EDRP lecture 8. Poisson process and its generalizations

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Digression on order statistics

Definition
Let \( \{X_1, \ldots, X_n\} \) be a simple random sample. Random vector \( \{Y_1, \ldots, Y_n\} \) satisfying conditions: \( Y_1 \leq Y_2 \leq \ldots \leq Y_n \) with probability 1 and defined in the following way:

\[
Y_i = \text{ the } i\text{-th (with respect to the magnitude) value of } \{X_1, \ldots, X_n\}
\]

will be called vector of order statistics of the vector \( \{X_1, \ldots, X_n\} \)

Fact
Coordinates of the vector \( \{Y_1, \ldots, Y_n\} \) will be denoted traditionally in the following way \( \{X_{1:n}, \ldots, X_{n:n}\} \).

We need also the following two simple lemmas:
Lemma

If the random sample \( \{X_1, \ldots, X_n\} \) is simple (i.e. random variables \( \{X_i\}_{i=1}^n \) are independent with identical distributions) and the density of \( X_1 \) is equal \( f(x) \), then the joint density of order statistics \( \{X_{1:n}, \ldots, X_{n:n}\} \) is equal:

\[
g(y_1, \ldots, y_n) = n! \prod_{i=1}^{n} f(y_i),
\]

for \( y_1 \leq y_2 \leq \ldots \leq y_n \).

Proof.

Let us notice, that in order to observe \( X_{1:n} = y_1, \ldots, X_{n:n} = y_n \) any of \( n! \) permutations \( \{X_{(1)}, \ldots, X_{(n)}\} \) of variables \( \{X_1, \ldots, X_n\} \) has to assume values \( \{y_1, \ldots, y_n\} \). \( \square \)
Proof. 
And moreover

\[
P \left( X_{(1)} \in (y_1, y_1 + dy_1), \ldots, X_{(n)} \in (y_n, y_n + dy_n) \right)
\]

\[
\approx \prod_{i=1}^{n} f(y_i) \, dy_1, \ldots, dy_n.
\]
Let random variables $S_1, \ldots, S_n$ be i.i.d and have uniform distribution $U(0, t)$ on the interval $(0, t)$. Then joint distribution of the vector of order statistics $(S_{1:n}, \ldots, S_{n:n})$ has the following density:

$$g(y_1, \ldots, y_n) = \frac{n!}{t^n},$$

(1)

for $y_1 \leq y_2 \leq \ldots \leq y_n$. 

Digression on order statistics
Theorem

Under condition that there were exactly $n$ call on the interval $<0,t>$ moments of successive calls $S_1, \ldots, S_n$ are distributed as the order statistics of $n$ i.i.d. random variables drawn from uniform distribution on $<0,t>$, i.e.

$$f_{S_1,\ldots,S_n}(s_1, \ldots, s_n) = \frac{n!}{t^n}; \text{ for } 0 < s_1 < \ldots, < s_n < t.$$ 

Fact

This result can be expressed also in the following way: Given that there were $n$ calls, moments of calls considered as unordered random variables are independent and have the same uniform distributions on the interval $<0,t>$. 
Proof.
In order to get joint density of the vector \((S_1, \ldots, S_n)\) under condition \(N(t) = n\) let us notice, that for \(0 < s_1 < \ldots, < s_n < t\) an event \(S_1 = s_1, S_2 = s_2, \ldots, S_n = s_n, N(t) = n\) means, that we have for the first \(n+1\) interarrival times \(T_1 = s_1, T_2 = s_2 - s_2, \ldots, T_n = s_n - s_{n-1}, T_{n+1} > t - s_n\). Hence making use of independence of interarrival times we have:

\[
f(s_1, \ldots, s_n | n) = \frac{P(T_1 = s_1, \ldots, T_n = s_n - s_{n-1}, T_n > t - s_n)}{P(N(t) = n)} = \frac{\lambda e^{-\lambda s_1} \lambda e^{-\lambda(s_2-s_1)} \ldots \lambda e^{-\lambda(s_n-s_{n-1})} e^{-\lambda(t-s_n)}}{e^{-\lambda t} (\lambda t)^n / n!} = \frac{n!}{t^n}.
\]
The above mentioned theorem is used to generalize proposition on summing up Poisson processes. Let us assume now, that upon the arrival every event is classified as being of one of $k$ types, and that the probability that an event is classified as type $i$ event, $i = 1, \ldots, k$, depends on the time the event occurs. Specifically, suppose that if an event occurs at time $y$ then it will be classified as type $i$ event, independently of anything that has previously occurred, with probability $P_i(y)$, $i = 1, \ldots, k$ where $\sum_{i=1}^{k} P_i(y) = 1$. We can prove the following useful Theorem:
Theorem
Let $N_t^i; t \geq 0, i = 1, \ldots, k$ denotes number elements of type $i$ that arrived by the time $t$ i.e. that occurred in $<0, t>$. Then $N_t^i$ are independent Poisson random variables with expectations respectively:

$$EN_t^i = \lambda \int_0^t P_i(s) \, ds.$$
Proof.
Let us compute joint probability $P \left( N^1_t = n_1, \ldots, N^k_t = n_k \right)$. To do so first note that in order for there to have been $n_i$ type $i$ events for $i = 1, \ldots, k$ there must have been a total of $\sum_{i=1}^{k} n_i$ events. Hence, conditioning on $N(t)$ yields

$$P \left( N^1_t = n_1, \ldots, N^k_t = n_k \right)$$

$$= P \left( N^1_t = n_1, \ldots, N^k_t = n_k \mid N_t = \sum_{i=1}^{k} n_i \right) \times P \left( N_t = \sum_{i=1}^{k} n_i \right).$$

Now consider an arbitrary call, that occurred in the interval $(0, t \rangle$. If it had occurred at time $s$, then the probability that it would be a type $i$ event would be $P_i(s)$. 

\[\square\]
Proof.
Hence since by appropriate Theorem this event will have occurred at some time uniformly distributed on \((0, t)\), it follows that the probability that this event will be a type \(i\) event is

\[
p_i = \frac{1}{t} \int_0^t P_i(s) \, ds,
\]

independently of the other events. \(\square\)
Proof.

Hence

\[
P \left( N_t^1 = n_1, \ldots, N_t^k = n_k \mid N_t = \sum_{i=1}^{k} n_i \right) \]

will just equal the multinomial probability of \( n_i \) type \( i \) outcomes for \( i = 1, \ldots, k \) when each of \( \sum_{i=1}^{k} n_i \) independent trials results in outcome \( i \) with probability \( p_1, \ldots, p_k \). That is

\[
P \left( N_t^1 = n_1, \ldots, N_t^k = n_k \mid N_t = \sum_{i=1}^{k} n_i \right) = \frac{\left( \sum_{i=1}^{k} n_i \right)!}{\prod_{i=1}^{k} n_i!} \prod_{i=1}^{k} p_i^{n_i}
\]
Proof.
Consequently

\[
P \left( N_t^1 = n_1, \ldots, N_t^k = n_k \right)
\]

\[
= \frac{\left( \sum_{i=1}^k n_i \right)!}{\prod_{i=1}^k n_i!} \prod_{i=1}^k p_i^{n_i} e^{-\lambda t} \frac{(\lambda t)^{\sum_{i=1}^k n_i}}{(\sum_{i=1}^k n_i)!}
\]

\[
= \prod_{i=1}^k \left[ e^{-\lambda t p_i} \frac{(\lambda t p_i)^{n_i}}{n_i!} \right].
\]
Example

(Minimizing the Number of Encounters): Suppose that cars enter a one-way highway in accordance with a Poisson process with rate $\lambda$. The cars enter at point $a$ and depart at point $b$. Each car travels at a constant speed that is randomly determined, independently from car to car, from the distribution $G$. When a faster car encounters a slower one, it passes it with no time being lost. If your car enters the highway at time $s$ and you are able to choose your speed, what speed minimizes the expected number of encounters you will have with other cars, where we say that an encounter occurs each time your car either passes or is passed by another car?
Solution
We will show that for large $s$ the speed that minimizes the expected number of encounters is the median of the speed distribution $G$. To see this, suppose that the speed $x$ is chosen. Let $d = b - a$ denote the length of the road. Upon choosing the speed $x$, it follows that your car will enter the road at time $s$ and will depart at time $s + t_0$, where $t_0 = d / x$ is the travel time.

Now, the other cars enter the road according to a Poisson process with rate $\lambda$. Each of them chooses a speed $X$ according to the distribution $G$, and this results in a travel time $T = d / X$. Let $F$ denote the distribution of travel time $T$. That is,

$$F(t) = P(T < t) = P(d/X < t) = P(X > d/t) = \bar{G}(d/t).$$

(where we denoted by $\bar{F}(t) = 1 - F(t)$ a tail of a distribution $F$)
Solution

Let us say that an event occurs at time $t$ if a car enters the highway at time $t$. Also, say that the event is a type 1 event if it results in an encounter with your car. Now, your car will enter the road at time $s$ and will exit at time $s + t_0$. Hence, a car will encounter your car if it enters before $s$ and exits after $s + t_0$ (in which case your car will pass it on the road) or if it enters after $s$ but exits before $s + t_0$ (in which case it will pass your car). As a result, a car that enters the road at time $t$ will encounter your car if its travel time $T$ is such that

$$t + T > s + t_0 \quad \text{if} \quad t < s$$

$$t + T < s + t_0 \quad \text{if} \quad s < t < s + t_0$$
Solution

From the preceding, we see that an event at time $t$ will, independently of other events, be a type 1 event with probability $p(t)$ given by

$$p(t) = \begin{cases} 
  P(t + T > s + t_0) = \bar{F}(s + t_0 - t) & \text{if } t < s \\
  P(t + T < s + t_0) = F(s + t_0 - t) & \text{if } s < t < s + t_0 \\
  0 & \text{if } t > s > t_0 
\end{cases}$$

Since events (that is, cars entering the road) are occurring according to Poisson process it follows, upon applying above mentioned Theorem, that the total number of type 1 events that ever occurs is Poisson with mean

$$\lambda \int_{0}^{\infty} p(t) \, dt = \lambda \int_{0}^{s} \bar{F}(s + t_0 - t) \, dt + \lambda \int_{0}^{s + t_0} F(s + t_0 - t) \, dt$$

$$= \lambda \int_{t_0}^{s + t_0} \bar{F}(y) \, dy + \lambda \int_{0}^{t_0} F(y) \, dy$$
Solution

To choose the value of $t_0$ that minimizes the preceding quantity, we differentiate. This gives

$$\frac{d}{dt_0} \left( \lambda \int_{t_0}^{s+t_0} \bar{F}(y) \, dy + \lambda \int_0^{t_0} F(y) \, dy \right) = \lambda (\bar{F}(s+t_0) - \bar{F}(t_0) + F(t_0))$$

Setting this equal to 0, and using that $\bar{F}(s+t_0) \equiv 0$ when $s$ is large, we see that the optimal travel time $t_0$ is such that

$$F(t_0) = \bar{F}(t_0) = 1 - F(t_0)$$

or

$$F(t_0) = \frac{1}{2}.$$
Solution

That is the optimal travel time is the median of the travel time distribution. Since the speed $X$ is equal to the distance $d$ divided by the travel time $T$, it follows that the optimal speed $x_0 = d/t_0$ is such $F(d/x_0) = \frac{1}{2}$. Since

$$F(d/x_0) = \bar{G}(x_0)$$

we see that $G(x_0) = \frac{1}{2}$, and so the optimal speed is the median of the distribution of speeds.

Solution

Summing up, we have shown that for any speed $x$ the number of encounters with other cars will be a Poisson random variable, and the mean of this Poisson will be smallest when the speed $s$ is taken to be the median of the distribution $G$. 
Nonhomogeneous Poisson process

**Definition**
A counting process \( \mathcal{N} = \{ N_t; t \geq 0 \} \) will be called nonhomogeneous Poisson process with intensity function \( \lambda (t), t \geq 0 \), if

i) \( N_0 = 0 \),

ii) \( \mathcal{N} \) has independent increments

iii) \( P (N_{t+h} - N_t \geq 2) = o_t(h) \)

iv) \( P (N_{t+h} - N_t = 1) = \lambda (t) h + o_1 (h) \).
Lemma
If we let \( m_t = \int_0^t \lambda(y) \, dy \), then we will show that

\[
P(N_{t+s} - N_t = n) = e^{-(m_{t+s} - m_t)} \frac{(m_{t+s} - m_t)^n}{n!}, \quad n \geq 0.
\]

Proof.
Let us denote

\[
g(t) = \text{Exp} (-\nu (N_{t+s} - N_s)),
\]

for fixed \( s \) and \( \nu \geq 0 \). We have

\[
g(t + h) = \text{Exp} (-\nu (N_{t+s+h} - N_s))
\]
\[= \text{Exp} (-\nu (N_{t+s+h} - N_{t+s}) - \nu (N_{t+s} - N_s))
\]
\[= g(t) \text{Exp} (-\nu (N_{t+s+h} - N_{t+s}))
\]
\[= g(t) \left(1 - \lambda(t+s)h + e^{-\nu} \lambda(s+t)h + o(h)\right).
\]
Proof.

Hence:

\[
\frac{g(t+h) - g(t)}{h g(t)} = \left( e^{-\nu} - 1 \right) \lambda (s + t) + \frac{o(h)}{h}.
\]

Passing with \( h \) to \( 0^+ \) yields:

\[
\frac{g'(t)}{g(t)} = \left( e^{-\nu} - 1 \right) \lambda (s + t).
\]

Thus

\[
g(t) = \text{Exp} \left( \left( e^{-\nu} - 1 \right) \int_s^{t+s} \lambda(\tau) \, d\tau \right).
\]

But this is a p.g.f of Poisson r.v. with parameter \( \int_s^{t+s} \lambda(\tau) \).
We have the following generalization of the theorem on conditional distribution of moments of arrivals of calls.

**Theorem**

If \( N(t) = n \), the moments of successive calls \( S_1, \ldots, S_n \) are distributed as the order statistics of \( n \) i.i.d. random variables drawn from distribution \( \Lambda(s)/\Lambda(t) \) on \( < 0, t > \), where \( \Lambda(s) = \int_0^s \lambda(x) \, dz \) and \( \lambda(t) \) is the intensity of the simulated process.

**Proof.**

Very similar to the proof for the uniform Poisson process.

**Fact**

*This results in special way of simulating nonhomogeneous Poisson process*
Compound Poisson process

Let \( \{Y_i\}_{s \geq 1} \) be a sequence of independent random variables with identical distributions (a i.i.d. sequence). A sequence \( \{Y_i\}_{s \geq 1} \) will be called a sequence of losses. Let further \( \{N_t; t \geq 0\} \) be independent of \( \{Y_i\} \) Poisson process with intensity \( \lambda \). A compound Poisson process will be called the following process:

\[
X_t = \sum_{i=1}^{N_t} Y_i.
\]

Examples of such processes are process of losses of an insurance company (process of events (accidents) is Poissonian, \( i \)-th loss \( Y_i \)), a process of incomes to supermarket cashier (clients leave at a Poissonian flow and an \( i \)-th leaves \( Y_i \) zł.). We have the following Lemma:

**Lemma**

1) \( EX_t = \lambda t EY_1 \)

2) \( \text{var} (X_t) = \lambda t EY_1^2 \).
Proof.
Using properties of conditional expectation we have:

$$EX_t = E \left( E \left( X_t \mid N_t \right) \right).$$

hence

$$E \left( X_t \mid N_t = n \right) = E \left( \sum_{i=1}^{N_t} Y_i \mid N_t = n \right) = E \left( \sum_{i=1}^{n} Y_i \mid N_t = n \right) = E \left( \sum_{i=1}^{n} Y_i \right) = nEY_1.$$ Generally we have

$$E \left( X_t \mid N_t \right) = N_tEY_1,$$

which leads immediately to the first assertion. ∎
Proof. 
To get the second we will use the following property of conditional variance:

\[ \text{var}(X) = E(\text{var}(X|A)) + \text{var}(E(X|A)) \]

that is true for any random variable $X$ such that, $EX^2 < \infty$ and arbitrary $\sigma-$ field $A$. 
Thus

\[ \text{var}(X_t) = E(\text{var}(X_t|N_t)) + \text{var}(E(X_t|N_t)). \]

\[ \text{var}(E(X_t|N_t = n)) = \text{var}\left(\sum_{i=1}^{N_t} Y_i|N_t = n\right) = \text{var}(\sum_{i=1}^{n} Y_i|N_t = n) = \text{var}(\sum_{i=1}^{n} Y_i) = n \text{var}(Y_1). \]

Hence

\[ \text{var}(X_t|N_t) = N_t \text{var}(Y_1). \]

Summarizing we get:

\[ \text{var}(X_t) = E(N_t \text{var}(Y_1)) + \text{var}(N_tEY_1) = \lambda t \text{var}(Y_1) + \lambda t (EY_1)^2 = \lambda t EY_1^2. \]