Ann Oper Res
DOI 10.1007/s10479-007-0234-9

Inequality measures and equitable locations

Włodzimierz Ogryczak

Abstract While making location decisions, the distribution of distances (outcomes) among the service recipients (clients) is an important issue. In order to comply with the minimization of distances as well as with an equal consideration of the clients, mean-equity approaches are commonly used. They quantify the problem in a lucid form of two criteria: the mean outcome representing the overall efficiency and a scalar measure of inequality of outcomes to represent the equity (fairness) aspects. The mean-equity model is appealing to decision makers and allows a simple trade-off analysis. On the other hand, for typical dispersion indices used as inequality measures, the mean-equity approach may lead to inferior conclusions with respect to the distances minimization. Some inequality measures, however, can be combined with the mean itself into optimization criteria that remain in harmony with both inequality minimization and minimization of distances. In this paper we introduce general conditions for inequality measures sufficient to provide such an equitable consistency. We verify the conditions for the basic inequality measures thus showing how they can be used in location models not leading to inferior distributions of distances.

Keywords Location · Multiple criteria · Efficiency · Equity · Fairness · Inequality measures

1 Introduction

The spatial distribution of public goods and services is influenced by facility location decisions and the issue of equity (or fairness) is important in many location decisions.

The research was supported by the Ministry of Science and Information Society Technologies under grant 3T11C 005 27 "Models and Algorithms for Efficient and Fair Resource Allocation in Complex Systems".

W. Ogryczak (✉)
Institute of Control and Computation Engineering, Warsaw University of Technology, Nowowiejska 15/19, 00-665 Warsaw, Poland
e-mail: W.Ogryczak@ia.pw.edu.pl

Especially, various public facilities (or public service delivery systems) like schools, libraries, health-service centers etc. require some spatial equity while making location-allocation decisions (Coulter 1980; Mayhew and Leonardi 1982; O'Brien 1969). Equity is usually quantified with the so-called inequality measures to be minimized. Inequality measures were primarily studied in economics (Atkinson 1970; Sen 1973; Young 1994). The simplest inequality measures are based on the absolute measurement of the spread of outcomes. Variance is the most commonly used inequality measure of this type and it was also widely analyzed within various location models (Maimon 1986; Berman 1990; Carrizosa 1999). However, many various measures have been proposed in the literature to gauge the level of equity in facility location alternatives (Marsh and Schilling 1994) or spatial equity in general (Hay 1995). In economics one usually considers relative inequality measures normalized by mean outcome. Among many inequality measures perhaps the most commonly accepted by economists is the Gini index (Lorenz measure), which has been also analyzed in the location context (Berman and Kaplan 1990; Erkut 1993; Maimon 1988; Mandell 1991). One can easily notice that a direct minimization of typical inequality measures (especially relative ones) contradicts the minimization of individual outcomes. As noticed by Erkut (1993), it is rather a common flaw of all the relative inequality measures that while moving away from the clients to be serviced one gets better values of the measure as the relative distances become closer to one-another. As an extreme, one may consider an unconstrained continuous (single-facility) location problem and find that the facility located at (or near) infinity will provide (almost) perfectly equal service (in fact, rather lack of service) to all the clients.

Although minimization of the inequality measures contradicts the minimization of individual outcomes, the inequality minimization itself can be consistently incorporated into locational models. The notion of equitable multiple criteria optimization (Kostreva and Ogryczak 1999b) introduces the preference structure that complies with both the outcomes minimization and with the inequality minimization rules (Sen 1973; Young 1994). The equitable efficient solutions represent a subset of all efficient (Pareto-optimal) solutions which takes into account the inequality minimization according to the Pigou–Dalton approach. The equitable optimization is well suited for the locational analysis (Kostreva and Ogryczak 1999a; Ogryczak 2000). It turns out that equitably efficient solution concepts may be modeled with the standard multiple criteria optimization applied to the cumulative ordered outcomes. The center solution concept represent only first criterion and in order to guarantee the equitable efficiency of a selected location pattern one needs to take into account all the ordered outcomes like in the lexicographic center (Ogryczak 1997a) which is a lexicographic refinement of the center solution concept. The entire multiple criteria ordered model is rich with various equitably efficient solution concepts (Ogryczak and Zawadzki 2002; Kostreva et al. 2004). Although the cumulated ordered outcomes can be expressed with linear programming models (Ogryczak and Tamir 2003), these approaches requires the disaggregation of location problem with the client weights which usually dramatically increases the problem size.

For typical inequality measures a simplified bicriteria mean-equity model is computationally very attractive since both the criteria are well defined directly for the weighted location problem without necessity of its disaggregation but it may result in solutions which are inefficient. Therefore, we are interested in a proper use of the mean-equity models in a way to guarantee the equitable efficiency of selected solutions. It turns out that, under the assumption of bounded trade-offs, the bicriteria mean-equity approaches for selected absolute inequality measures (maximum upper deviation, mean semideviation or mean absolute difference) comply with the rules of equitable multiple criteria optimization (Ogryczak 2000).

In other words, several inequality measures can be combined with the mean itself into the optimization criteria generalizing the concept of the worst outcome and generating equitably consistent underachievement measures. We generalize those findings by introducing simple sufficient conditions for inequality measures to keep this consistency property. It allows us to identify more inequality measures which can be effectively used to incorporate equity factors into various location while preserving the consistency with distance minimization. Among others the standard upper semideviation turns out to be such a consistent inequality measure.

The paper is organized as follows. In the next section we introduce the problem and the basic inequality measures. In Sect. 3 the equitable optimization with the preference structure that complies with both the efficiency (Pareto-optimality) principle and with the Pigou–Dalton principle of transfers is discussed and the underachievement criteria are introduced. Further, in Sect. 4, the equitable consistency of the underachievement criteria is analyzed and sufficient conditions for the inequality measures to keep this consistency property are introduced. There is shown that properties of convexity and positive homogeneity together with some boundedness condition is sufficient for a typical inequality measure to guarantee the corresponding equitable consistency. We verify the properties for the basic inequality measures used in location problem thus showing how they can be applied not leading to inferior distributions of distances.

2 Efficiency and inequality measures

The generic location problem that we consider may be stated as follows. There is given a set $I = \{1, 2, \ldots, m\}$ of m clients (service recipients). Each client is represented by a specific point in the geographical space. There is also given a set Q of location patterns (location decisions). For each client i ($i \in I$) a function $f_i(\mathbf{x})$ of the location pattern $\mathbf{x}$ has been defined. This function, called the individual objective function, measures the outcome (effect) $y_i = f_i(\mathbf{x})$ of the location pattern for client i (Marsh and Schilling 1994). In the simplest problems an outcome usually expresses the distance. However, we emphasize to the reader that we do not restrict our considerations to the case of outcomes measured as distances. They can be measured (modeled) as travel time, travel costs as well as in a more subjective way as relative travel costs (e.g., travel costs by clients incomes) or ultimately as the levels of clients dissatisfaction (individual disutility) of location decisions. Several specialized measures of proximity or accessibility have been developed for public service delivery systems (Abernathy and Hershey 1972; Malczewski 2000; Mayhew and Leonardi 1982; Tsou et al. 2005). In typical formulations of location problems related to desirable facilities a smaller value of the outcome (distance) means a better effect (higher service quality or client satisfaction). This remains valid for location of obnoxious facilities if the distances are replaced with their complements to some large number or other (decreasing) disutility function of distances. Therefore, without loss of generality, we can assume that each individual outcome y_i is to be minimized. This allows us to consider the generic location problem as the multiple criteria minimization (Ogryczak, 1997a, 1999):

$$\min\{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in Q\} = \min\{\mathbf{y} : \mathbf{y} \in A\}, \tag{1}$$

where $\mathbf{f} = (f_1, \ldots, f_m)$ is a vector-function that maps feasible decisions (locations) $\mathbf{x} \in Q$ into the outcome space $Y = R^m$ and $A = \{\mathbf{y} \in Y : \mathbf{y} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} \in Q\}$ denotes the set of attainable outcome vectors.

We do not assume any special form of the problem constraints allowing the feasible set to be a general, possibly discrete (nonconvex), set. Similarly, we do not assume any special form of the individual objective functions nor their special properties (like convexity) while analyzing properties of the solution concepts. We have only assumed a finite set of clients for the minimization of the individual outcomes. Therefore, the results of our analysis apply to various classes of location problems covering continuous as well as discrete and special network models (c.f., Love et al. 1988; Francis et al. 1992; Current et al. 1990; Mirchandani and Francis 1990; Labbé et al. 1996).

Model (1) only says that we are interested in the minimization of all outcome functions f_i for $i \in I$. In order to make it operational, one needs to assume some solution concept. Typical solution concepts for locations problems are based on the minimization of some scalar achievement function $C(\mathbf{y})$ of outcome vectors $\mathbf{y}$. Most classical location studies focus on the minimization of the mean (or total) distance (the median concept) or the minimization of the maximum distance (the center concept) to the service facilities (Morrill and Symons 1977).

Since for each outcome the smaller value is preferred, some outcome vectors are clearly dominated by others. We say that outcome vector $\mathbf{y}'$ *(Pareto) dominates* $\mathbf{y}''$, iff $y'_i \leq y''_i$ for all $i \in I$ where at least one strict inequality holds. We say that a location pattern $\mathbf{x} \in Q$ is an *Pareto-efficient* solution of the multiple criteria problem (1), iff $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is nondominated. The latter refers to the commonly used definition of the efficient solutions as feasible solutions for which one cannot improve any criterion without worsening another (e.g., Vincke 1992).

Frequently, one may be interested in putting into location models some additional client weights $v_i > 0$. Typically the model of distribution weights is introduced to represent the service demand thus defining distribution of outcomes $y_i = f_i(\mathbf{x})$ according to measures defined by the weights v_i for $i = 1, \dots, m$. Note that such distribution weights allows us for a clear interpretation of weights as the client repetitions at the same place. Splitting a client into two clients sharing the demand at the same geographical point does not cause any change of the final distribution of outcomes. For theoretical considerations one may assume that the problem is transformed (disaggregated) to the unweighted one (that means all the client weights are equal to 1). Note that such a disaggregation is possible for integer as well as rational client weights, but it usually dramatically increases the problem size. Therefore, we are interested in solution concepts which can be applied directly to the weighted problem.

Alternatively, scaling weights could be used as client importance factors thus defining outcomes $y_i = v_i f_i(\mathbf{x})$ uniformly distributed for $i = 1, \dots, m$. Such an usage of weights represents actually redefinition of outcome values. Recall that we consider the outcome values $f_i(\mathbf{x})$ as distance dependent but allowing any specific form of this function thus any weighted scaling is already taken into account within the outcomes definition. Actually, the distance scaling model means the use of unweighted location problem with a very simple modification of distances. Therefore, our analysis is focused on the model of distribution (demand) weights.

As mentioned, for some theoretical considerations it might be convenient to disaggregate the weighted problems into the unweighted one. Therefore, to simplify the analysis we will assume integer weights v_i, although while discussing solution concepts we will use the normalized client weights

$$\bar{v}_i = v_i \Big/ \sum_{i=1}^{m} v_i \quad \text{for } i = 1, 2, \dots, m$$

rather than the original quantities v_i. Note that, in the case of unweighted problem (all $v_i = 1$), all the normalized weights are given as $\bar{v}_i = 1/m$. Furthermore, to avoid possible misunderstandings between the weighted outcomes and the corresponding unweighted form of outcomes we will use the following notation. Index set I will always denote unweighted clients (with possible repetitions if originally weighted) and vector $\mathbf{y} = (y_i)_{i \in I} = (y_1, y_2, \dots, y_m)$ will denote the unweighted outcomes. While directly dealing with the weighted problem (without its disaggregation to the unweighted one) we will use I_v to denote the set of clients and the corresponding outcomes will be represented by vector $\mathbf{y} = (y_{v_i})_{i \in I_v}$. We illustrate this with the following small example.

Example 1 Let us consider a weighted single facility location problem with two clients C1 and C2 having assigned demand weights $v_1 = 1$ and $v_2 = 9$, respectively. Their distances to two potential locations P1 and P2 are given as follows:

	C1	C2
P1	0	10
P2	10	10

Hence, $I_v = \{1, 2\}$ and the potential locations generate two outcome vectors $\mathbf{y}' = (0_1, 10_9)$ and $\mathbf{y}'' = (10_1, 10_9)$, respectively. The demand weights are understood as clients repetitions. Thus, the problem is understood as equivalent to the unweighted problem with 10 clients ($I = \{1, 2, \dots, 10\}$) where the first one corresponds to C1 and the further nine unweighted clients correspond to single demand units of the original client C2. In this disaggregated form, the outcome vectors generated by two locations P1 and P2 are given as $\mathbf{y}' = (0, 10, 10, 10, 10, 10, 10, 10, 10, 10)$ and $\mathbf{y}'' = (10, 10, 10, 10, 10, 10, 10, 10, 10, 10)$, respectively. Note that outcome vector $\mathbf{y}''$ with all the distances 10 is obviously worse than unequal vector $\mathbf{y}'$ with one distance reduced to 0. Actually, $\mathbf{y}'$ Pareto dominates $\mathbf{y}''$.

The classical solution concepts of median and center are well defined for aggregated location models using (distribution) demand weights $v_i > 0$. Exactly, the *median* solution concept is defined by minimization the *mean* outcome

$$\mu(\mathbf{y}) = \frac{1}{m} \sum_{i \in I} y_i = \sum_{i \in I_v} \bar{v}_i y_{v_i}, \tag{2}$$

i.e., by the optimization problem

$$\min\{\mu(\mathbf{f}(\mathbf{x})) : \mathbf{x} \in Q\}. \tag{3}$$

The *center* solution concept is defined by minimization of the maximum (worst) outcome

$$M(\mathbf{y}) = \max_{i \in I} y_i = \max_{i \in I_v} y_{v_i}, \tag{4}$$

thus resulting in the optimization problem

$$\min\{M(\mathbf{f}(\mathbf{x})) : \mathbf{x} \in Q\}. \tag{5}$$

Note the maximum outcome $M(\mathbf{y})$ is not affected by the distribution weights at all and the same applies to center solution itself. The weighted center solution concepts considered in some location models (Labbé et al. 1996) represent distance scaling weights rather than the distribution weights. In our analysis such scaling weights are considered as included within the outcome functions $f_i(\mathbf{x})$.

The individual outcomes in our multiple criteria location model express the same quantity (usually the distance) for various clients. Thus the outcomes are uniform in the sense of the scale used and their values are directly comparable. Moreover, especially when locating public facilities, we want to consider all the clients impartially and equally. Thus the distribution of distances (outcomes) among the clients is more important than the assignment of several distances (outcomes) to the specific clients. Both the center and the median solution concepts minimize only simple scalar characteristics of the distribution: the maximum (the worst) outcome and the mean outcome, respectively.

Equity is, essentially, an abstract socio-political concept that implies fairness and justice (Young 1994). Nevertheless, equity is usually quantified with the so-called inequality measures to be minimized. Inequality measures were primarily studied in economics (Sen 1973). However, Marsh and Schilling (1994) described twenty different measures proposed in the literature to gauge the level of equity in facility location alternatives. Typical inequality measures are some deviation type dispersion characteristics. They are *translation invariant*

$$\varrho(\mathbf{y} + a\mathbf{e}) = \varrho(\mathbf{y}) \quad \text{for any outcome vector } \mathbf{y} \text{ and real number } a \tag{6}$$

where $\mathbf{e}$ denotes the vector of units $(1, \dots, 1)$, thus being not affected by any shift of the outcome scale. Moreover, the inequality measures are also *inequality relevant* which means that they are equal to 0 in the case of a perfectly equal outcomes while taking positive values for any unequal one.

The simplest inequality measures are based on the absolute measurement of the spread of outcomes, like the *mean absolute difference* also called the Gini's mean difference (López-de-los-Mozos and Mesa 2003; Ogryczak 2000)

$$D(\mathbf{y}) = \frac{1}{2m^2} \sum_{i \in I} \sum_{j \in I} |y_i - y_j| = \frac{1}{2} \sum_{i \in I_v} \sum_{j \in I_v} |y_{v_i} - y_{v_j}| \bar{v}_i \bar{v}_j \tag{7}$$

or the *maximum (absolute) difference*

$$S(\mathbf{y}) = \max_{i,j \in I} |y_i - y_j| = \max_{i,j \in I_v} |y_{v_i} - y_{v_j}|. \tag{8}$$

In the location framework better intuitive appeal may have inequality measures related to deviations from the mean outcome (Mulligan 1991) like the *mean (absolute) deviation*

$$\delta(\mathbf{y}) = \frac{1}{m} \sum_{i \in I} |y_i - \mu(\mathbf{y})| = \sum_{i \in I_v} |y_{v_i} - \mu(\mathbf{y})| \bar{v}_i \tag{9}$$

or the *maximum (absolute) deviation* (López-de-los-Mozos and Mesa 2001)

$$R(\mathbf{y}) = \max_{i \in I} |y_i - \mu(\mathbf{y})| = \max_{i \in I_v} |y_{v_i} - \mu(\mathbf{y})|. \tag{10}$$

Note that the *standard deviation* σ (or the *variance* σ^2) represents both the deviations and the spread measurement as

$$\sigma(\mathbf{y}) = \sqrt{\frac{1}{m} \sum_{i \in I} (y_i - \mu(\mathbf{y}))^2} = \sqrt{\frac{1}{2m^2} \sum_{i \in I} \sum_{j \in I} (y_i - y_j)^2}$$

$$= \sqrt{\sum_{i \in I_v} (y_{v_i} - \mu(\mathbf{y}))^2 \bar{v}_i} = \sqrt{\frac{1}{2} \sum_{i \in I_v} \sum_{j \in I_v} (y_{v_i} - y_{v_j})^2 \bar{v}_i \bar{v}_j}. \tag{11}$$

Deviational measures may be focused on the upper semideviations as related to worsening of outcome while ignoring downside semideviations related to improvement of outcome. One may define the *maximum (upper) semideviation*

$$\Delta(\mathbf{y}) = \max_{i \in I} (y_i - \mu(\mathbf{y})) = \max_{i \in I_v} (y_{v_i} - \mu(\mathbf{y})), \tag{12}$$

and the *mean (upper) semideviation*

$$\bar{\delta}(\mathbf{y}) = \frac{1}{m} \sum_{i \in I} (y_i - \mu(\mathbf{y}))_+ = \sum_{i \in I_v} (y_{v_i} - \mu(\mathbf{y}))_+ \bar{v}_i, \tag{13}$$

where $(\cdot)_+$ denotes the nonnegative part of a number. Similarly, the *standard (upper) semideviation* is given as

$$\bar{\sigma}(\mathbf{y}) = \sqrt{\frac{1}{m} \sum_{i \in I} (y_i - \mu(\mathbf{y}))_+^2} = \sqrt{\sum_{i \in I_v} (y_{v_i} - \mu(\mathbf{y}))_+^2 \bar{v}_i}. \tag{14}$$

One may notice that, due to the mean definition, the mean absolute semideviation is always equal to half of the mean absolute deviation ($\bar{\delta}(\mathbf{y}) = \frac{1}{2}\delta(\mathbf{y})$) but similar symmetry property does not apply to the maximum semideviation or the standard semideviation.

In income economics, relative inequality measures (normalized by mean outcome) are commonly used with the Gini coefficient $D(\mathbf{y})/\mu(\mathbf{y})$ as a typical example. The latter is a relative measure of the mean absolute difference and has been also analyzed in the location context (Mandell 1991; Mulligan 1991; Erkut 1993). One can easily notice that direct minimization of relative inequality measures contradicts the minimization of individual outcomes (Erkut 1993). Unfortunately, the same applies to all dispersion type inequality measures, including the upper semideviations. This can be illustrated by a simple example of discrete location problem.

Recall the single facility location problem from Example 1 where the perfectly equal outcome vector $\mathbf{y}''$ with all the distances 10 is Pareto dominated by $\mathbf{y}'$ with one distance reduced to 0. Actually, $\mathbf{y}''$ is obviously worse than $\mathbf{y}'$ as it allows us to reduce the distance to one client without worsening the others. Typical needs for equity of outcomes are caused by the necessity to guarantee a good quality of service to all clients, like in emergency systems. There is no justification, however, to enforce worse quality of service for one client if it does not allow us to improve the service for any other. Nevertheless, $\varrho(\mathbf{y}'') > 0$ for any dispersion type inequality measure ϱ while $\varrho(\mathbf{y}') = 0$ for each such a measure. Hence, the second location pattern is clearly optimal while directly minimizing the inequality measure $\min\{\varrho(\mathbf{f}(\mathbf{x})) : \mathbf{x} \in Q\}$.

In order to overcome the flaws of direct minimization of inequality measures, following Mandell (1991), the bicriteria mean-equity model:

$$\min\{(\mu(\mathbf{f}(\mathbf{x})), \varrho(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\} \tag{15}$$

is usually considered. The model takes into account both the efficiency with minimization of the mean outcome $\mu(\mathbf{y})$ and the equity with minimization of an inequality measure $\varrho(\mathbf{y})$.

For typical inequality measures bicriteria model (15) is computationally very attractive since both the criteria are well defined directly for the weighted location problem without necessity of its disaggregation. Unfortunately, the bicriteria mean-equity model still does not completely eliminate contradiction to the minimization of individual outcomes. When considering the discrete location problem from Example 1, for any dispersion type inequality measure one gets $\varrho(\mathbf{y}'') > 0 = \varrho(\mathbf{y}')$ while $\mu(\mathbf{y}'') = 9 < 10 = \mu(\mathbf{y}')$. Hence, $\mathbf{y}''$ is not bicriteria dominated by $\mathbf{y}'$ and vice versa. Nevertheless, one must accept that for any dispersion type inequality measure ϱ, location P2 with obviously worse outcome vector than that for location P1 is a Pareto-optimal solution in the corresponding bicriteria mean-equity model. This leads us a crucial question how to prevent the model from possible selection of such an obviously worse location. We answer this question in two steps. First, we recall the concept of equitable efficiency which is based on extension of the standard Pareto-efficiency concept with the Pigou–Dalton equity theory and when applied to location problems it remains in harmony with both inequality minimization and distances (outcomes) minimization, Next, we introduce general conditions under which various inequality measures can be used together with the mean in bicriteria optimization to maintain the equitably efficiency of selected locations.

3 Equitable efficiency and underachievement criteria

As pointed out in the previous section, direct use of the inequality measure minimization may result in locations strictly worsening all the distances. In other words, the inequality measures minimization may contradict the outcomes minimization. It does not mean, however, that the inequality minimization itself cannot be consistently incorporated into the location models. There exist models of equitable optimization based on the majorization theory (Hardy et al. 1934; Marshall and Olkin 1979) which are consistent both with the Pareto-efficiency and theories of inequality measurement (in particular the Pigou–Dalton approach). Namely, the Pareto dominance relation is transitive and can be transitively extended with additional relations representing inequality minimization (Ogryczak 1997b; Kostreva and Ogryczak 1999b). The resulting notion of equitable multiple criteria optimization is based on the preference structure that complies with both the Pareto-efficiency and with the inequality measurement rules, and it is well suited for the locational analysis (Kostreva and Ogryczak 1999a; Ogryczak 2000).

The preference model consists of three binary relations (Vincke 1992): strict preference $\prec$, indifference $\cong$ and weak preference $\preceq$ representing the union of two former ($\mathbf{y}' \preceq \mathbf{y}'' \Leftrightarrow \mathbf{y}' \prec \mathbf{y}'' \vee \mathbf{y}' \cong \mathbf{y}''$). The indifference relation is reflexive ($\mathbf{y} \cong \mathbf{y}$) while the strict preference relation is asymmetric ($\mathbf{y}' \prec \mathbf{y}'' \Rightarrow \mathbf{y}'' \nprec \mathbf{y}'$). We assume that the preference model is transitive which means that $\mathbf{y}' \prec \mathbf{y}'' \wedge \mathbf{y}'' \prec \mathbf{y}''' \Rightarrow \mathbf{y}' \prec \mathbf{y}'''$, and $\mathbf{y}' \cong \mathbf{y}'' \wedge \mathbf{y}'' \cong \mathbf{y}''' \Rightarrow \mathbf{y}' \cong \mathbf{y}'''$, as well as, $\mathbf{y}' \cong \mathbf{y}'' \wedge \mathbf{y}'' \prec \mathbf{y}''' \Rightarrow \mathbf{y}' \prec \mathbf{y}'''$, and $\mathbf{y}' \prec \mathbf{y}'' \wedge \mathbf{y}'' \cong \mathbf{y}''' \Rightarrow \mathbf{y}' \prec \mathbf{y}'''$. Actually, the preference model is completely characterized by the relation of weak preference, as $\mathbf{y}' \prec \mathbf{y}'' \Leftrightarrow \mathbf{y}' \preceq \mathbf{y}'' \wedge \mathbf{y}'' \npreceq \mathbf{y}'$, and similarly, $\mathbf{y}' \cong \mathbf{y}'' \Leftrightarrow \mathbf{y}' \preceq \mathbf{y}'' \wedge \mathbf{y}'' \preceq \mathbf{y}'$. Therefore, it is commonly identified with the weak preference relation $\preceq$ (Vincke 1992). However, for clear understanding of equitable preference we introduce this model with directly given properties of relations $\prec$ and $\cong$.

Let us focus on the location problem with the unweighted outcomes (disaggregated if necessary). A transitive preference model $\preceq$ is called *equitably rational* if it fulfills the following requirements (axioms):

(i) strict monotonicity

$$\mathbf{y} - \varepsilon \mathbf{e}_i \prec \mathbf{y} \quad \text{for } \varepsilon > 0;\ i \in I \tag{16}$$

where $\mathbf{e}_i$ denotes the i-th unit vector;

(ii) impartiality (anonymity, symmetry)

$$(y_{\tau(1)}, y_{\tau(2)}, \dots, y_{\tau(m)}) \cong (y_1, y_2, \dots, y_m) \quad \text{for any } \tau \in \Pi(I) \tag{17}$$

where $\Pi(I)$ is the set of all permutations of the set I;

(iii) principle of transfers

$$y_{i'} > y_{i''} \quad \Rightarrow \quad \mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''} \prec \mathbf{y} \quad \text{for } 0 < \varepsilon < y_{i'} - y_{i''};\ i', i'' \in I. \tag{18}$$

The requirement of strict monotonicity (16) represents the basic rationality assumption that for any outcome smaller value is preferred (minimization). When considered for a transitive preference model it guarantees its consistency with Pareto-dominance in the sense that $\mathbf{y}' \leq \mathbf{y}'' \wedge \mathbf{y}' \neq \mathbf{y}'' \;\Rightarrow\; \mathbf{y}' \prec \mathbf{y}''$. Note that the median solution concept fulfills the strict monotonicity requirement as $\mu(\mathbf{y} - \varepsilon \mathbf{e}_i) < \mu(\mathbf{y})$ while the center concept meets it only for a specific case of decreasing the uniquely defined largest outcome and generally it satisfies only the weak monotonicity condition $M(\mathbf{y} - \varepsilon \mathbf{e}_i) \leq M(\mathbf{y})$. Therefore, the median model always generates Pareto-efficient locations whereas the center solution can be Pareto dominated by some alternative optimum to the center problem. However, the center model can be regularized to a strictly monotonic solution concept of the lexicographic center (Ogryczak 1997a).

The next two requirements are related to the inequality minimization (Young 1994). The equity or inequality are considered as properties of a distribution of outcomes. Hence, any two outcome vectors characterized by the same distribution of outcomes must be indifferent with respect to the inequality measurement. Within the class of problems with unweighted outcomes this means ignoring their ordering or individual assignment as expressed with the impartiality axiom (17) which guarantees impartial treatment of all the clients. For instance, two outcome vectors (0, 3, 0, 5) and (5, 0, 3, 0) represent the same distribution of outcomes: two outcomes 0, one 3 and one 5. Therefore, they are indifferent with respect to the inequality measurement, though, quite different from the perspective of any individual client. Note that impartiality axiom is satisfied by all typical inequality measures as enumerated in Sect. 2 as well as by the standard location concepts of the center and the median. A transfer of any small amount from an outcome to any other relatively worse-off outcome (so-called equitable transfer) represents the simplest (basic) construction decreasing inequality among outcomes while preserving their mean (total). The principle of transfers (18) requiring that such a transfer results in a more preferred outcome vector is the commonly accepted Pigou–Dalton axiom for inequality minimization. Unfortunately, not all widely used inequality measures fulfill completely this axiom in the sense that $\varrho(\mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''}) < \varrho(\mathbf{y})$. Actually, among the measures enumerated in Sect. 2 only the mean absolute difference $D(\mathbf{y})$ and the standard deviation $\sigma(\mathbf{y})$ (or the variance) satisfy completely the axiom (for any equitable transfer). Upper semideviations, generally, do not fulfill the axiom for equitable transfers between two outcomes below the mean. The mean absolute deviation $\delta(\mathbf{y})$ reacts only to transfers from an outcome above the mean to an outcome below the mean while the maximum upper deviation $\Delta(\mathbf{y})$ is only sensitive to transfers from the largest outcome (if unique). Nevertheless, no inequality measure contradicts the principle of transfers, as for any equitable transfer they all satisfy the weak condition $\varrho(\mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''}) \leq \varrho(\mathbf{y})$. This weak condition is also satisfied by solution concepts of the median and the center. Actually, the median is completely neutral with respect to any transfer $\mu(\mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''}) = \mu(\mathbf{y})$ while the center concept, similar to the maximum deviation $\Delta(\mathbf{y})$ is sensitive to transfers from the largest outcome (if unique).

The requirements of impartiality (17) and equitability (expressed with the principle of transfers (18)) themselves do not contradict monotonicity (16). Therefore, they could be unified by the transitivity rule to a consistent concept of the equitably rational preference model. The corresponding equitable dominance model ($\preceq_e$) is defined as the relation valid for all equitably rational preference models thus representing the weakest relation satisfying axioms (16–18). That means $\mathbf{y}' \prec_e \mathbf{y}''$ if and only if $\mathbf{y}' \prec \mathbf{y}''$ is valid for all equitably rational preference models $\preceq$, and $\mathbf{y}' \cong_e \mathbf{y}''$ if and only if $\mathbf{y}' \cong \mathbf{y}''$ for all equitably rational preference models. This leads to the following definition of the equitable dominance relation (Ogryczak 1997b; Kostreva and Ogryczak 1999b).

Definition 1 *We say that outcome vector* $\mathbf{y}'$ *equitably dominates* $\mathbf{y}''$ ($\mathbf{y}' \prec_e \mathbf{y}''$) *if and only if, there exists a finite sequence of vectors* $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \dots, \mathbf{y}^t$ *such that* $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'_k} + \varepsilon_k \mathbf{e}_{i''_k}$, $0 \le \varepsilon_k \le y^{k-1}_{i'_k} - y^{k-1}_{i''_k}$ *for* $k = 1, 2, \dots, t$ *and there exists a permutation* τ *such that* $y'_{\tau(i)} \le y^t_i$ *for all* $i \in I$, *where at least one* $\varepsilon_k > 0$ *or at least one inequality* $y'_{\tau(i)} \le y^t_i$ *is strict.*

Outcome vectors $\mathbf{y}'$ and $\mathbf{y}''$ are equitably indifferent ($\mathbf{y}' \cong_e \mathbf{y}''$) if and only if, there exists a permutation $\tau \in \Pi(I)$ such that $y'_{\tau(i)} = y''_i$ for all $i \in I$. Hence, the relation of weak equitable dominance $\mathbf{y}' \preceq_e \mathbf{y}''$ denotes that either there exists a permutation $\tau \in \Pi(I)$ such that either $(y'_{\tau(1)}, \dots, y'_{\tau(m)}) = \mathbf{y}''$ or $(y'_{\tau(1)}, \dots, y'_{\tau(m)}) \le \mathbf{y}^t$ where sequence of vectors $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \dots, \mathbf{y}^t$ satisfies all the requirements of Definition 1. Thus, $\mathbf{y}' \preceq_e \mathbf{y}''$ if and only if, there exists a finite sequence of vectors $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \dots, \mathbf{y}^t$ such that $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'_k} + \varepsilon_k \mathbf{e}_{i''_k}$, $0 \le \varepsilon_k \le y^{k-1}_{i'_k} - y^{k-1}_{i''_k}$ for $k = 1, 2, \dots, t$ and there exists a permutation τ such that $y'_{\tau(i)} \le y^t_i$ for all $i \in I$.

Example 2 Let us consider a simple single facility location problem with three clients (C1, C2, and C3) and three potential locations (P1, P2 and P3). The distances (say in kilometers) between several clients and potential locations are given as follows:

	C1	C2	C3
P1	4	2	12
P2	6	6	6
P3	5	3	1

Hence, the potential locations generate the outcome vectors $\mathbf{y}' = (4, 2, 12)$, $\mathbf{y}'' = (6, 6, 6)$ and $\mathbf{y}''' = (5, 3, 1)$, respectively. Note that the perfectly equal outcome vector $\mathbf{y}''$ with all the distances 6 equitably dominates that for location P1 since $\mathbf{y}'' = (6, 6, 6)$ can be obtained from $\mathbf{y}' = (4, 2, 12)$ by application of two equitable transfers. On the other hand, outcome vector $\mathbf{y}''$ is obviously worse than unequal vector $\mathbf{y}'''$ with all the distances smaller than 6. Actually, the perfectly equal outcome vector $\mathbf{y}''$ is equitably dominated by unequal vector $\mathbf{y}'''$ since $\mathbf{y}''' = (5, 3, 1) \le (6, 6, 6) = \mathbf{y}''$. One may also notice that despite location P1 is not Pareto dominated by P3, it is equitably dominated since vector $\mathbf{y}'''$ appropriately rearranged (permuted) to (3,1,5) gets all the outcomes smaller than the corresponding outcomes of $\mathbf{y}' = (4, 2, 12)$.

Figure 1 presents the structure of equitable dominance for two-dimensional outcome vectors. For any outcome vector $\mathbf{y} = (y_1, y_2)$, the symmetric (permuted) vector (y_2, y_1) is

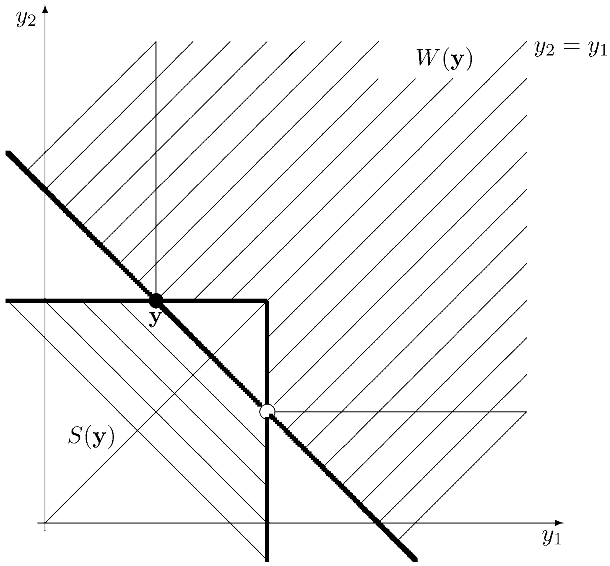

Fig. 1 Structure of the equitable dominance: $W(\mathbf{y})$—the set equitably dominated by $\mathbf{y}$, $S(\mathbf{y})$—the set of outcomes equitably dominating $\mathbf{y}$

equitably indifferent with $\mathbf{y}$. Set $W(\mathbf{y})$ of outcome vectors dominated by $\mathbf{y}$ contains all vectors Pareto dominated by $\mathbf{y}$ or by its symmetric copy, as well as vectors lying further from the perfect equity line $y_2 = y_1$. Set $S(\mathbf{y})$ of outcome vectors dominating $\mathbf{y}$ consists of vectors Pareto dominating $\mathbf{y}$ or its symmetric copy, as well as vectors closer to the perfect equity line. Note that set $S(\mathbf{y})$ is always convex.

We say that a location pattern $\mathbf{x} \in Q$ is *equitably efficient*, if and only if there does not exist any $\mathbf{x}' \in Q$ such that $\mathbf{f}(\mathbf{x}')$ equitably dominates $\mathbf{f}(\mathbf{x})$. In other words, a location pattern is equitably efficient if one cannot improve the distribution of its outcomes either by decreasing some of them or by any sequence of equitable transfers. Note that every equitably efficient location pattern is also Pareto-efficient but not vice versa. In Example 2 both locations P1 and P3 are Pareto-efficient but only P3 is equitably efficient.

The relation of equitable dominance $\preceq_e$ can be expressed as a vector inequality on the cumulative ordered outcomes. For the unweighted problem this can be mathematically formalized as follows. First, we introduce the ordering map $\Theta : R^m \rightarrow R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \geq \theta_2(\mathbf{y}) \geq \cdots \geq \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, 2, \ldots, m$. This allows us to focus on distributions of outcomes impartially. Next, we apply cumulation to the ordered outcome vectors to get quantities

$$\bar{\theta}_i(\mathbf{y}) = \sum_{j=1}^{i} \theta_j(\mathbf{y}) \quad \text{for } i = 1, 2, \ldots, m \tag{19}$$

expressing, respectively, the largest outcome, the total of the two largest outcomes, the total of the three largest outcomes, etc. Pointwise comparison of the cumulated ordered outcomes $\bar{\Theta}(\mathbf{y})$ was extensively analyzed within the theory of majorization (Marshall and Olkin 1979), where it is called the relation of weak submajorization. The theory of majorization includes the results which allows one to derive the following theorem (Kostreva and Ogryczak 1999b).

Theorem 1 *Outcome vector* $\mathbf{y}' \in Y$ *equitably dominates* $\mathbf{y}'' \in Y$, *if and only if* $\bar{\theta}_i(\mathbf{y}') \leq \bar{\theta}_i(\mathbf{y}'')$ *for all* $i \in I$ *where at least one strict inequality holds.*

The equitable dominance for general weighted problems can be derived by their disaggregation to the unweighted ones. It can be mathematically formalized as follows. First, we introduce the left-continuous right tail cumulative distribution function (cdf):

$$F_{\mathbf{y}}(d) = \sum_{i \in I_v} \bar{v}_i \delta_i(d) \quad \text{where } \delta_i(d) = \begin{cases} 1 & \text{if } y_{v_i} \geq d, \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

which for any real (outcome) value d provides the measure of outcomes greater or equal to d. Note that the requirement of impartiality means that two outcome vectors $\mathbf{y}'$ and $\mathbf{y}''$ resulting in identical cdf are indifferent. Next, we introduce the quantile function $F_{\mathbf{y}}^{(-1)}$ as the right-continuous inverse of the cumulative distribution function $F_{\mathbf{y}}$:

$$F_{\mathbf{y}}^{(-1)}(\beta) = \sup\{\eta : F_{\mathbf{y}}(\eta) \geq \beta\} \quad \text{for } 0 < \beta \leq 1.$$

By integrating $F_{\mathbf{y}}^{(-1)}$ one gets:

$$F_{\mathbf{y}}^{(-2)}(0) = 0 \quad \text{and} \quad F_{\mathbf{y}}^{(-2)}(\beta) = \int_0^{\beta} F_{\mathbf{y}}^{(-1)}(\alpha) d\alpha \quad \text{for } 0 < \beta \leq 1, \tag{21}$$

where $F_{\mathbf{y}}^{(-2)}(1) = \mu(\mathbf{y})$. Graphs of functions $F_{\mathbf{y}}^{(-2)}(\beta)$ (with respect to β) take the form of concave curves, the *(upper) absolute Lorenz curves*. The absolute Lorenz curve defines the relation (partial order) equivalent to the equitable dominance. Exactly, outcome vector $\mathbf{y}'$ equitably dominates $\mathbf{y}''$, if and only if $F_{\mathbf{y}'}^{(-2)}(\beta) \leq F_{\mathbf{y}''}^{(-2)}(\beta)$ for all $\beta \in (0, 1]$ where at least one strict inequality holds. Note that for the expanded form to the unweighted outcomes, the absolute Lorenz curve is completely defined by the values of the (cumulated) ordered outcomes. Hence, $\bar{\theta}_i(\mathbf{y}) = m F_{\mathbf{y}}^{(-2)}(i/m)$ for $i = 1, 2, \ldots, m$, and pointwise comparison of cumulated ordered outcomes is enough to justify equitable dominance. Figure 2 presents the absolute Lorenz curves for distance distributions of three locations from Example 2. One can easily see that vector of perfectly equal distances $\mathbf{y}'' = (6, 6, 6)$ dominates vector $\mathbf{y}' = (4, 2, 12)$, but it is further dominated by vector $\mathbf{y}''' = (5, 3, 1)$ of unequal smaller distances.

Alternatively, the equitable dominance can be expressed on the cumulative distribution functions. Having introduced the left-continuous right tail cumulative distribution function (20), one may further integrate it to get the second order cumulative distribution function $F_{\mathbf{y}}^{(2)}(\eta) = \int_{\eta}^{\infty} F_{\mathbf{y}}(\xi) d\xi$ for $\eta \in R$, representing average exceed over any real target η. Graphs of functions $F_{\mathbf{y}}^{(2)}(\eta)$ (with respect to η) take the form of convex decreasing curves (Ogryczak 1997b). By the theory of convex conjugate functions (Rockafellar 1970), the pointwise comparison of the second order cumulative distribution functions provides an alternative characterization of the equitable dominance relation (Ogryczak and Ruszczyński 2002). Exactly, $\mathbf{y}'$ equitably dominates $\mathbf{y}''$, if and only if $F_{\mathbf{y}'}^{(2)}(\eta) \leq F_{\mathbf{y}''}^{(2)}(\eta)$ for all η where at least one strict inequality holds.

Furthermore, the classical results of Hardy et al. (1934) allow us to refer the equitable dominance to the mean utility. For any convex, increasing utility function $u : R \to R$, if outcome vector $\mathbf{y}'$ equitably dominates $\mathbf{y}''$, then

$$\frac{1}{m} \sum_{i=1}^{m} u(y_i') = \sum_{i \in I_v} \bar{v}_i u(y_{v_i}') \leq \frac{1}{m} \sum_{i=1}^{m} u(y_i'') = \sum_{i \in I_v} \bar{v}_i u(y_{v_i}'').$$

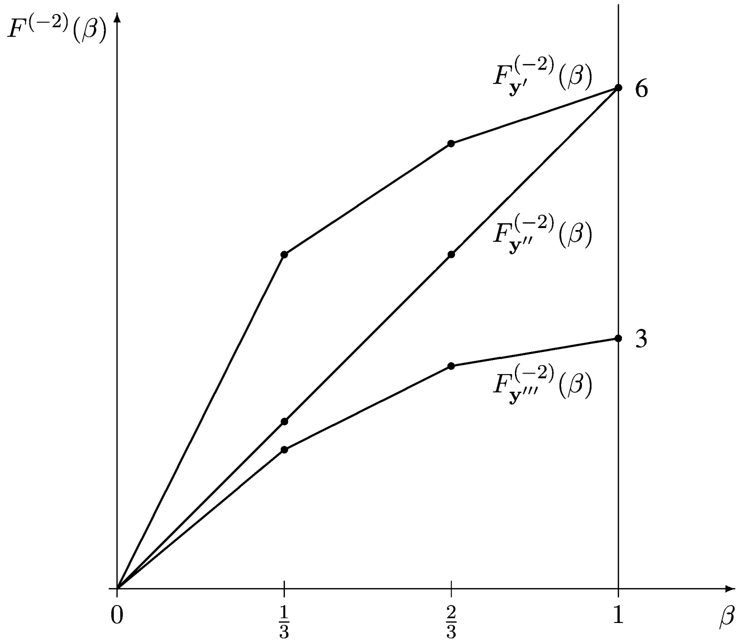

Fig. 2 Equitable dominance and the absolute Lorenz curves: $\mathbf{y}''' \prec_e \mathbf{y}'' \prec_e \mathbf{y}'$ for Example 2

Finally, there are three alternative analytical characterizations of the relation of equitable dominance as specified in the following theorem. Note that according to condition (iii), the equitable dominance is actually the so-called increasing convex order which is more commonly known as the second degree stochastic dominance (SSD) or stop loss order (Mueller and Stoyan 2002).

Theorem 2 *For any outcome vectors* $\mathbf{y}', \mathbf{y}'' \in A$ *each of the three following conditions is equivalent to the* (*weak*) *equitable dominance* $\mathbf{y}' \preceq_e \mathbf{y}''$:

(i) $F^{(-2)}_{\mathbf{y}'}(\beta) \leq F^{(-2)}_{\mathbf{y}''}(\beta)$ *for all* $\beta \in (0, 1]$;
(ii) $F^{(2)}_{\mathbf{y}'}(\eta) \leq F^{(2)}_{\mathbf{y}''}(\eta)$ *for all real* η;
(iii) $\sum_{i \in I_v} \bar{v}_i u(y'_i) \leq \sum_{i \in I_v} \bar{v}_i u(y''_i)$ *for any convex*, *increasing function* u.

We say that a solution concept (achievement function) $C(\mathbf{y})$ is *equitably consistent* if

$$\mathbf{y}' \preceq_e \mathbf{y}'' \quad \Rightarrow \quad C(\mathbf{y}') \leq C(\mathbf{y}''). \tag{22}$$

The relation of equitable consistency is called *strong* if, in addition, the following holds $\mathbf{y}' \prec_e \mathbf{y}'' \Rightarrow C(\mathbf{y}') < C(\mathbf{y}'')$. Note that an equitably consistent solution concept has to assign equal values to vectors representing identical distribution of outcomes, i.e. it must be a symmetric function of unweighted outcomes $C(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}) = C(y_1, y_2, \ldots, y_m)$ for any permutation $\tau \in \Pi(I)$. Further, for strong equitably consistency a solution concept should assign larger (worse) values $C(\mathbf{y}') > C(\mathbf{y})$ to all equitably dominated vectors $\mathbf{y}' \in W(\mathbf{y})$, as well as smaller (better) values $C(\mathbf{y}') < C(\mathbf{y})$ to all equitably dominating vectors $\mathbf{y}' \in S(\mathbf{y})$. Although various equitably consistent solution concepts may differently classify other outcomes vectors from white areas in Fig. 1.

Note that from the strong equitable consistency of a solution concept $C(\mathbf{y})$ it follows $C(\mathbf{y}') \leq C(\mathbf{y}'') \Rightarrow \mathbf{y}'' \not\prec_e \mathbf{y}'$. Hence, every optimal location generated by the solution concept $C(\mathbf{y})$ is equitably efficient. In the case of (weak) equitable consistency (22) only a weaker implication $C(\mathbf{y}') < C(\mathbf{y}'') \Rightarrow \mathbf{y}'' \not\prec_e \mathbf{y}'$ holds. Thus, any optimal location $\bar{\mathbf{y}}$ generated by the solution concept $C(\mathbf{y})$ cannot be equitably dominated by a nonoptimal location with a larger value $C(\mathbf{y})$ but it may be dominated by an alternative optimal location with the same value of $C(\mathbf{y})$. Certainly, the strong consistency is more desirable property but a very few solution concepts are strongly equitably consistent, similarly to a very few inequality measures satisfying completely the principle of transfers. Moreover, any equitably consistent solution concept may be regularized to reach the strong consistency and thereby to guarantee the equitable efficiency of the optimal location. Therefore, we do not restrict our analysis to strong consistency, though, paying special attention to identify this relation whenever possible.

According to condition (iii) of Theorem 2, for any convex, increasing function $u : R \to R$, the solution concept defined by achievement function $C(\mathbf{y}) = \sum_{i=1}^{m} u(y_i)$ is equitably consistent. In the case of strictly increasing and strictly convex function u the consistency is strong. Various convex functions u can be used to define such equitable solution concepts. In the case of the outcomes restricted to positive values, any p-power y^p is a strictly increasing and convex function for $p > 1$. This justifies the l_p norms as a source of equitable solution concepts, since the minimization of any such norm $\|\mathbf{y}\|_p$ is then equivalent to the minimization of $\|\mathbf{y}\|_p^p = \sum_{i=1}^{m} y_i^p$.

Condition (i) of Theorem 2 (or directly Theorem 1) permits one to seek equitably efficient location patterns as efficient solutions of the multiple criteria problem with objectives $\bar{\Theta}(\mathbf{f}(\mathbf{x}))$ (c.f. Kostreva and Ogryczak 1999a):

$$\min\{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}. \tag{23}$$

The worst outcome (4) and the mean outcome (2) correspond, respectively, to the first and to the last (m-th) criterion in problem (23). Thus both the center and the median concepts use only a single objective from the multiple criteria problem (23). It means that both concepts are equitably consistent in the sense of (22)

$$\mathbf{y}' \preceq_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') \leq \mu(\mathbf{y}'') \quad \text{and} \quad M(\mathbf{y}') \leq M(\mathbf{y}'').$$

However, they are not strongly consistent and the solutions can be equitably dominated by some alternative center or median solutions, respectively. In order to guarantee the equitable efficiency of a selected location pattern one need to take into account all the criteria of (23) like in the *lexicographic center* (Ogryczak 1997a). The lexicographic center is a refinement of the center solution concept which corresponds to the lexicographic approach to multicriteria optimization in (23) (Kostreva and Ogryczak 1999a). The cumulated ordered outcomes (19) are convex piecewise linear functions of $\mathbf{y}$ (Kostreva and Ogryczak 1999b) and they can be expressed with linear programming models (Ogryczak and Tamir 2003). Nevertheless, the multicriteria ordered model (23) is, in general, rather hard to implement as it requires the disaggregation of a location problem with client weights v_i which usually dramatically increases the problem size.

As a simplified approach one may consider a bicriteria mean-equity model (Mandell 1991): (15) taking into account both the efficiency with minimization of the mean outcome $\mu(\mathbf{y})$ and the equity with minimization of an inequality measure $\varrho(\mathbf{y})$. For typical inequality measures bicriteria model (15) is computationally very attractive since both the criteria are well defined directly for the weighted location problem without necessity of its disaggregation. Unfortunately, as pointed out in the previous section, for any dispersion type inequality measures the bicriteria mean-equity model is not consistent with the outcomes minimization, and therefore is not consistent with the equitable dominance. Note that the lack of consistency with the equitable dominance applies also to the maximum semideviation $\Delta(\mathbf{y})$ (12) whereas adding this measure to the mean $\mu(\mathbf{y}) + \Delta(\mathbf{y}) = M(\mathbf{y}) = \bar{\theta}_1(\mathbf{y})$ results in the worst outcome and thereby the first criterion of the ordered multicriteria model (23). In other words, although a direct use of the maximum semideviation contradicts the efficiency, the measure can be used complementary to the mean leading to the worst outcome criterion which is equitably consistent. This construction can be generalized for various (dispersion type) inequality measures. For any inequality measure ϱ we introduce the corresponding underachievement function defined as the sum of the mean outcome and the inequality measure itself, i.e.

$$M_{\varrho}(\mathbf{y}) = \mu(\mathbf{y}) + \varrho(\mathbf{y}). \tag{24}$$

In the case of maximum semideviation the corresponding underachievement $M_{\Delta}(\mathbf{y})$ function represents the worst outcome $M(\mathbf{y})$. Similarly, in the case of mean semideviation one gets the underachievement function

$$M_{\bar{\delta}}(\mathbf{y}) = \mu(\mathbf{y}) + \bar{\delta}(\mathbf{y}) = \frac{1}{m}\sum_{i\in I}\max\{y_i, \mu(\mathbf{y})\} = \sum_{i\in I_v}\bar{v}_i \max\{y_{v_i}, \mu(\mathbf{y})\}$$

representing the mean underachievement. Further, due to $|y_i - y_j| = 2\max\{y_i, y_j\} - y_i - y_j$, one gets an alternative formula for the mean absolute difference

$$D(\mathbf{y}) = \frac{1}{m^2}\sum_{i\in I}\sum_{j\in I}\max\{y_i, y_j\} - \mu(\mathbf{y}) = \sum_{i\in I_v}\sum_{j\in I_v}\bar{v}_i\bar{v}_j \max\{y_{v_i}, y_{v_j}\} - \mu(\mathbf{y}) \tag{25}$$

and the corresponding underachievement function

$$M_D(\mathbf{y}) = \mu(\mathbf{y}) + D(\mathbf{y}) = \frac{1}{m^2}\sum_{i\in I}\sum_{j\in I}\max\{y_i, y_j\} = \sum_{i\in I_v}\sum_{j\in I_v}\bar{v}_i\bar{v}_j \max\{y_{v_i}, y_{v_j}\}$$

representing the mean pairwise worse outcome. Both the above underachievement measures $M_{\bar{\delta}}(\mathbf{y})$ and $M_D(\mathbf{y})$ are equitably consistent (Ogryczak 2000). This leads us to a very important problem of identification of some clear conditions for inequality measures ϱ sufficient to guarantee that the corresponding underachievement measures are equitably consistent.

4 Consistency results

Inequality measures in mean-equity models are translation invariant (6) and inequality relevant deviation type measures (dispersion parameters). Thus, they are not affected by any shift of the outcome scale and they are equal to 0 in the case of perfectly equal outcomes while taking positive values for any unequal one. Moreover, they depend only on the distribution of outcomes thus in terms of the unweighted location model they are impartial, i.e., $\varrho(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}) = \varrho(y_1, y_2, \ldots, y_m)$ for any permutation τ. Unfortunately, as discussed earlier, such inequality measures are not consistent with the equitable optimization or axiomatic models of equitable preferences (Marshall and Olkin 1979;

Kostreva and Ogryczak 1999a). Indeed, in the bicriteria mean-equity model its efficient set may contain equitably inferior locations characterized by a small inequality but also very high distances.

This flaw can be overcome by replacing the original mean-equity bicriteria optimization (15) with the following bicriteria problem:

$$\min\{(\mu(\mathbf{f}(\mathbf{x})), \mu(\mathbf{f}(\mathbf{x})) + \varrho(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\} \tag{26}$$

where the second objective represents the corresponding underachievement measure $M_\varrho(\mathbf{y})$ (24). Note that for any inequality measure $\varrho(\mathbf{y}) \geq 0$ one gets $M_\varrho(\mathbf{y}) \geq \mu(\mathbf{y})$ thus really expressing underachievements (comparing to mean) from the perspective of outcomes being minimized.

The equitable consistency of inequality measures may be formalized as follows. We say that inequality measure $\varrho(\mathbf{y})$ is *mean-complementary equitably consistent* if the corresponding underachievement measure $M_\varrho(\mathbf{y})$ is equitably consistent, i.e.,

$$\mathbf{y}' \preceq_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') + \varrho(\mathbf{y}') \leq \mu(\mathbf{y}'') + \varrho(\mathbf{y}''). \tag{27}$$

The relation of equitable (mean-complementary) consistency is called *strong* if, in addition to (27), the following holds

$$\mathbf{y}' \prec_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') + \varrho(\mathbf{y}') < \mu(\mathbf{y}'') + \varrho(\mathbf{y}''). \tag{28}$$

Theorem 3 *If the inequality measure $\varrho(\mathbf{y})$ is mean-complementary equitably consistent* (27), *then except for outcomes with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$, every efficient solution of the bicriteria problem* (26) *is an equitably efficient location. In the case of strong consistency* (28), *every location $\mathbf{x} \in Q$ efficient to* (26) *is, unconditionally, equitably efficient.*

Proof Let $\mathbf{x}^0 \in Q$ be an efficient solution of (26). Suppose that $\mathbf{x}^0$ is not equitably efficient. This means, there exists $\mathbf{x} \in Q$ such that $\mathbf{y} = \mathbf{f}(\mathbf{x}) \prec_e \mathbf{y}^0 = \mathbf{f}(\mathbf{x}^0)$. Then, it follows $\mu(\mathbf{y}) \leq \mu(\mathbf{y}^0)$, and simultaneously $\mu(\mathbf{y}) + \varrho(\mathbf{y}) \leq \mu(\mathbf{y}^0) + \varrho(\mathbf{y}^0)$, by virtue of the mean-complementary equitable consistency (27). Since $\mathbf{x}^0$ is efficient to (26) no inequality can be strict, which implies $\mu(\mathbf{y}) = \mu(\mathbf{y}^0)$ and $\mu(\mathbf{y}) + \varrho(\mathbf{y}) = \mu(\mathbf{y}^0) + \varrho(\mathbf{y}^0)$ (and thereby $\varrho(\mathbf{y}) = \varrho(\mathbf{y}^0)$).

In the case of the strong mean-complementary equitable consistency (28), the supposition $\mathbf{y} = \mathbf{f}(\mathbf{x}) \prec_e \mathbf{y}^0 = \mathbf{f}(\mathbf{x}^0)$ implies $\mu(\mathbf{y}) \leq \mu(\mathbf{y}^0)$ and $\mu(\mathbf{y}) + \varrho(\mathbf{y}) < \mu(\mathbf{y}^0) + \varrho(\mathbf{y}^0)$ which contradicts the efficiency of $\mathbf{x}^0$ with respect to (26). Hence, $\mathbf{x}^0$ is equitably efficient. □

An important advantage of mean-equity approaches is the possibility of a pictorial trade-off analysis. Having assumed a trade-off coefficient λ between the inequality measure $\varrho(\mathbf{y})$ and the mean outcome, one may directly compare real values of $\mu(\mathbf{y}) + \lambda\varrho(\mathbf{y})$. Note that $(1 - \lambda)\mu(\mathbf{y}) + \lambda(\mu(\mathbf{y}) + \varrho(\mathbf{y})) = \mu(\mathbf{y}) + \lambda\varrho(\mathbf{y})$. Hence, the complete weighting parameterization of the mean-underachievement model (26) with $0 < \lambda < 1$ is equivalent to the bounded trade-off analysis of the bicriteria mean-equity model (15). This allows us to use Theorem 3 to derive the consistency results for the trade-off approach defined by solving the optimization problem

$$\min\{\mu(\mathbf{f}(\mathbf{x})) + \lambda\varrho(\mathbf{f}(\mathbf{x})) : \mathbf{x} \in Q\}. \tag{29}$$

Corollary 1 *If the inequality measure $\varrho(\mathbf{y})$ is mean-complementary equitably consistent* (27), *then except for location patterns with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$, every optimal solution of problem* (29) *with $0 < \lambda < 1$ is an equitably efficient solution. In the case of strong consistency* (28), *every location pattern $\mathbf{x} \in Q$ optimal to* (29) *with $0 < \lambda < 1$ is, unconditionally, equitably efficient.*

Typical dispersion type inequality measures are convex, i.e.

$$\varrho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \le \lambda\varrho(\mathbf{y}') + (1-\lambda)\varrho(\mathbf{y}'') \quad \text{for any } \mathbf{y}', \mathbf{y}'' \text{ and } 0 \le \lambda \le 1.$$

Actually, convexity of an inequality measure on equally distributed outcomes is necessary for its mean-complementary equitable consistency. Note, that for any two vectors $\mathbf{y}'$ and $\mathbf{y}''$ representing the same distribution of outcomes as $\mathbf{y}$ (i.e., $\mathbf{y}' = (y_{\tau'(1)}, \ldots, y_{\tau'(m)})$ for some permutation τ' and $\mathbf{y}'' = (y_{\tau''(1)}, \ldots, y_{\tau''(m)})$ for some permutation τ'') due to convexity of $\bar{\theta}_i(\mathbf{y})$, one gets $\bar{\theta}_i(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \le \lambda\bar{\theta}_i(\mathbf{y}') + (1-\lambda)\bar{\theta}_i(\mathbf{y}'') = \bar{\theta}_i(\mathbf{y})$ for all $i \in I$ and any $0 \le \lambda \le 1$. Hence, $\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'' \preceq_e \mathbf{y}$ and $M_\varrho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \le M_\varrho(\mathbf{y})$ is necessary for the equitable consistency. Thus, due to equal means $\mu(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') = \mu(\mathbf{y}') = \mu(\mathbf{y}'') = \mu(\mathbf{y})$, the inequality measure depending only on distribution $\varrho(\mathbf{y}') = \varrho(\mathbf{y}'') = \varrho(\mathbf{y})$ must satisfy $\varrho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \le \varrho(\mathbf{y}) = \lambda\varrho(\mathbf{y}') + (1-\lambda)\varrho(\mathbf{y}'')$ which represents the convexity of $\varrho(\mathbf{y})$. Certainly, the underachievement function $M_\varrho(\mathbf{y})$ must be also monotonic for the equitable consistency which enforces more restrictions on the inequality measures. We will show further that convexity together with positive homogeneity and some boundedness of an inequality measure is sufficient to guarantee monotonicity of the corresponding underachievement measure and thereby to guarantee the mean-complementary equitable consistency of the inequality measure itself.

We say that (dispersion type) inequality measure $\varrho(\mathbf{y}) \ge 0$ is *Δ-bounded* if it is upper bounded by the maximum upper deviation, i.e.,

$$\varrho(\mathbf{y}) \le \Delta(\mathbf{y}) \quad \forall\mathbf{y}. \tag{30}$$

Moreover, we say that $\varrho(\mathbf{y}) \ge 0$ is strictly Δ-bounded if inequality (30) is a strict bound, except from the case of perfectly equal outcomes, i.e.,

$$\varrho(\mathbf{y}) < \Delta(\mathbf{y}) \quad \text{for any } \mathbf{y} \text{ such that } \Delta(\mathbf{y}) > 0. \tag{31}$$

Theorem 4 *Let $\varrho(\mathbf{y}) \ge 0$ be a convex, positively homogeneous and translation invariant (dispersion type) inequality measure. If the measure is additionally Δ-bounded* (30), *then the corresponding underachievement function $M_\varrho(\mathbf{y}) = \mu(\mathbf{y}) + \varrho(\mathbf{y})$ is*:

(i) *monotonous*: $\mathbf{y}' \le \mathbf{y}''$ *implies* $M_\varrho(\mathbf{y}') \le M_\varrho(\mathbf{y}'')$,
(ii) *convex*: $M_\varrho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \le \lambda M_\varrho(\mathbf{y}') + (1-\lambda)M_\varrho(\mathbf{y}'')$ *for any* $0 \le \lambda \le 1$,
(iii) *positively homogeneous*: $M_\varrho(h\mathbf{y}) = hM_\varrho(\mathbf{y})$ *for positive real number* h,
(iv) *translation equivariant*: $M_\varrho(\mathbf{y} + a\mathbf{e}) = M_\varrho(\mathbf{y}) + a$, *for any real number* a.

If the inequality measure $\varrho(\mathbf{y})$ is strictly Δ-bounded (31), *then the corresponding underachievement function $M_\varrho(\mathbf{y})$ is*:

(i$'$) *strictly monotonous*: $\mathbf{y}' \le \mathbf{y}''$ *and* $\mathbf{y}' \ne \mathbf{y}''$ *implies* $M_\varrho(\mathbf{y}') < M_\varrho(\mathbf{y}'')$.

Proof If $\varrho(\mathbf{y}) \ge 0$ is a convex, positively homogeneous and translation invariant (dispersion type) inequality measure, then the underachievement function $M_\varrho(\mathbf{y}) = \mu(\mathbf{y}) + \varrho(\mathbf{y})$ does

satisfy the requirements of translation equivariance, positive homogeneity, and convexity. Further, if $\mathbf{y}' \leq \mathbf{y}''$, then $\mathbf{y}' = \mathbf{y}'' + (\mathbf{y}' - \mathbf{y}'')$ and $\mathbf{y}' - \mathbf{y}'' \leq 0$. Hence, due to convexity and positive homogeneity, $M_\varrho(\mathbf{y}') \leq M_\varrho(\mathbf{y}'') + M_\varrho(\mathbf{y}' - \mathbf{y}'')$. Moreover, due to the bound (30), $M_\varrho(\mathbf{y}' - \mathbf{y}'') \leq \mu(\mathbf{y}' - \mathbf{y}'') + \Delta(\mathbf{y}' - \mathbf{y}'') \leq \mu(\mathbf{y}' - \mathbf{y}'') + 0 - \mu(\mathbf{y}' - \mathbf{y}'') = 0$. Thus, $M_\varrho(\mathbf{y})$ satisfies also the requirement of monotonicity.
Note that strict upper bound (31) causes that $M_\varrho(\mathbf{y}' - \mathbf{y}'') < 0$ for $\mathbf{y}' \neq \mathbf{y}''$, thus showing strict monotonicity of $M_\varrho(\mathbf{y})$. □

Monotonicity and convexity of the underachievement function turns out to be sufficient for its equitable consistency. Therefore, the following assertion is valid.

Theorem 5 *Let $\varrho(\mathbf{y}) \geq 0$ be a convex and Δ-bounded positively homogeneous inequality measure. Then $\varrho(\mathbf{y})$ is mean-complementary equitably consistent in the sense of* (27), *i.e.*

$$\mathbf{y}' \preceq_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') + \varrho(\mathbf{y}') \leq \mu(\mathbf{y}'') + \varrho(\mathbf{y}'').$$

Proof The relation of equitable dominance $\mathbf{y}' \preceq_e \mathbf{y}''$ denotes that there exists a finite sequence of vectors $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \ldots, \mathbf{y}^t$ such that $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'_k} + \varepsilon_k \mathbf{e}_{i''_k}$, $0 \leq \varepsilon_k \leq y^{k-1}_{i'_k} - y^{k-1}_{i''_k}$ for $k = 1, 2, \ldots, t$ and there exists a permutation τ such that $y'_{\tau(i)} \leq y^t_i$ for all $i \in I$. Note that the underachievement function $M_\varrho(\mathbf{y})$, similar as $\varrho(\mathbf{y})$ depends only on the distribution of outcomes and, due to Theorem 4, is monotonous. Hence, $M_\varrho(\mathbf{y}') \leq M_\varrho(\mathbf{y}^t)$. Further, let us notice that $\mathbf{y}^k = \lambda \bar{\mathbf{y}}^{k-1} + (1-\lambda)\mathbf{y}^{k-1}$ where $\bar{\mathbf{y}}^{k-1} = \mathbf{y}^{k-1} - (y_{i'_k} - y_{i''_k})\mathbf{e}_{i'_k} + (y_{i'_k} - y_{i''_k})\mathbf{e}_{i''_k}$ and $\lambda = \varepsilon / (y_{i'_k} - y_{i''_k})$. Vector $\bar{\mathbf{y}}^{k-1}$ has the same distribution of coefficients as $\mathbf{y}^{k-1}$ (actually it represents results of swapping $y_{i'}$ and $y_{i''}$). Hence, due to convexity of $M_\varrho(\mathbf{y})$, one gets $M_\varrho(\mathbf{y}^k) \leq \lambda M_\varrho(\bar{\mathbf{y}}^{k-1}) + (1-\lambda) M_\varrho(\mathbf{y}^{k-1}) = M_\varrho(\mathbf{y}^{k-1})$. Thus, $M_\varrho(\mathbf{y}') \leq M_\varrho(\mathbf{y}'')$ which justifies the mean-complementary equitable consistency of the inequality measure $\varrho(\mathbf{y})$. □

For strict equitable consistency some strict monotonicity and convexity properties of the achievement function are needed. Obviously, there does not exist any inequality measure which is positively homogeneous and simultaneously strictly convex. However, one may notice from the proof of Theorem 5 that only convexity properties on equally distributed outcome vectors are important for monotonous achievement functions. We say that function $C(\mathbf{y})$ is strictly convex on equally distributed outcome vectors, if

$$C(\lambda \mathbf{y}' + (1-\lambda)\mathbf{y}'') < \lambda C(\mathbf{y}') + (1-\lambda) C(\mathbf{y}'') \quad \text{for } 0 < \lambda < 1$$

for any two vectors $\mathbf{y}' \neq \mathbf{y}''$ but representing the same outcomes distribution as some $\mathbf{y}$, i.e., $\mathbf{y}' = (y_{\tau'(1)}, \ldots, y_{\tau'(m)})$ for some permutation τ' and $\mathbf{y}'' = (y_{\tau''(1)}, \ldots, y_{\tau''(m)})$ for some permutation τ''.

Theorem 6 *Let $\varrho(\mathbf{y}) \geq 0$ be a convex and strictly Δ-bounded positively homogeneous inequality measure. If $\varrho(\mathbf{y})$ is also strictly convex on equally distributed outcomes, then it is mean-complementary equitably strongly consistent in the sense that of* (28), *i.e.*

$$\mathbf{y}' \prec_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') + \varrho(\mathbf{y}') < \mu(\mathbf{y}'') + \varrho(\mathbf{y}'').$$

Proof The relation of weak equitable dominance $\mathbf{y}' \preceq_e \mathbf{y}''$ denotes that there exists a finite sequence of vectors $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \ldots, \mathbf{y}^t$ such that $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'_k} + \varepsilon_k \mathbf{e}_{i''_k}$, $0 \leq \varepsilon_k \leq y^{k-1}_{i'_k} -$

$y^{k-1}_{i''_k}$ for $k = 1, 2, \dots, t$ and there exists a permutation τ such that $y'_{\tau(i)} \le y^t_i$ for all $i \in I$. The strict equitable dominance $\mathbf{y}' \prec_e \mathbf{y}''$ means that $y'_{\tau(i)} < y^t_i$ for some $i \in I$ or at least one ε_k is strictly positive. Note that the underachievement function $M_\varrho(\mathbf{y})$ is strictly monotonous and strictly convex on equally distributed outcome vectors. Hence, $M_\varrho(\mathbf{y}') < M_\varrho(\mathbf{y}'')$ which justifies the mean-complementary equitable strong consistency of the inequality measure $\varrho(\mathbf{y})$. □

Corollary 2 *Let $\varrho(\mathbf{y}) \ge 0$ be a convex, positively homogeneous and Δ-bounded (dispersion type) inequality measure. Then except for location patterns with identical mean $\mu(\mathbf{y})$ and inequality measure $\varrho(\mathbf{y})$, every efficient solution to the bicriteria problem (26) is an equitably efficient solution of the location problem (1). If the measure is also strictly Δ-bounded and strictly convex on equally distributed outcome vectors, then every location $\mathbf{x} \in Q$ efficient to (26) is, unconditionally, equitably efficient.*

As mentioned, typical inequality measures are convex and many of them are positively homogeneous. Moreover, the measures such as the mean absolute (upper) semideviation $\bar{\delta}(\mathbf{y})$ (13), the standard upper semideviation $\bar{\sigma}(\mathbf{y})$ (14), and the mean absolute difference $D(\mathbf{y})$ (7) are Δ-bounded. Indeed, one may easily notice that $y_i - \mu(\mathbf{y}) \le \Delta(\mathbf{y})$ and therefore

$$\bar{\delta}(\mathbf{y}) \le \frac{1}{m}\sum_{i\in I}\Delta(\mathbf{y}) = \Delta(\mathbf{y}),$$

$$\bar{\sigma}(\mathbf{y}) \le \sqrt{\Delta(\mathbf{y})^2} = \Delta(\mathbf{y}),$$

$$D(\mathbf{y}) = \frac{1}{m^2}\sum_{i\in I}\sum_{j\in I}(\max\{y_i, y_j\} - \mu(\mathbf{y})) \le \Delta(\mathbf{y})$$

where the last formula is due to (25). Actually, all these inequality measures are strictly Δ-bounded since for any unequal outcome vector at least one outcome must be below the mean thus leading to strict inequalities in the above bounds. Obviously, Δ-bounded (but not strictly) is also the maximum absolute upper deviation $\Delta(\mathbf{y})$ itself. The same applies to the quantile generalization of the maximum upper deviations, i.e. to the worst conditional k/m-semideviation defined as the average of k largest semideviations (Ogryczak and Zawadzki 2002):

$$\Delta_{k/m}(\mathbf{y}) = \frac{1}{k}\sum_{i=1}^{k}(\theta_i(\mathbf{y}) - \mu(\mathbf{y})) \tag{32}$$

for unweighted problems while generalized to $\Delta_\beta(\mathbf{y}) = \frac{1}{\beta}\int_0^\beta (F^{(-1)}_{\mathbf{y}}(\alpha) - \mu(\mathbf{y}))d\alpha$ for weighted problems and any real $0 < \beta \le 1$. Thus, the following assertion is valid.

Corollary 3 *The following inequality measures $\varrho(\mathbf{y})$ are mean-complementary equitably consistent in the sense of (27):*

1. *the maximum upper deviation $\Delta(\mathbf{y})$ (12),*
2. *the mean absolute (upper) semideviation $\bar{\delta}(\mathbf{y})$ (13),*
3. *the standard upper semideviation $\bar{\sigma}(\mathbf{y})$ (14),*
4. *the mean absolute difference $D(\mathbf{y})$ (7),*
5. *the worst conditional k/m-semideviation $\Delta_{k/m}(\mathbf{y})$ (32).*

We emphasize that, despite the standard semideviation is a mean-complementary equitably consistent inequality measure, the consistency is not valid for variance, semivariance and even for the standard deviation. These measures, in general, do not satisfy all assumptions of Theorem 5. Corollary 3 enumerates only the simplest inequality measures studied in the locational context which satisfy the assumptions of Theorem 5 and thereby they are mean-complementary equitably consistent. Theorem 5 allows one to show this property for many other measures. In particular, one may easily find out that any convex combination of mean-complementary equitably efficient inequality measures remains also consistent. On the other hand, among typical inequality measures the mean absolute difference seems to be the only one meeting the stronger assumptions of Theorem 6 and thereby maintaining the strong consistency.

Corollary 4 *The mean absolute difference $D(\mathbf{y})$ (7) is mean-complementary equitably strongly consistent in the sense of* (28).

Note that the mean absolute semideviations are symmetric in the sense that the upper semideviation is always equal to the downside one. In other words, $\bar{\delta}(\mathbf{y}) = \frac{1}{2}\delta(\mathbf{y})$ and thereby Theorem 3 justifies also equitable consistency of the half mean absolute deviation. In general, one may just consider $\alpha\varrho(X)$ as a basic inequality measure, like the mean absolute semideviation equal to the half of the mean absolute deviation itself. In order to avoid creation of new inequality measures by simple scaling we rather parameterize the equitable consistency concept. We will say that an inequality measure ϱ is equitably α-consistent if

$$\mathbf{y}' \preceq_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') + \alpha\varrho(\mathbf{y}') \le \mu(\mathbf{y}'') + \alpha\varrho(\mathbf{y}''). \tag{33}$$

The relation of equitable α-consistency will be called *strong* if, in addition to (33), the following holds

$$\mathbf{y}' \prec_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') + \alpha\varrho(\mathbf{y}') < \mu(\mathbf{y}'') + \alpha\varrho(\mathbf{y}''). \tag{34}$$

Note that the equitable 1-consistency represent our basic relation of the mean-complementary equitable consistency. On the other hand, the equitable α-consistency of measure $\varrho(\mathbf{y})$ is equivalent to the mean-complementary equitable consistency of measure $\alpha\varrho(\mathbf{y})$. Thus the following assertion is valid.

Corollary 5 *If the inequality measure $\varrho(\mathbf{y})$ is equitably α-consistent* (33), *then except for outcomes with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$, every efficient solution of the bicriteria problem*

$$\min\{(\mu(\mathbf{f}(\mathbf{x})), \mu(\mathbf{f}(\mathbf{x})) + \alpha\varrho(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\} \tag{35}$$

is an equitably efficient location. In the case of strong consistency (28), *every location $\mathbf{x} \in Q$ efficient to* (35) *is, unconditionally, equitably efficient.*

In terms of the trade-off approach it leads us to the following statement.

Corollary 6 *If the inequality measure $\varrho(\mathbf{y})$ is equitably α-consistent* (33), *then except for location patterns with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$, every optimal solution of problem* (29) *with $0 < \lambda < \alpha$ is an equitably efficient solution. In the case of strong α-consistency* (34), *every location pattern $\mathbf{x} \in Q$ optimal to* (29) *with $0 < \lambda < \alpha$ is, unconditionally, equitably efficient.*

Directly from Theorems 5 and 6 applied to the measures $\alpha\varrho(\mathbf{y})$ as in definition of α-consistency one gets the following sufficient conditions.

Theorem 7 *Let $\varrho(\mathbf{y}) \geq 0$ be a convex, positively homogeneous and translation invariant (dispersion type) inequality measure. If $\alpha\varrho(\mathbf{y})$ is Δ-bounded, then $\varrho(\mathbf{y})$ is equitably α-consistent in the sense of (33), i.e.*

$$\mathbf{y}' \preceq_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') + \alpha\varrho(\mathbf{y}') \leq \mu(\mathbf{y}'') + \alpha\varrho(\mathbf{y}'').$$

Theorem 8 *Let $\varrho(\mathbf{y}) \geq 0$ be a convex and positively homogeneous inequality measure. If $\varrho(\mathbf{y})$ is also strictly convex on equally distributed outcomes and $\alpha\varrho(\mathbf{y})$ is strictly Δ-bounded, then $\varrho(\mathbf{y})$ is equitably strongly α-consistent in the sense of (34), i.e.*

$$\mathbf{y}' \prec_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') + \alpha\varrho(\mathbf{y}') < \mu(\mathbf{y}'') + \alpha\varrho(\mathbf{y}'').$$

Note that the equitable $\bar{\alpha}$-consistency of measure $\varrho(\mathbf{y})$ actually guarantees the mean-complementary equitable consistency of measure $\alpha\varrho(\mathbf{y})$ for all $0 < \alpha \leq \bar{\alpha}$, and the same remain valid for the strong consistency properties. It follows from a possible expression of $\mu(\mathbf{y}) + \alpha\varrho(\mathbf{y})$ as the convex combination

$$\mu(\mathbf{y}) + \alpha\varrho(\mathbf{y}) = \frac{\alpha}{\bar{\alpha}}(\mu(\mathbf{y}) + \bar{\alpha}\varrho(\mathbf{y})) + \left(1 - \frac{\alpha}{\bar{\alpha}}\right)\mu(\mathbf{y})$$

with $0 < \alpha/\bar{\alpha} \leq 1$. Hence, for any $\mathbf{y}' \preceq_e \mathbf{y}''$, due to $\mu(\mathbf{y}') \leq \mu(\mathbf{y}'')$ one gets $\mu(\mathbf{y}') + \alpha\varrho(\mathbf{y}') \leq \mu(\mathbf{y}'') + \alpha\varrho(\mathbf{y}'')$ in the case of the equitable $\bar{\alpha}$-consistency of measure $\varrho(\mathbf{y})$ (or respective strict inequality in the case of strong consistency). Therefore, while analyzing specific inequality measures we seek the largest values α guaranteeing the corresponding equitable efficiency.

As mentioned, the mean absolute semideviation is twice the mean absolute upper semideviation which means that $\alpha\delta(\mathbf{y})$ is Δ-bounded for any $0 < \alpha \leq 0.5$. The symmetry of mean absolute semideviations $\bar{\delta}(\mathbf{y}) = \sum_{i\in I}(y_i - \mu(\mathbf{y}))_+ = \sum_{i\in I}(\mu(\mathbf{y}) - y_i)_+$ can be also used to derive some Δ-boundedness relations for other inequality measures. In particular, one may find out that for m-dimensional outcome vectors of unweighted location problem, any downside semideviation from the mean cannot be larger than $m-1$ upper semideviations. Hence, the maximum absolute deviation satisfies the inequality $\frac{1}{m-1}R(\mathbf{y}) \leq \Delta(\mathbf{y})$, while the maximum absolute difference fulfills $\frac{1}{m}S(\mathbf{y}) \leq \Delta(\mathbf{y})$. In the case of weighted problems these bounds take the forms $\min_{i\in I_v} \bar{v}_i/(1 - \min_{i\in I_v} \bar{v}_i)R(\mathbf{y}) \leq \Delta(\mathbf{y})$ and $\min_{i\in I_v} \bar{v}_i S(\mathbf{y}) \leq \Delta(\mathbf{y})$, respectively. Similarly, for the standard deviation one gets

$$\frac{1}{\sqrt{m-1}}\delta(\mathbf{y}) \leq \Delta(\mathbf{y}) \quad \text{or} \quad \sqrt{\frac{\min_{i\in I_v} \bar{v}_i}{1 - \min_{i\in I_v} \bar{v}_i}}\delta(\mathbf{y}) \leq \Delta(\mathbf{y})$$

for unweighted or weighted problems, respectively. Actually, $\alpha\sigma(\mathbf{y})$ is strictly Δ-bounded for any $0 < \alpha \leq 1/\sqrt{m-1}$ since for any unequal outcome vector at least one outcome must be below the mean thus leading to strict inequalities in the above bounds. These leads us to the following corollary.

Corollary 7 *The following inequality measures $\varrho(\mathbf{y})$ are equitably α-consistent within the specified intervals of α:*

1. *the mean absolute deviation with* $0 < \alpha \leq 0.5$,
2. *the maximum absolute deviation with* $0 < \alpha \leq \frac{1}{m-1}$, *or* $0 < \alpha \leq \frac{\min_{i \in I_v} \bar{v}_i}{1 - \min_{i \in I_v} \bar{v}_i}$ *in the weighted case*,
3. *the maximum absolute difference with* $0 < \alpha \leq \frac{1}{m}$, *or* $0 < \alpha \leq \min_{i \in I_v} \bar{v}_i$ *in the weighted case*,
4. *the standard deviation with* $0 < \alpha \leq \frac{1}{\sqrt{m-1}}$, *or* $0 < \alpha \leq \sqrt{\frac{\min_{i \in I_v} \bar{v}_i}{1 - \min_{i \in I_v} \bar{v}_i}}$ *in the weighted case.*

Moreover, the α*-consistency of the standard deviation is strong.*

The equitable consistency results for basic dispersion type inequality measures considered in location problems are summarized in Table 1 where α values for unweighted as well as weighted problems are given and the strong consistency is indicated. One may easily notice that all the inequality measures with corresponding values of α result in $\mu(\mathbf{y}''') + \alpha\varrho(\mathbf{y}''') < \mu(\mathbf{y}'') + \alpha\varrho(\mathbf{y}'') < \mu(\mathbf{y}') + \alpha\varrho(\mathbf{y}')$ when applied to $\mathbf{y}''' \prec_e \mathbf{y}'' \prec_e \mathbf{y}'$ from Example 2. Table 1 points out how the inequality measures can be used in location models to guarantee their harmony both with distance minimization (Pareto-efficiency) and with inequalities minimization (Pigou–Dalton equity theory). Exactly, for each inequality measure applied with the corresponding value α from Table 1 (or smaller positive value), every efficient solution of the bicriteria problem (35), i.e. $\min\{(\mu(\mathbf{f}(\mathbf{x})), \mu(\mathbf{f}(\mathbf{x})) + \alpha\varrho(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}$, is an equitably efficient location, except for outcomes with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$. In the case of strong consistency (as for mean absolute difference or standard deviation), every location $\mathbf{x} \in Q$ efficient to (35) is, unconditionally, equitably efficient.

To illustrate further the results of Table 1 let us consider Example 1. Recall that the perfectly equal outcome vector $\mathbf{y}''$ with all the distances 10 is obviously worse than unequal vector $\mathbf{y}'$ with one distance smaller than 10. Actually, $\mathbf{y}'$ Pareto dominates $\mathbf{y}''$ and therefore it is the only equitably efficient solution to this location problem. While calculating several inequality measures for those outcome vectors one gets the results presented in Table 2. One may easily notice that all the inequality measures satisfy corresponding inequalities $\mu(\mathbf{y}') + \alpha\varrho(\mathbf{y}') \leq \mu(\mathbf{y}'') + \alpha\varrho(\mathbf{y}'')$ when using α from Table 1 or smaller, e.g. $\mu(\mathbf{y}') + 1/3\sigma(\mathbf{y}') = 9 + 1 \leq \mu(\mathbf{y}'') + 1/3\sigma(\mathbf{y}'') = 10$. Due to those inequalities, location P1 is the only Pareto-optimal solution of the bicriteria problem $\min\{(\mu(\mathbf{f}(\mathbf{x})), \mu(\mathbf{f}(\mathbf{x})) + \alpha\varrho(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}$. On the other hand, $\mu(\mathbf{y}') + \sigma(\mathbf{y}') = 9 + 3 > \mu(\mathbf{y}'') + \sigma(\mathbf{y}'') = 10$ and location P2 is a Pareto-optimal solution to the bicriteria problem $\min\{(\mu(\mathbf{f}(\mathbf{x})), \mu(\mathbf{f}(\mathbf{x})) + \sigma(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}$, as

Table 1 Equitable consistency results for the basic dispersion type inequality measures

Measure			α-consistency		
Standard upper semideviation	$\bar{\sigma}(\mathbf{y})$	(14)	1	1	
Standard deviation	$\sigma(\mathbf{y})$	(11)	$\frac{1}{\sqrt{m-1}}$	$\sqrt{\frac{\min_{i \in I_v} \bar{v}_i}{1 - \min_{i \in I_v} \bar{v}_i}}$	strong
Mean absolute semideviation	$\bar{\delta}(\mathbf{y})$	(13)	1	1	
Mean absolute deviation	$\delta(\mathbf{y})$	(9)	$\frac{1}{2}$	$\frac{1}{2}$	
Maximum upper semideviation	$\Delta(\mathbf{y})$	(12)	1	1	
Maximum absolute deviation	$R(\mathbf{y})$	(10)	$\frac{1}{m-1}$	$\frac{\min_{i \in I_v} \bar{v}_i}{1 - \min_{i \in I_v} \bar{v}_i}$	
Conditional k/m-semideviation	$\Delta_{k/m}(\mathbf{y})$	(32)	1	1	
Mean absolute difference	$D(\mathbf{y})$	(7)	1	1	strong
Maximum absolute difference	$S(\mathbf{y})$	(8)	$\frac{1}{m}$	$\min_{i \in I_v} \bar{v}_i$	

Table 2 Values of inequality measures for Example 1

$\mathbf{y}$	$\mu(\mathbf{y})$	$\bar{\sigma}(\mathbf{y})$	$\sigma(\mathbf{y})$	$\bar{\delta}(\mathbf{y})$	$\delta(\mathbf{y})$	$\Delta(\mathbf{y})$	$R(\mathbf{y})$	$\Delta_{3/10}(\mathbf{y})$	$D(\mathbf{y})$	$S(\mathbf{y})$
$\mathbf{y}'$	9	0.95	3	0.9	1.8	1	9	1	0.9	10
$\mathbf{y}''$	10	0	0	0	0	0	0	0	0	0

well as for the standard mean-equity model $\min\{(\mu(\mathbf{f}(\mathbf{x})), \sigma(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}$. Similar effects may be observed for the mean absolute deviation, the maximum absolute deviation or the maximum absolute difference if not respecting the corresponding bounds α.

The consistency results summarized in Table 1 are sufficient conditions. This means that whenever the α limit is observed the corresponding consistency relation is valid for any location problem. It may happen that for a specific location problem and a specific inequality measure the equitable consistency is valid for larger values of α. Nevertheless, we have provided strict bounds in the sense that for a larger value of α there exists a location problem on which the equitable consistency is not valid, and the bicriteria problem (35) may generate equitably dominated solution. This can be usually demonstrated with a simple counterexample of a weighted discrete single facility location problems with 2 clients and 2 potential locations P1 and P2, similar to Example 1. Let us assume that the location P1 generates distance 1 to both the clients while P2 generates distance 0 to the first client and distance 1 to the second. For any positive client weights $\bar{v}_1$ and $\bar{v}_2$ the second location equitably dominates the first one (actually, it is obviously better). Nevertheless, while using the mean absolute semideviation with $\alpha = 1 + \varepsilon$ ($\varepsilon > 0$) one get the value of $\mu(\mathbf{y}) + \alpha\bar{\delta}(\mathbf{y})$ smaller for P1 than that for P2 whenever $\bar{v}_2 = 1/(1+\omega)$ with $0 < \omega < \varepsilon$ and $\bar{v}_1 = 1 - \bar{v}_2$. While using the standard semideviation the same effect can be demonstrated with $\bar{v}_2 = 1/(1+\omega)^2$. Similar counterexamples may easily be built for other measures.

It follows from Table 1 that all basic inequality measures maintain some equitable consistency when appropriately used in location models. It does not mean, however, that any such measure allows one to model all possible equitable preferences. The bicriteria model (35) is only a simplified approximation to the multiple criteria model (23) representing the entire gamut of equitable preferences. Hence, for any inequality measure specific equitable efficient locations may exist which cannot be identified with the bicriteria problem (35). Nevertheless, the bicriteria models are just commonly accepted due to their simplicity in modeling the mean-equity preferences. Our analysis summarized in Table 1 shows how various inequality measures can be used for this purpose not contradicting the equitable dominance and thereby not leading to solutions obviously inefficient with respect to distances (outcomes) minimization.

Among numerous measures listed in Table 1 only the mean absolute difference and the standard deviation fulfill the strong consistency which is crucial to guarantee the equitable efficiency of any selected solution. The mean absolute difference being mean-complementary equitably strongly consistent may be used to regularize other consistent but not strongly consistent inequality measures, Namely, if $\varrho(\mathbf{y})$ is a mean-complementary equitably consistent inequality measure, then for any $0 < \varepsilon < 1$ the convex combination $(1-\varepsilon)\varrho(\mathbf{y}) + \varepsilon D(\mathbf{y})$ satisfies strict forms of both Δ-boundedness and convexity requirements and therefore it is mean-complementary equitably strongly consistent. By using arbitrary small positive ε, this approach allows one to build strongly consistent forms (actually regularizations) of maximum semideviation $(1-\varepsilon)\Delta(\mathbf{y}) + \varepsilon D(\mathbf{y})$, of the mean absolute semideviation $(1-\varepsilon)\bar{\delta}(\mathbf{y}) + \varepsilon D(\mathbf{y})$, or other equitably consistent inequality measures.

When used with a very small coefficient, the standard deviation also maintains strong mean-complementary equitable consistency which allows one to consider it as a regularization term transforming any mean-complementary equitably consistent inequality measure $\varrho(\mathbf{y})$ into a strongly consistent measure $(1-\varepsilon)\varrho(\mathbf{y}) + \varepsilon\sigma(\mathbf{y})$. Usage of the mean absolute difference for this purpose seems to be simpler, though, due to its linear programming computability (Kostreva and Ogryczak 1999a).

5 Concluding remarks

When making location decisions related to public service facilities, such as schools, nurseries, libraries, health-services and other urban public facilities (Abernathy and Hershey 1972; Benito Alonso and Devaux 1981; Coulter 1980; Malczewski 2000; Mandell 1991; O'Brien 1969; Tsou et al. 2005) the distribution of distances among the service recipients (clients) and equity or fairness in treatment of the population is an important issue. Problems of fair location-allocation decisions arise also in technical systems like in telecommunication networks which must serve various users (Pióro and Medhi 2004). In order to take into account both the overall efficiency and equity the bicriteria mean-equity approaches are usually applied, where the mean distance as well as some inequality measure are minimized. Quantification of the equity in a scalar inequality measure is well appealing to system designers and not complicating too much the decision model still allowing for consideration of multiple criteria spatial, economic, and others which is a common need while dealing with public services (Current et al. 1990; Malczewski and Ogryczak 1988, 1990). Unfortunately, for typical inequality measures, the mean-equity approach may lead to inferior conclusions with respect to distances minimization. The class of preference models complying with the minimization of distances as well as with an equal consideration of the clients is mathematically formalized with the concept of equitable dominance. Solution concepts equitably consistent (consistent with the equitable dominance) do not contradict the minimization of distances or the inequality minimization. Therefore, the achievement of equitable consistency by the mean-equity models has a paramount importance.

In this paper we have analyzed how scalar inequality measures can be used to guarantee the equitable consistency. It turns out that several inequality measures can be combined with the mean itself into the optimization criteria generalizing the concept of the worst outcome and generating equitably consistent underachievement measures. We have introduced general conditions for inequality measures sufficient to provide the equitable consistency of the corresponding underachievement measures. We have shown that properties of convexity and positive homogeneity together with being bounded by the maximum upper semideviation are sufficient for a typical inequality measure to guarantee the corresponding equitable consistency. It allows us to identify various inequality measures which can be effectively used to incorporate equity factors into various location problems while preserving the consistency with outcomes minimization. Among others the standard upper semideviation turns out to be such a consistent inequality measure while the mean absolute difference is strongly consistent.

Our analysis is related to the properties of solutions to location models. It has been shown how inequality measures can be included into the location models avoiding contradiction to the minimization of distances. We do not analyze algorithmic issues of the models. Generally, the requirement of convexity necessary for the consistency, guarantees that the corresponding optimization criteria belong to the class of convex optimization, not complicating the original location model with any additional discrete structure. Many of the inequality measures, we analyzed, can be implemented with auxiliary linear programming constraints. Nevertheless, further research on efficient computational algorithms for solving the corresponding equitable location models (Nickel and Puerto 2005) is necessary.

This paper is focused on location problems. However, the location decisions are analyzed from the perspective of their effects for individual clients. Therefore, the general concept of the proposed approaches can be used for optimization of various systems which serve many users.

References

Abernathy, W. J., & Hershey, J. C. (1972). A spatial-allocation model for regional health-services planning. *Operations Research*, *20*, 629–642.

Atkinson, A. B. (1970). On the measurement of inequality. *Journal of Economic Theory*, *2*, 244–263.

Benito Alonso, M. A., & Devaux, P. (1981). Location and size of day nurseries. *European Journal of Operational Research*, *6*, 195–198.

Berman, O. (1990). Mean-variance location problems. *Transportation Science*, *24*, 287–293.

Berman, O., & Kaplan, E. H. (1990). Equity maximizing facility location schemes. *Transportation Science*, *24*, 137–144.

Carrizosa, E. (1999). Minimizing the variance of euclidean distances. *Studies in Locational Analysis*, *12*, 101–118.

Coulter, P. B. (1980). Measuring the inequality of urban public services: a methodological discussion with applications. *Policy Studies Journal*, *8*, 693–698.

Current, J., Min, H., & Schilling, D. (1990). Multiobjective analysis of facility location decisions. *European Journal of Operational Research*, *49*, 295–307.

Erkut, E. (1993). Inequality measures for location problems. *Location Science*, *1*, 199–217.

Francis, R. L., McGinnis, L. F., & White, J. A. (1992). *Facility layout and location: an analytical approach*. Englewood Cliffs: Prentice-Hall.

Hardy, G. H., Littlewood, J. E., & Pólya, G. (1934). *Inequalities*. Cambridge: Cambridge University Press.

Hay, A. M. (1995). Concepts of equity, fairness and justice in geographical studies. *Transactions of the Institute of British Geographer*, *20*, 500–508.

Kostreva, M. M., & Ogryczak, W. (1999a). Equitable approaches to location problems. In J.-C. Thill (Ed.), *Spatial multicriteria decision making and analysis: a geographic information sciences approach* (pp. 103–126). Brookfield: Ashgate.

Kostreva, M. M., & Ogryczak, W. (1999b). Linear optimization with multiple equitable criteria. *RAIRO Recherche Opérationnelle*, *33*, 275–297.

Kostreva, M. M, Ogryczak, W., & Wierzbicki, A. (2004). Equitable aggregations and multiple criteria analysis. *European Journal of Operational Research*, *158*, 362–377.

Labbé, M., Peeters, D., & Thisse, J.-F. (1996). Location on networks. In M. O. Ball, T. L. Magnanti, C. L. Monma & G. L. Nemhauser (Eds.), *Handbook in operations research and management science: network routing* (pp. 551–624). Amsterdam: North-Holland.

López-de-los-Mozos, M. C., & Mesa, J. A. (2001). The maximum absolute deviation measure in location problems on networks. *European Journal of Operational Research*, *135*, 184–194.

López-de-los-Mozos, M. C., & Mesa, J. A. (2003). The sum of absolute differences on a network: algorithm and comparison with other equality measures. *INFOR*, *41*, 195–210.

Love, R. F., Morris, J. G., & Wesolowsky, G. O. (1988). *Facilities location: models & methods*. New York: North-Holland.

Maimon, O. (1986). The variance equity measure in locational decision theory. *Annals of Operations Research*, *6*, 147–160.

Maimon, O. (1988). An algorithm for the Lorenz measure in locational decisions on trees. *Journal of Algorithms*, *9*, 583–596.

Malczewski, J. (2000). Multicriteria spatial allocation of educational resources: an overview. *Socio-Economic Planning Sciences*, *34*, 219–235.

Malczewski, J., & Ogryczak, W. (1988). A multiobjective approach to reorganization of health service areas: a case study. *Environment and Planning A*, *20*, 1461–1470.

Malczewski, J., & Ogryczak, W. (1990). An interactive approach to the central facility location problem: locating pediatric hospitals in Warsaw. *Geographical Analysis*, *22*, 244–258.

Mandell, M. B. (1991). Modelling effectiveness-equity trade-offs in public service delivery systems. *Management Science*, *37*, 467–482.

Marsh, M. T., & Schilling, D. A. (1994). Equity measurement in facility location analysis: a review and framework. *European Journal of Operational Research*, *74*, 1–17.
Marshall, A. W., & Olkin, I. (1979). *Inequalities: theory of majorization and its applications*. New York: Academic Press.
Mayhew, L. D., & Leonardi, G. (1982). Equity, efficiency, and accessibility in urban and regional health-care systems. *Environment and Planning A*, *14*, 1479–1507.
Mirchandani, P. B., & Francis, R. L. (1990). *Discrete location theory*. New York: Wiley.
Morrill, R. L., & Symons, J. (1977). Efficiency and equity aspects of optimum location. *Geographical Analysis*, *9*, 215–225.
Mueller, A., & Stoyan, D. (2002). *Comparison methods for stochastic models and risks*. New York: Wiley.
Mulligan, G. F. (1991). Equity measures and facility location. *Papers in Regional Science*, *70*, 345–365.
Nickel, S., & Puerto, J. (2005). *Location theory: a unified approach*. Berlin: Springer.
O'Brien, R. J. (1969). Model for planning the location and size of urban schools. *Socio-Economic Planning Sciences*, *2*, 141–153.
Ogryczak, W. (1997a). On the lexicographic minimax approach to location problems. *European Journal of Operational Research*, *100*, 566–585.
Ogryczak, W. (1997b). *Linear and discrete optimization with multiple criteria: preference models and applications to decision support*. Warsaw: Warsaw University Press (in Polish)
Ogryczak, W. (1999). On the distribution approach to location problems. *Computers and Industrial Engineering*, *37*, 595–612.
Ogryczak, W. (2000). Inequality measures and equitable approaches to location problems. *European Journal of Operational Research*, *122*, 374–391.
Ogryczak, W., & Ruszczyński, A. (2002). Dual stochastic dominance and related mean-risk models. *SIAM Journal on Optimization*, *13*, 60–78.
Ogryczak, W., & Tamir, A. (2003). Minimizing the sum of the *k* largest functions in linear time. *Information Processing Letters*, *85*, 117–122.
Ogryczak, W., & Zawadzki, M. (2002). Conditional median: a parametric solution concept for location problems. *Annals of Operations Research*, *110*, 167–181.
Pióro, M., & Medhi, D. (2004). *Routing, flow and capacity design in communication and computer networks*. San Francisco: Kaufmann.
Rockafellar, R. T. (1970). *Convex analysis*. Princeton: Princeton University Press.
Sen, A. (1973). *On economic inequality*. Oxford: Clarendon.
Tsou, K.-W., Hung, Y.-T., & Chang, Y.-L. (2005). An accessibility-based integrated measure of relative spatial equity in urban public facilities. *Cities*, *22*, 424–435.
Vincke, P. (1992). *Multicriteria decision-aid*. New York: Wiley.
Young, H. P. (1994). *Equity in theory and practice*. Princeton: Princeton University Press.