-0.5}{T}) \cdot \cos(\omega t)$*; select* ω *such that an integer number of periods fits in* T*.*

Fourier transform of a periodic (limited mean power) signal

Limited mean power condition: $\frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2 < \infty$
Periodicity with period N $\longrightarrow$ no component that is nonperiodic or periodic with different period.
Conclusion: only N-periodic components (this includes N/k: N/2, N/3,...) $\longrightarrow e^{j2\pi nk/N}$

Fourier spectrum (forwar transform) definition:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N},$$

defined for $-\infty < k < \infty$
but periodic with period N
(e.g. $0 < k < N-1$)

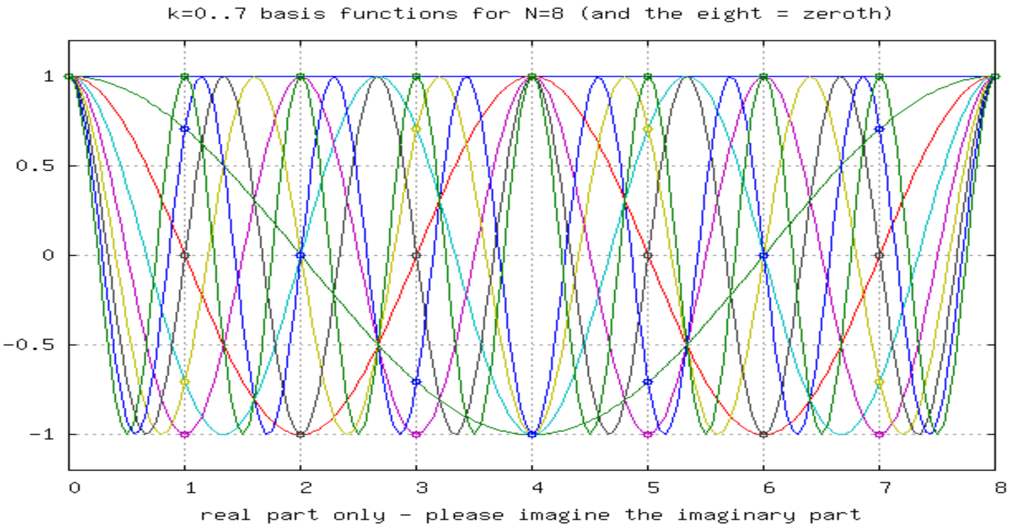
reconstruction $\longrightarrow$ inverse transform

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}$$

defined for $-\infty < n < \infty$
but periodic with period N
(e.g. $0 < n < N-1$)

$\longrightarrow$ We represent $x[n]$ as a sum of N complex cosines with angular frequencies
$\theta_k = \frac{2\pi}{N} \cdot k\,, \quad k = 0,\, 1,\, \ldots,\, N-1$

8 basis functions for N=8 (real part only)

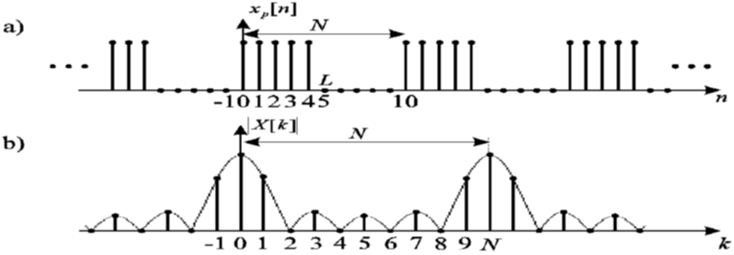


Example

Discrete-time unipolar square wave:
$x_p[n]$ with period $N = 10$ has $L = 5$ nonzero samples $(n = 0, 1, \ldots L-1)$ in each period

$$X(k) = \sum_{n=0}^{N-1} x_p(n) \mathrm{e}^{-\mathrm{j}\,2\pi kn/N} = \sum_{n=0}^{L-1} \mathrm{e}^{-\mathrm{j}\,2\pi kn/N} = \mathrm{e}^{-\mathrm{j}(L-1)\pi k/N} \frac{\sin(L\pi k/N)}{\sin(\pi k/N)}, \qquad k = 0, 1, \ldots$$

The amplitude spectrum $|X[k]| = \left| \frac{\sin(L\theta_{k/2})}{\sin(\theta_{k/2})} \right|, \qquad \theta_k = 2\pi k/N$ is shown



Discrete Fourier Transform (DFT)

It is discrete both ways - forward and inverse!

- A signal $x[n]$ defined for $-\infty < n < \infty$
- Its spectrum $X(e^{j\theta})$ defined for continuous $0 \leq \theta < 2\pi$
- Life is short ...

$\longrightarrow$ Let us take a fragment of $x[n]$: $x_0[n]$, $n = 0, 1, \ldots, N-1$

$$x_0[n] = x[n]g[n], \text{ where } g[n] = \begin{cases} 1 & \text{for} \quad n = 0, 1, \ldots, N-1 \\ 0 & \text{for} \quad \text{others } n \end{cases}$$

$g[n]$ – *window (gate?)* function (here: a *rectangular window*) ($w[n]$ we reserve for *white* noise)

$\longrightarrow$ We take only N values of $\theta_k = \frac{2\pi}{N} k, \quad k = 0, 1, \ldots, N-1$

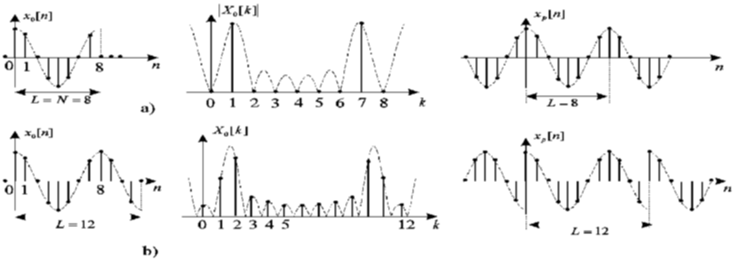
$$X_0\left(e^{j\theta_k}\right) = \sum_{n=0}^{N-1} x_0(n)\, e^{-jn\theta_k} = \sum_{n=0}^{N-1} x_0(n) e^{-j2\pi nk/N}$$

DFT properties

extras w.r.t. FT properties

Orthogonality of basis functions over N samples – (see next slide)
Periodicity As we sample the spectrum, the reconstructed signal is periodic with period N. If we compute IDFT for $-\infty < n < \infty \ldots$

- A non-periodic signal was reconstructed as periodic
- A periodic signal was reconstructed as N-periodic



DFT as an orthogonal transform

An orthogonal transform (e.g. DFT) is a decomposition of a function (signal) on a set of orthogonal basis functions $\phi_k[n]$.

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} A(k)\cdot\phi_k[n]$$

Because of $\phi_k[n]$ orthogonality, $A(k)$ are easy to calculate:

$$A(k) = \sum_{n=0}^{N-1} x(n)\cdot\phi_k^*(n)$$

Basis sequences (transform kernel) have to be orthogonal:

$$\frac{1}{N}\sum_{k=0}^{N-1}\phi_k(n)\cdot\phi_m^*(n) = \begin{cases} 1 & m = k \\ 0 & otherwise \end{cases}$$ Scalar product is zero = orthogonal!

DFT basis functions $\phi_k(n) = \mathrm{e}^{-\mathrm{j}n\theta_k} = \mathrm{e}^{-\mathrm{j}2\pi nk/N}$ are orthogonal – we chose θ_k so that they be!

Inverse DFT

a nice math exercise

Take forward DFT definition as a linear equation set, with $x_0[n]$ as unknowns.

$$X_0(k) = \sum_{n=0}^{N-1} x_0(n) \mathrm{e}^{-\mathrm{j}2\pi nk/N}$$

Inverse DFT

a nice math exercise

Take forward DFT definition as a linear equation set, with $x_0[n]$ as unknowns. When we multiply both sides by $\mathrm{e}^{\mathrm{j}2\pi rk/N}$, $r = 0, 1, \ldots, N-1$ and sum for $k = 0, 1, \ldots, N-1$

$$\sum_{k=0}^{N-1} X_0(k)\, \mathrm{e}^{\mathrm{j}\,2\pi rk/N} = \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_0(n) \mathrm{e}^{-\mathrm{j}\,2\pi nk/N} \right] \mathrm{e}^{\mathrm{j}\,2\pi rk/N} =$$

Will we get $x_0(n)$?

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$$= \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x_0(n)\, \mathrm{e}^{\mathrm{j}\,2\pi k(r-n)/N} = \sum_{n=0}^{N-1} x_0(n) \sum_{k=0}^{N-1} \mathrm{e}^{\mathrm{j}\,2\pi k(r-n)/N}$$

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Will we get $x_0(n)$? Simple! Orthogonality of basis helps – the second sum is mainly 0:

$$\sum_{k=0}^{N-1}\mathrm{e}^{\mathrm{j}\,2\pi k(r-n)/N} = \begin{cases} N, & r = n \\ 0, & r \neq n \end{cases} \Rightarrow \sum_{k=0}^{N-1} X_0(k)\mathrm{e}^{\mathrm{j}\,2\pi r\,k/N} = N\;x_0(r), \quad r = 0, 1, \ldots, N-1$$

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Take forward DFT definition as a linear equation set, with $x_0[n]$ as unknowns. When we multiply both sides by $\mathrm{e}^{\mathrm{j}2\pi rk/N}$, $r = 0, 1, \ldots, N-1$ and sum for $k = 0, 1, \ldots, N-1$

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$$x_0(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_0(k)\mathrm{e}^{\mathrm{j}2\pi nk/N}, \quad n = 0, 1, \ldots, N-1$$

So IDFT is almost the same as DFT – remove minus and scale by $1/N$.

Forward and Inverse DFT - transformation matrix

x and $\mathcal{F}(x)$ are sets of N numbers – i.e. vectors in N-dimensional space.

$$\begin{aligned} \mathcal{F}(x) &= F \cdot x \\ F^{-1} \cdot \mathcal{F}(x) &= F^{-1} \cdot F \cdot x \\ F^{-1} \cdot \mathcal{F}(x) &= x \end{aligned}$$

algebraic trivia:
How to construct F matrix? $F_{kn} = e^{-j2\pi nk/N}$
What is F^{-1}? (not-so-trivial, but see IDFT slide)
Note nice properties of F matrix...

Entertainment maths

When we define $\mathcal{F}$ as mathematicians like it – with factor of $1/\sqrt{N}$ at both forward and inverse transform, then:

- $\mathcal{F}(\mathcal{F}(x))$ gives $x(-n)$
- $\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(x))))$ gives again x
- $\mathcal{F}(\mathcal{F}(\mathcal{F}(x)))$ gives $\mathcal{F}^{-1}(x)$

The same happens with F matrix, of course.

Simple DFT matrices

- $N = 1 \ldots$
- $N = 2 \longrightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- $N = 4 \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$

If you remember this, you may implement DFT with order 1, 2 or 3 without any multiplication (with pencil and paper or with a digital circuit).

Problems to be studied

What happens to the spectrum (and DFT) when

- Signal is inverted in time $x_1(n) = x(-n)$
- Signal is upsampled by factor of two $x_1(2n) = x(n);\ x_1(2n+1) = 0$ (or three, or more)
- Signal is extended with zeros (zero-padded) to a double/triple/... length
- Signal is decimated (downsampled) $x_1(n) = x(2n);$ odd samples $x(2n+1)$ are discarded
- Signal is modulated with $(-1)^n$, with a complex cosine $e^{j\theta_0 n}$, real cosine $cos(\theta_0 n)$ or real sine
- ...

Decimation

When Roman soldiers ran away chased by Spartakus, Marcus Licinius Krassus ordered to kill *one out of each ten* – deci – of them. The morale has risen, and next time they were more eager to be killed in the battlefield. In three years, they defeated Spartakus at the Silarus river.

The decimation factor (defined as $N_{original}/N_{left}$) was $10/9$ in that case – in signal practice the factor is usually 2, 4, or even 2^K