Importance weighted max-min fairness and related compensatory models for network resource allocation

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Short paper

The problem of telecommunications network design with the objective to maximize service data flows and provide fair treatment of all services is very up-to-date. In this application, the so-called Max-Min Fair (MMF) solution concept is widely used to formulate the resource allocation scheme [1,6,13]. It assumes that the worst service performance is maximized and the solution is additionally regularized with the lexicographic maximization of the second worst performance, the third one, etc. The MMF approach is an extreme and very stiff and it can be compromised by some compensatory models based on the ordered weighted averaging (OWA). In the OWA aggregation preferential weights are allocated to the ordered outcomes. Such OWA aggregations are sometimes called Ordered Ordered Weighted Averages [5]. When differences between weights tend to infinity, the OWA model becomes Lexicographic Max-Min [16], thus extending the MMF approach for nonconvex problems.

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For modeling various fair preferences one may use some combinations can be expressed with weights $\omega_i = \sum_{j=i}^{m} s_j$ $(i = 1, \ldots, m)$ allocated to coordinates of the ordered outcome vector, i.e., as the so-called Ordered Weighted Average (OWA) [15]:

$$(1) \qquad \max \{\sum_{i=1}^{m} \omega_i \theta_i(\mathbf{f}(\mathbf{x})) \ : \ \mathbf{x} \in Q\}$$

If weights ω_i are strictly decreasing and positive, i.e. $\omega_1 > \omega_2 > \cdots > \omega_{m-1} > \omega_m > 0$, then each optimal solution of the OWA problem (1) is a fairly efficient solution. Such OWA aggregations are sometimes called Ordered Ordered Weighted Averages (OOWA) [5].

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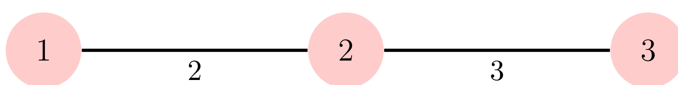

Consider the bandwidth allocation for multi-commodity multi-path networks as presented above. The demands generate elastic traffic, i.e., each of them can consume any bandwidth assigned to its path. There are 3 demand pairs: $d_1 = \{1, 2\}, d_2 = \{2, 3\}, d_3 = \{1, 3\}$ and corresponding bandwidth/flows: y_1, y_2, y_3. The OOWA maximization: $\max(0.4y_{\langle 1\rangle} + 0.35y_{\langle 2\rangle} + 0.25y_{\langle 3\rangle})$ (where $y_{\langle k\rangle}$ the k-th smallest value) results in $y_1 = 2, y_2 = 3, y_3 = 0$ which is maximum throughput solution. On the other hand, the OOWA maximization: $\max(0.6y_{\langle 1\rangle} + 0.3y_{\langle 2\rangle} + 0.1y_{\langle 3\rangle})$ results in $y_1 = y_3 = 1, y_2 = 2$ which is the MMF solution.

Frequently, one may be interested in putting into allocation models some additional service importance weights. Typically the model of distribution weights is introduced to represent the service importance thus defining distribution of outcomes according to measures defined by the weights. Such distribution weights allow one for a clear interpretation of weights as the service repetitions. Splitting a service into two services does not cause any change of the final distribution of outcomes.

Consider again the bandwidth allocation for multi-commodity multi-path networks with 3 demand pairs: $d_1 = \{1, 2\}, d_2 = \{2, 3\}, d_3 = \{1, 3\}$ and corresponding flows: y_1, y_2, y_3. The standard MMF solution lexmax $(y_{\langle 1\rangle}, y_{\langle 2\rangle}, y_{\langle 3\rangle})$ is $y_1 = y_3 = 1, y_2 = 2$ Consider importance weights $v_1 = 1, v_2 = 1, v_3 = 2$ interpreted as demand repetitions: $d'_1 = \{1, 2\}, d'_2 = \{2, 3\}, d'_3 = \{1, 3\}, d'_4 = \{1, 3\}$ with y'_1, y'_2, y'_3, y'_4. Then, lexmax $(y'_{\langle 1\rangle}, y'_{\langle 2\rangle}, y'_{\langle 3\rangle}, y'_{\langle 4\rangle})$ generate solution $y'_1 = y'_3 = y'_4 = 2/3, y'_2 = 5/3$ and the importance weighted MMF results in $y_1 = 2/3, y_2 = 4/3, y_3 = 5/3$.

We will use the normalized weights $\bar{v}_i = v_i / \sum_{i\in I} v_i$, rather than the original quantities v_i. Note that, in the case of unweighted problem (all $v_i = 1$), all the normalized weights are given as $\bar{v}_i = 1/m$. The importance weights can be easily accommodated in solution concept of the mean outcome $\mu(\mathbf{y}) = \sum_{i\in I} \bar{v}_i y_i$ as well as in solution concepts based on the mean utility, like the Proportional Fairness (PF) [4]. For any utility function $u : R \rightarrow R$ one gets $\mu(u(\mathbf{y})) = \sum_{i\in I} \bar{v}_i u(y_i)$. In the case of logarithmic utility function one gets the PF formula $PF_{\bar{v}}(\mathbf{y}) = \sum_{i\in I} \bar{v}_i \log(y_i)$.

Similar, approach may be applied to the OOWA solution concepts resulting in the so-called Weighted OWA (WOWA) aggregations. The OWA aggregation (1) is built for equally important outcomes where only distribution of outcome values is evaluated. For instance, considering two outcomes with the OWA weights $w_1 = 0.9$ and $w_2 = 0.1$ both symmetric outcome vectors $\mathbf{y}^1 = (0, 1)$ and $\mathbf{y}^2 = (1, 0)$ result in the same OWA aggregation $OWA_1 = OWA_2 = 0.9 \cdot 0 + 0.1 \cdot 1 = 0.1$. Nevertheless, the importance weights of outcomes can be introduced into the OWA aggregation following the rule that the importance weights v_i define a repetition measure within the distribution (population) of outcome values while the OWA weights w_i are applied to averages within specific quantiles of size $1/m$ for this distribution. For instance, introducing importance weights $v_1 = 0.75$ and $v_2 = 0.25$ we replace $\mathbf{y}^1 = (0, 1)$ with the distribution taking value 0 with the repetition measure 0.75 and taking value 1 with the repetition measure 0.25 while $\mathbf{y}^2 = (1, 0)$ is replaced with the distribution taking value 1 with the repetition measure 0.75 and taking value 0 with the repetition measure 0.25. In this specific case, the distributions may easily be equivalently interpreted in terms of four dimensional space of equally important outcomes (measure 1/4 each) where the original first outcome has been triplicated, thus $\mathbf{y}^1 = (0, 0, 0, 1)$ and $\mathbf{y}^2 = (1, 1, 1, 0)$. The OWA aggregation with weights $s_1 = 0.9$ and $s_2 = 0.1$ applied to the corresponding averages within quantiles of size $1/2$ results then in aggregation values $0.9 \cdot 0 + 0.1 \cdot (0 + 1)/2 = 0.05$ for $\mathbf{y}^1$ and $0.9 \cdot (0 + 1)/2 + 0.1 \cdot 1 = 0.55$ for $\mathbf{y}^2$, respectively. Certainly, one do not need to transform all the cases to equally important outcomes in order to calculate appropriate OWA value. Such an importance weighting OWA formula was introduced as the WOWA aggregation formally defined as follows [14]

$$WOWA(\mathbf{y}) = \sum_{i=1}^{m} \omega_i \theta_i(\mathbf{y}) \tag{2}$$

with

$$\omega_i = \omega^*(\sum_{k=1}^{i} \bar{v}_{\tau(k)}) - \omega^*(\sum_{k=1}^{i-1} \bar{v}_{\tau(k)}) \tag{3}$$

where ω^* is piecewise linear function interpolating points $(\frac{i}{m}, \sum_{k=1}^{i} w_k)$ together with (0.0) and τ representing the ordering permutation for $\mathbf{y}$ (i.e. $y_{\tau(i)} = \theta_i(\mathbf{y})$). Function w^* can be defined by its generation function

$$g(\xi) = m w_k \quad \text{for } (k-1)/m < \xi \leq k/m, \quad k = 1, \dots, m$$

with the formula $w^*(\alpha) = \int_0^\alpha g(\xi)\, d\xi$. Introducing breakpoints $\alpha_i = \sum_{k \leq i} v_{\tau(k)}$ and $\alpha_0 = 0$ allows us to express

$$\omega_i = \int_0^{\alpha_i} g(\xi)\, d\xi - \int_0^{\alpha_{i-1}} g(\xi)\, d\xi = \int_{\alpha_{i-1}}^{\alpha_i} g(\xi)\, d\xi$$

and the entire WOWA aggregation as

$$WOWA(\mathbf{y}) = \sum_{i=1}^{m} \theta_i(\mathbf{y}) \int_{\alpha_{i-1}}^{\alpha_i} g(\xi)\, d\xi = \int_0^1 g(\xi) F_{\mathbf{y}}^{(-1)}(\xi)\, d\xi$$

where $F_{\mathbf{y}}^{(-1)}$ is the inverse of the cumulative distribution function, i.e. a stepwise function $F_{\mathbf{y}}^{(-1)}(\xi) = \theta_i(\mathbf{y})$ for $\alpha_{i-1} < \xi \leq \alpha_i$. Hence,

$$WOWA(\mathbf{y}) = \sum_{k=1}^{m} w_k m \int_{(k-1)/m}^{k/m} F_{\mathbf{y}}^{(-1)}(\xi)\, d\xi \tag{4}$$

Note that $m \int_{(k-1)/m}^{k/m} F_{\mathbf{y}}^{(-1)}(\xi)\, d\xi$ represents the average within the k-th portion of $1/m$ smallest outcomes, the corresponding conditional mean. Hence, the formula (4) defines WOWA aggregations with preferential weights $\mathbf{w}$ as the corresponding OWA aggregation but applied to the conditional means calculated according to the importance weights $\bar{\mathbf{v}}$ instead of the original outcomes [10].

For theoretical considerations one may assume that the problem is transformed (disaggregated) to the unweighted one (that means all the importance weights are equal to 1). Such a disaggregation is possible for integer as well as rational weights, but it usually dramatically increases the problem size. Therefore, we are interested in solution concepts and solution algorithms which can be applied directly to the weighted problem. In this paper we study both exact and approximate solution algorithms.

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