Chapter 2
Wiener Models

Please cite the book:
Maciej Ławryńczuk: Nonlinear Predictive Control Using Wiener Models: Computationally Efficient Approaches for Polynomial and Neural Structures. Studies in Systems, Decision and Control, vol. 389, Springer, Cham, 2022.

Abstract This Chapter is concerned with Wiener models. At first, input-output structures are described: one SISO case and five MIMO ones. Next, state-space models are detailed: one SISO case and two MIMO ones. A short review of identification methods of Wiener models is given, possible internal structures of both model parts are discussed and example applications of Wiener models are reported. Finally, other structures of cascade models are shortly mentioned.

2.1 Structures of Input-Output Wiener Models

For prediction in MPC, i.e. to calculate the quantities $\hat{y}(k+1|k), \ldots, \hat{y}(k+N|k)$ used in the minimised MPC cost-function (1.7) or (1.13), a dynamical model of the process is necessary. In this work, Wiener models are used for this purpose. As far as input-output models are concerned, one SISO structure and as many as five MIMO model configurations are described.

2.1.1 SISO Wiener Model

The structure of the SISO input-output Wiener model [57] is depicted in Fig. 2.1. It consists of a linear dynamic block followed by a nonlinear static one. The linear dynamic part of the model is described by the equation

$$\boldsymbol{A}(q^{-1})v(k) = \boldsymbol{B}(q^{-1})u(k) \tag{2.1}$$

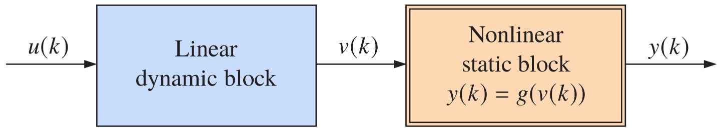

Fig. 2.1 The structure of the SISO Wiener model

where the polynomials are

$$\boldsymbol{A}(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{n_\mathrm{A}} q^{-n_\mathrm{A}} \tag{2.2}$$

$$\boldsymbol{B}(q^{-1}) = b_1 + \ldots + b_{n_\mathrm{B}} q^{-n_\mathrm{B}} \tag{2.3}$$

The auxiliary signal v is the output of the first block and the input of the second block. All signals u, v and y are scalars. The backward shift operator (the unit time delay) is denoted by q^{-1}, the integers n_A and n_B define the order of dynamics, the constant parameters of the linear dynamic part are denoted by the real numbers a_j ($j = 1, \ldots, n_\mathrm{A}$) and b_j ($j = 1, \ldots, n_\mathrm{B}$). From Eqs. (2.1), (2.2)-(2.3), the output of the linear part of the model is

$$v(k) = \sum_{i=1}^{n_\mathrm{B}} b_i u(k-i) - \sum_{i=1}^{n_\mathrm{A}} a_i v(k-i) \tag{2.4}$$

The nonlinear static part of the model is described by the general equation

$$y(k) = g(v(k)) \tag{2.5}$$

where the function $g \colon \mathbb{R} \to \mathbb{R}$ is required to be differentiable (for implementation of the computationally efficient nonlinear MPC algorithms described in Chapter 3). It means that polynomials, neural networks, fuzzy systems (with differentiable membership functions) or Support Vector Machines (SVM) may be used in the second model block. The output of the SISO Wiener model can be explicitly expressed as a function of the input signal and the auxiliary signal of the model at some previous sampling instants. Taking into account Eqs. (2.4) and (2.5), we obtain

$$y(k) = g\left(\sum_{i=1}^{n_\mathrm{B}} b_i u(k-i) - \sum_{i=1}^{n_\mathrm{A}} a_i v(k-i)\right) \tag{2.6}$$

Let us stress that the signal v is used in the model, but in general, we assume that it does not exist in the process. Hence, measurement of that signal is impossible, but its value may be assessed from the model for the current operating point of the process. Similarly, predictions of the signals v over the prediction horizon may also be computed.

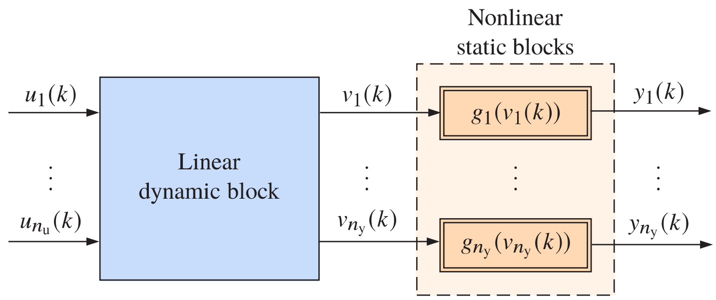

Fig. 2.2 The structure of the MIMO Wiener model I

2.1.2 MIMO Wiener Model I

The structure of the MIMO Wiener model I [57] is depicted in Fig. 2.2. It consists of one linear dynamic MIMO block and n_y SISO nonlinear static ones. The linear dynamic part of the model is described by Eq. (2.1) but now $u \in \mathbb{R}^{n_u}$ and $v \in \mathbb{R}^{n_y}$. Because SISO nonlinear static blocks are used, $n_v = n_y$. The polynomial model matrices are

$$\boldsymbol{A}(q^{-1}) = \begin{bmatrix} 1 + a_1^1 q^{-1} + \ldots + a_{n_A}^1 q^{-n_A} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 + a_1^{n_y} q^{-1} + \ldots + a_{n_A}^{n_y} q^{-n_A} \end{bmatrix} \tag{2.7}$$

$$\boldsymbol{B}(q^{-1}) = \begin{bmatrix} b_1^{1,1} q^{-1} + \ldots + b_{n_B}^{1,1} q^{-n_B} & \ldots & b_1^{1,n_u} q^{-1} + \ldots + b_{n_B}^{1,n_u} q^{-n_B} \\ \vdots & \ddots & \vdots \\ b_1^{n_y,1} q^{-1} + \ldots + b_{n_B}^{n_y,1} q^{-n_B} & \ldots & b_1^{n_y,n_u} q^{-1} + \ldots + b_{n_B}^{n_y,n_u} q^{-n_B} \end{bmatrix} \tag{2.8}$$

The constant parameters of the linear dynamic part are denoted by the real numbers a_j^m ($j = 1, \ldots, n_A$, $m = 1, \ldots, n_y$) and $b_j^{m,n}$ ($j = 1, \ldots, n_B$, $m = 1, \ldots, n_y$, $n = 1, \ldots, n_u$). From Eqs. (2.1) and (2.7)-(2.8), we can calculate the consecutive outputs of the linear dynamic part of the model

$$v_1(k) = \sum_{n=1}^{n_u} \sum_{i=1}^{n_B} b_i^{1,n} u_n(k-i) - \sum_{i=1}^{n_A} a_i^1 v_1(k-i) \tag{2.9}$$

$$\vdots$$

$$v_{n_y}(k) = \sum_{n=1}^{n_u} \sum_{i=1}^{n_B} b_i^{n_y,n} u_n(k-i) - \sum_{i=1}^{n_A} a_i^{n_y} v_{n_y}(k-i) \tag{2.10}$$

which may be compactly expressed as

$$v_m(k) = \sum_{n=1}^{n_{\mathrm{u}}} \sum_{i=1}^{n_{\mathrm{B}}} b_i^{m,n} u_n(k-i) - \sum_{i=1}^{n_{\mathrm{A}}} a_i^m v_m(k-i),\ m = 1, \ldots, n_{\mathrm{y}} \tag{2.11}$$

The nonlinear static parts of the model are described by the general equations

$$y_1(k) = g_1(v_1(k)) \tag{2.12}$$

$$\vdots$$

$$y_{n_{\mathrm{y}}}(k) = g_{n_{\mathrm{y}}}(v_{n_{\mathrm{y}}}(k)) \tag{2.13}$$

which may be compactly expressed as

$$y_m(k) = g_m(v_m(k)),\ m = 1, \ldots, n_{\mathrm{y}} \tag{2.14}$$

where the functions $g_m \colon \mathbb{R} \to \mathbb{R}$ are required to be differentiable. From Eqs. (2.9)-(2.10) and (2.12)-(2.13), we obtain model outputs

$$y_1(k) = g_1\left(\sum_{n=1}^{n_{\mathrm{u}}} \sum_{i=1}^{n_{\mathrm{B}}} b_i^{1,n} u_n(k-i) - \sum_{i=1}^{n_{\mathrm{A}}} a_i^1 v_1(k-i)\right) \tag{2.15}$$

$$\vdots$$

$$y_{n_{\mathrm{y}}}(k) = g_{n_{\mathrm{y}}}\left(\sum_{n=1}^{n_{\mathrm{u}}} \sum_{i=1}^{n_{\mathrm{B}}} b_i^{n_{\mathrm{y}},n} u_n(k-i) - \sum_{i=1}^{n_{\mathrm{A}}} a_i^{n_{\mathrm{y}}} v_{n_{\mathrm{y}}}(k-i)\right) \tag{2.16}$$

which may be compactly expressed as

$$y_m(k) = g_m\left(\sum_{n=1}^{n_{\mathrm{u}}} \sum_{i=1}^{n_{\mathrm{B}}} b_i^{m,n} u_n(k-i) - \sum_{i=1}^{n_{\mathrm{A}}} a_i^m v_m(k-i)\right),\ m = 1, \ldots, n_{\mathrm{y}} \tag{2.17}$$

2.1.3 MIMO Wiener Model II

The structure of the MIMO Wiener model II is depicted in Fig. 2.3. Similarly to the MIMO Wiener model I shown in Fig. 2.2, it consists of one linear dynamic MIMO block and n_{y} static ones. On the other hand, there are two important differences. Firstly, the number of auxiliary signals between two model parts (n_{v}) may be, in general, different from the number of outputs (n_{y}). The number of auxiliary signals may be treated as an additional model parameter, but it is straightforward to choose $n_{\mathrm{v}} = n_{\mathrm{y}}$. Secondly, in the MIMO Wiener model II, the nonlinear static blocks are of the Multiple-Input Single-Output (MISO) type, each of them has n_{v} inputs and

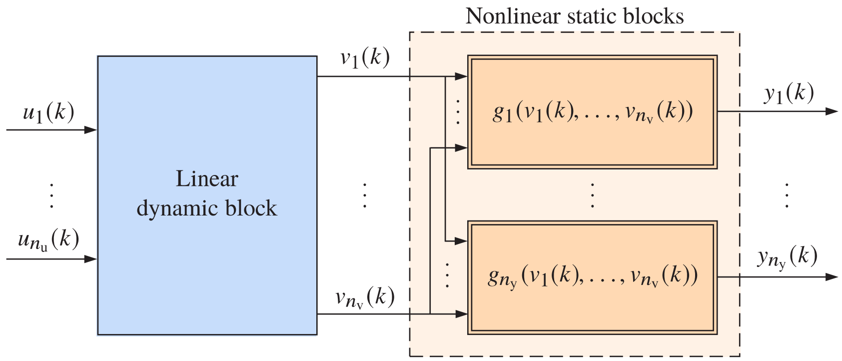

Fig. 2.3 The structure of the MIMO Wiener model II

one output. The linear dynamic part of the model is described by Eq. (2.1) but now $u \in \mathbb{R}^{n_u}$ and $v \in \mathbb{R}^{n_v}$. The polynomial model matrices are

$$\boldsymbol{A}(q^{-1}) = \begin{bmatrix} 1 + a_1^1 q^{-1} + \ldots + a_{n_A}^1 q^{-n_A} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 + a_1^{n_v} q^{-1} + \ldots + a_{n_A}^{n_v} q^{-n_A} \end{bmatrix} \tag{2.18}$$

$$\boldsymbol{B}(q^{-1}) = \begin{bmatrix} b_1^{1,1} q^{-1} + \ldots + b_{n_B}^{1,1} q^{-n_B} & \ldots & b_1^{1,n_u} q^{-1} + \ldots + b_{n_B}^{1,n_u} q^{-n_B} \\ \vdots & \ddots & \vdots \\ b_1^{n_v,1} q^{-1} + \ldots + b_{n_B}^{n_v,1} q^{-n_B} & \ldots & b_1^{n_v,n_u} q^{-1} + \ldots + b_{n_B}^{n_v,n_u} q^{-n_B} \end{bmatrix} \tag{2.19}$$

where the constant parameters of the linear dynamic part are denoted by the real numbers a_j^m ($j = 1, \ldots, n_A$, $m = 1, \ldots, n_v$) and $b_j^{m,n}$ ($j = 1, \ldots, n_B$, $m = 1, \ldots, n_v$, $n = 1, \ldots, n_u$). Taking into account Eqs. (2.1), (2.18)-(2.19), the consecutive outputs of the linear dynamic part of the model are calculated from

$$v_1(k) = \sum_{n=1}^{n_u} \sum_{i=1}^{n_B} b_i^{1,n} u_n(k-i) - \sum_{i=1}^{n_A} a_i^1 v_1(k-i) \tag{2.20}$$

$$\vdots$$

$$v_{n_v}(k) = \sum_{n=1}^{n_u} \sum_{i=1}^{n_B} b_i^{n_v,n} u_n(k-i) - \sum_{i=1}^{n_A} a_i^{n_v} v_{n_v}(k-i) \tag{2.21}$$

which may be compactly expressed as

$$v_m(k) = \sum_{n=1}^{n_u} \sum_{i=1}^{n_B} b_i^{m,n} u_n(k-i) - \sum_{i=1}^{n_A} a_i^m v_m(k-i), \ m = 1, \ldots, n_v \tag{2.22}$$

The nonlinear static parts of the model are described by the general equations

$$y_1(k) = g_1(v_1(k), \ldots, v_{n_\mathrm{v}}(k)) \tag{2.23}$$

$$\vdots$$

$$y_{n_\mathrm{y}}(k) = g_{n_\mathrm{y}}(v_1(k), \ldots, v_{n_\mathrm{v}}(k)) \tag{2.24}$$

which may be compactly expressed as

$$y_m(k) = g_m(v_1(k), \ldots, v_{n_\mathrm{v}}(k)),\ m = 1, \ldots, n_\mathrm{y} \tag{2.25}$$

where the functions $g_m \colon \mathbb{R}^{n_\mathrm{v}} \rightarrow \mathbb{R}$ are required to be differentiable. From Eqs. (2.20)-(2.21), (2.23)-(2.24), we obtain model outputs

$$\begin{aligned} y_1(k) = g_1\Bigg(&\sum_{n=1}^{n_\mathrm{u}} \sum_{i=1}^{n_\mathrm{B}} b_i^{1,n} u_n(k-i) - \sum_{i=1}^{n_\mathrm{A}} a_i^1 v_1(k-i), \ldots, \\ &\sum_{n=1}^{n_\mathrm{u}} \sum_{i=1}^{n_\mathrm{B}} b_i^{n_\mathrm{v},n} u_n(k-i) - \sum_{i=1}^{n_\mathrm{A}} a_i^{n_\mathrm{v}} v_{n_\mathrm{v}}(k-i)\Bigg) \end{aligned} \tag{2.26}$$

$$\vdots$$

$$\begin{aligned} y_{n_\mathrm{y}}(k) = g_{n_\mathrm{y}}\Bigg(&\sum_{n=1}^{n_\mathrm{u}} \sum_{i=1}^{n_\mathrm{B}} b_i^{1,n} u_n(k-i) - \sum_{i=1}^{n_\mathrm{A}} a_i^1 v_1(k-i), \ldots, \\ &\sum_{n=1}^{n_\mathrm{u}} \sum_{i=1}^{n_\mathrm{B}} b_i^{n_\mathrm{v},n} u_n(k-i) - \sum_{i=1}^{n_\mathrm{A}} a_i^{n_\mathrm{v}} v_{n_\mathrm{v}}(k-i)\Bigg) \end{aligned} \tag{2.27}$$

which may be compactly expressed as

$$\begin{aligned} y_m(k) = g_m\Bigg(&\sum_{n=1}^{n_\mathrm{u}} \sum_{i=1}^{n_\mathrm{B}} b_i^{1,n} u_n(k-i) - \sum_{i=1}^{n_\mathrm{A}} a_i^1 v_1(k-i), \ldots, \\ &\sum_{n=1}^{n_\mathrm{u}} \sum_{i=1}^{n_\mathrm{B}} b_i^{n_\mathrm{v},n} u_n(k-i) - \sum_{i=1}^{n_\mathrm{A}} a_i^{n_\mathrm{v}} v_{n_\mathrm{v}}(k-i)\Bigg),\ m = 1, \ldots, n_\mathrm{y} \end{aligned} \tag{2.28}$$

2.1.4 MIMO Wiener Model III

The structure of the MIMO Wiener model III is depicted in Fig. 2.4. In general, similarly to the MIMO Wiener model I, it consists of n_y SISO nonlinear static blocks defined by Eq. (2.14), but the linear dynamic part of the model is different, it is not represented by one MIMO block defined by Eq. (2.1). As a result of a model identification procedure or from fundamental knowledge of the process, the transfer functions of the consecutive input-output channels are typically found. They

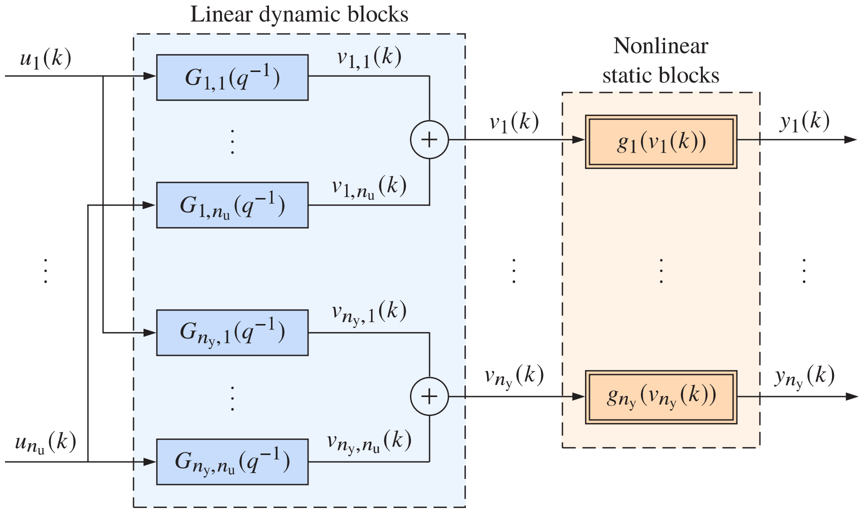

Fig. 2.4 The structure of the MIMO Wiener model III

comprise the linear dynamic part of the Wiener model. The first block of the model is described by the array of transfer functions

$$\begin{bmatrix} v_1(k) \\ \vdots \\ v_{n_\mathrm{y}}(k) \end{bmatrix} = \begin{bmatrix} G_{1,1}(q^{-1}) & \ldots & G_{1,n_\mathrm{u}}(q^{-1}) \\ \vdots & \ddots & \vdots \\ G_{n_\mathrm{y},1}(q^{-1}) & \ldots & G_{n_\mathrm{y},n_\mathrm{u}}(q^{-1}) \end{bmatrix} \begin{bmatrix} u_1(k) \\ \vdots \\ u_{n_\mathrm{u}}(k) \end{bmatrix} \tag{2.29}$$

The transfer functions have the general form

$$G_{m,n}(q^{-1}) = \frac{N_{m,n}(q^{-1})}{D_{m,n}(q^{-1})} \tag{2.30}$$

for all inputs and outputs of the linear dynamic block, i.e. for $m = 1, \ldots, n_\mathrm{y}$, $n = 1, \ldots, n_\mathrm{u}$. The numerators and the denominators of the transfer functions (2.30) are polynomials

$$N_{m,n}(q^{-1}) = b_1^{m,n} q^{-1} + \ldots + b_{n_\mathrm{B}^{m,n}}^{m,n} q^{-n_\mathrm{B}^{m,n}} \tag{2.31}$$

$$D_{m,n}(q^{-1}) = 1 + a_1^{m,n} q^{-1} + \ldots + a_{n_\mathrm{A}^{m,n}}^{m,n} q^{-n_\mathrm{A}^{m,n}} \tag{2.32}$$

The integer numbers $n_\mathrm{A}^{m,n}$ and $n_\mathrm{B}^{m,n}$ denote the order of dynamics of the consecutive denominators and nominators, respectively. Let us stress the fact that order of dynamics of the consecutive transfer functions (2.30) may be different. In the MIMO Wiener model I, all input-output channels have the same order of dynamics, defined by n_A and n_B.

The MIMO Wiener model I is usually considered in the literature [57, 74]. Even though, initially, we may have the simple rudimentary model comprised of SISO transfer functions as in Eq. (2.29), it is transformed to the MIMO Wiener model I. More specifically, the linear dynamic block of the model III is transformed. From Eq. (2.29) and Fig. 2.4, we have

$$v_1(k) = \sum_{i=1}^{n_u} G_{1,i}(q^{-1})u_i(k) \tag{2.33}$$

$$\vdots$$

$$v_{n_y}(k) = \sum_{i=1}^{n_u} G_{n_y,i}(q^{-1})u_i(k) \tag{2.34}$$

Taking into account Eq. (2.30), the linear part of the model (2.33)-(2.34) becomes

$$v_1(k) = \sum_{i=1}^{n_u} \frac{N_{1,i}(q^{-1})}{D_{1,i}(q^{-1})}u_i(k) \tag{2.35}$$

$$\vdots$$

$$v_{n_y}(k) = \sum_{i=1}^{n_u} \frac{N_{n_y,i}(q^{-1})}{D_{n_y,i}(q^{-1})}u_i(k) \tag{2.36}$$

Multiplying the consecutive equations of the linear block (2.35)-(2.36) by the common denominators $\prod_{i=1}^{n_u} D_{1,i}(q^{-1}), \ldots, \prod_{i=1}^{n_u} D_{n_y,i}(q^{-1})$, respectively, we obtain

$$\prod_{i=1}^{n_u} D_{1,i}(q^{-1})v_1(k) = \sum_{j=1}^{n_u} N_{1,j}(q^{-1}) \prod_{\substack{i=1 \\ i \neq j}}^{n_u} D_{1,i}(q^{-1})u_j(k) \tag{2.37}$$

$$\vdots$$

$$\prod_{i=1}^{n_u} D_{n_y,i}(q^{-1})v_{n_y}(k) = \sum_{j=1}^{n_u} N_{n_y,j}(q^{-1}) \prod_{\substack{i=1 \\ i \neq j}}^{n_u} D_{n_y,i}(q^{-1})u_j(k) \tag{2.38}$$

Equations (2.37)-(2.38) may be rewritten in such a way that we obtain the linear dynamic block used in the MIMO Wiener model I (Eq. (2.1)) where the entries of the matrices $\boldsymbol{A}(q^{-1})$ and $\boldsymbol{B}(q^{-1})$ (Eqs. (2.7)-(2.8)) are

$$A_{1,1}(q^{-1}) = \prod_{i=1}^{n_\mathrm{u}} D_{1,i}(q^{-1}) \tag{2.39}$$

$$\vdots$$

$$A_{n_\mathrm{y},n_\mathrm{y}}(q^{-1}) = \prod_{i=1}^{n_\mathrm{u}} D_{n_\mathrm{y},i}(q^{-1}) \tag{2.40}$$

and

$$B_{1,1}(q^{-1}) = N_{1,1}(q^{-1}) \prod_{i=2}^{n_\mathrm{u}} D_{1,i}(q^{-1}) \tag{2.41}$$

$$\vdots$$

$$B_{n_\mathrm{y},n_\mathrm{u}}(q^{-1}) = N_{n_\mathrm{y},n_\mathrm{u}}(q^{-1}) \prod_{i=1}^{n_\mathrm{u}-1} D_{n_\mathrm{y},i}(q^{-1}) \tag{2.42}$$

As a result of multiplication in Eqs. (2.39)-(2.40) and (2.41)-(2.42), the linear part of the classical MIMO block (2.1) used in the MIMO Wiener model I is likely to be of a high-order, even though the transfer functions (2.30) used in the rudimentary MIMO Wiener model III are of a low order. As it is demonstrated in Chapter 4.5, it may lead to serious numerical problems and make predictive control difficult or completely impossible. Hence, when the process has really multiple inputs and outputs, it is strongly recommended to use the MIMO Wiener model III, not the classical model I.

In the case of the MIMO Wiener model III, in order to explicitly express model outputs as functions of the input signals of the process and the auxiliary signals of the model at some previous sampling instants, we use Fig. 2.4, Eqs. (2.29) and (2.30) which give

$$v_{1,1}(k) = G_{1,1}(q^{-1})u_1(k) = \frac{N_{1,1}(q^{-1})}{D_{1,1}(q^{-1})}u_1(k) \tag{2.43}$$

$$\vdots$$

$$v_{n_\mathrm{y},n_\mathrm{u}}(k) = G_{n_\mathrm{y},n_\mathrm{u}}(q^{-1})u_{n_\mathrm{u}}(k) = \frac{N_{n_\mathrm{y},n_\mathrm{u}}(q^{-1})}{D_{n_\mathrm{y},n_\mathrm{u}}(q^{-1})}u_{n_\mathrm{u}}(k) \tag{2.44}$$

Taking into account Eqs. (2.31)-(2.32), we obtain

$$v_{1,1}(k) = \sum_{i=1}^{n_{\mathrm{B}}^{1,1}} b_i^{1,1} u_1(k-i) - \sum_{i=1}^{n_{\mathrm{A}}^{1,1}} a_i^{1,1} v_{1,1}(k-i) \tag{2.45}$$

$$\vdots$$

$$v_{n_{\mathrm{y}},n_{\mathrm{u}}}(k) = \sum_{i=1}^{n_{\mathrm{B}}^{n_{\mathrm{y}},n_{\mathrm{u}}}} b_i^{n_{\mathrm{y}},n_{\mathrm{u}}} u_{n_{\mathrm{u}}}(k-i) - \sum_{i=1}^{n_{\mathrm{A}}^{n_{\mathrm{y}},n_{\mathrm{u}}}} a_i^{n_{\mathrm{y}},n_{\mathrm{u}}} v_{n_{\mathrm{y}},n_{\mathrm{u}}}(k-i) \tag{2.46}$$

Eqs. (2.45)-(2.46) may be rewritten in the compact form

$$v_{m,n}(k) = \sum_{i=1}^{n_{\mathrm{B}}^{m,n}} b_i^{m,n} u_n(k-i) - \sum_{i=1}^{n_{\mathrm{A}}^{m,n}} a_i^{m,n} v_{m,n}(k-i),\ m = 1, \ldots, n_{\mathrm{y}},\ n = 1, \ldots, n_{\mathrm{u}} \tag{2.47}$$

From Fig. 2.4, we have

$$v_1(k) = \sum_{i=1}^{n_{\mathrm{u}}} v_{1,n}(k) \tag{2.48}$$

$$\vdots$$

$$v_{n_{\mathrm{y}}}(k) = \sum_{i=1}^{n_{\mathrm{u}}} v_{n_{\mathrm{y}},n}(k) \tag{2.49}$$

which may be rewritten compactly

$$v_m(k) = \sum_{i=1}^{n_{\mathrm{u}}} v_{m,n}(k),\ m = 1, \ldots, n_{\mathrm{y}} \tag{2.50}$$

Using Eqs. (2.14), (2.48)-(2.49), the consecutive model outputs are

$$y_1(k) = g_1\left(\sum_{n=1}^{n_{\mathrm{u}}} v_{1,n}(k)\right) \tag{2.51}$$

$$\vdots$$

$$y_{n_{\mathrm{y}}}(k) = g_{n_{\mathrm{y}}}\left(\sum_{n=1}^{n_{\mathrm{u}}} v_{n_{\mathrm{y}},n}(k)\right) \tag{2.52}$$

which may be rewritten compactly

$$y_m(k) = g_m\left(\sum_{n=1}^{n_{\mathrm{u}}} v_{m,n}(k)\right),\ m = 1, \ldots, n_{\mathrm{y}} \tag{2.53}$$