Preemptive Reference Point Method

Włodzimierz Ogryczak

Service de Mathématiques de la Gestion, Université Libre de Bruxelles, B–1050 Bruxelles, Belgium
on leave from Institute of Informatics, Warsaw University, Warsaw, Poland

Abstract. The reference point method for solving multi–criteria optimization problems is an interactive technique where the decision maker specifies requirements, similar to goal programming, in terms of aspiration levels. Ogryczak [5] showed how the reference point method could be modeled within goal programming methodology provided that the nonnegativity restrictions on weights were dropped. It allows us to consider the reference point approach as an extension of goal programming. However, in most of real–life applications of goal programming the goals are grouped according to the predefined priorities (the so–called preemptive goal programming) whereas in the reference point method all the deviations are considered to be equally important. In this paper we show how the priorities can be incorporated into the reference point method.

1 Introduction

Consider a decision problem defined as an optimization problem with k objective functions. For simplification of the formal presentation we assume, without loss of generality, that all the objective functions are to be minimized. The problem can be formulated then as follows

$$\min \{ \mathbf{F}(\mathbf{x}) \ : \ \mathbf{x} \in Q \} \tag{1}$$

where

$\mathbf{F} = (F_1, \dots, F_k)$	represents a vector of k objective functions,
Q	denotes the feasible set of the problem,
$\mathbf{x}$	is a vector of decision variables.

Consider further an achievement vector $\mathbf{q} = \mathbf{F}(\mathbf{x})$ which measures achievement of decision $\mathbf{x}$ with respect to the specified set of k objectives $F_1, \dots, F_k$. It is clear that an achievement vector is better than another if all of its individual achievements are better or at least one individual achievement is better whereas no other one is worse. Such a relation is called domination of achievement vectors. Unfortunately, there usually does not exist an achievement vector that dominates all others with respect to all the criteria. Thus in

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terms of strict mathematical relations we cannot distinguish the best achievement vector. The nondominated vectors are incomparable on the basis of the specified set of objective functions. The feasible solutions (decisions) that generate nondominated achievement vectors are called efficient or Pareto–optimal solutions to the multi–criteria problem.

It seems clear that the solution of multi–criteria optimization problems should simply depend on identification of the efficient solutions. However, even finite characteristic of the efficient set for a real-life problem is usually so large that it cannot be considered a solution to the decision problem. So, there arises a need for further analysis, or rather decision support, to help the decision maker (DM) to select one efficient solution for implementation. Of course, the original objective functions do not allow one to select any efficient solution as better than any other one. Therefore this analysis depends usually on additional information about the DM's preferences. The DM, working interactively with a decision support system (DSS), specifies the preferences in terms of some control parameters, and the DSS at each interactive step provides one efficient solution that meets the current preferences. Such a DSS can be used for analysis of decision problems with finite as well as infinite efficient sets. There is important, however, that the control parameters provide the completeness of the control, i.e., that varying the control parameters the DM can identify every nondominated achievement vector.

Goal programming (GP), originally proposed by Charnes & Cooper [2], seems to be a convenient generating technique for a DSS. It is, in fact, commonly used in real–life applications (see [8]). Goal programming requires one to transform objectives into goals by specification of an aspiration level for each objective. An optimal solution is then the one that minimizes deviations from the aspiration levels. Various measures of multidimensional deviations have been proposed. They are expressed as achievement functions. Depending on the type of the achievement function we distinguish (compare [4]): weighted (minsum) GP, fuzzy (minmax) GP, preemptive (lexicographic) GP. If a GP model is used as the basis of a DSS the aspiration levels can be changed during the decision analysis as the DM preferences evolve. One of the most important advantages of the interactive GP approach is that it does not require the DM to be consistent and coherent in the preferences.

Goal programming, unfortunately, does not satisfy the efficiency (Pareto–optimality) principle. Simply, the GP approach does not suggest decisions that optimize the objective functions. It only yields decisions that have outcomes closest to the specified aspiration levels. This weakness of goal programming has led to the development of the so–called quasisatisficing approach which always generates efficient solutions. The quasisatisficing approach also deals with the aspiration levels but they are understood in a different way than in GP approaches. In goal programming the vector of aspiration levels is (weakly) preferred to any other achievement vector whereas in the quasisatisficing approach the aspiration vector is preferred to any other achievement vector which does not dominate the aspiration vector.

The best formalization of the quasisatisficing approach to multi–criteria optimization was proposed and developed mainly by Wierzbicki [9] as the reference point method. The reference point method is an interactive technique where the DM specifies requirements, as in GP, in terms of aspiration levels. Depending on the specified aspiration levels a scalarizing achievement function is built which, when minimized, generates an efficient solution to the problem. The computed efficient solution is presented to the DM as the current solution allowing comparison with previous solutions and modifications of the aspiration levels if necessary. The scalarizing achievement function not only guarantees efficiency of the solution but also reflects the DM's preferences as specified via the aspiration levels. In building the function it is assumed that the DM prefers outcomes that satisfy all the aspiration levels to any outcome that does not reach one or more of the aspiration levels.

One of the simplest scalarizing functions takes the following form (see [7])

$$s(\mathbf{q}, \mathbf{a}, \lambda) = \max_{1 \le i \le k} \{\lambda_i (q_i - a_i)\} + \varepsilon \sum_{i=1}^{k} \lambda_i (q_i - a_i) \tag{2}$$

where

- $\mathbf{a}$ denotes the vector of aspiration levels,
- λ is a scaling vector, $\lambda_i > 0$,
- ε is an arbitrarily small positive number.

Minimization of the scalarizing achievement function (2) over the attainable set $Y = \{ \mathbf{q} = \mathbf{F}(\mathbf{x}) \ : \ \mathbf{x} \in Q \}$ generates an efficient solution. The selection of the solution within the efficient set depends on two vector parameters: an aspiration vector $\mathbf{a}$ and a scaling vector λ. In practical implementations the former is usually designated as a control tool for use by the DM whereas the latter is automatically calculated on the basis of some predecision analysis. The small scalar ε is introduced only to guarantee efficiency in the case of a nonunique optimal solution.

The reference point method although using the same main control parameters (aspiration levels) always generates an efficient solution to the multi–criteria problem whereas goal programming does not. Ogryczak [5] has shown that the implementation techniques of goal programming can be used to model the reference point approach. The proposed reference GP problem takes the following form

$$\text{RGP:} \quad \text{lexmin } \mathbf{g}(\mathbf{d}^-, \mathbf{d}^+) = [g_1(\mathbf{d}^-, \mathbf{d}^+), g_2(\mathbf{d}^-, \mathbf{d}^+)] \tag{3}$$

subject to

$$F_i(\mathbf{x}) + d_i^- - d_i^+ = a_i \quad \text{for } i = 1, 2, \ldots, k \tag{4}$$

$$d_i^- \ge 0, \quad d_i^+ \ge 0 \quad \text{for } i = 1, 2, \ldots, k \tag{5}$$

$$d_i^- d_i^+ = 0 \quad \text{for } i = 1, 2, \ldots, k \tag{6}$$

$$\mathbf{x} \in Q \tag{7}$$

where

$$g_1(\mathbf{d}^-, \mathbf{d}^+) = \max_{i=1,\ldots,k} (-w_i^- d_i^- + w_i^+ d_i^+) \quad (8)$$

$$g_2(\mathbf{d}^-, \mathbf{d}^+) = \sum_{i=1}^{k} (-w_i^- d_i^- + w_i^+ d_i^+) \quad (9)$$

d_i^- and d_i^+ are negative and positive goal deviations, respectively, i.e., nonnegative state variables which measure deviations of the current value of the i-th objective function from the corresponding aspiration level;

w_i^- and w_i^+ are positive weights corresponding to several goal deviations, e.g., $w_i^- = w_i^+ = \lambda_i$ for the exact model of (2).

The main specificity which differentiates the RGP model from the standard GP approaches depends on the use of negative weight coefficients $-w_i^-$ associated with the negative deviations d_i^-. Further, the RGP problem uses both the minmax and the minsum achievement functions. The minmax achievement functions are not very common in lexicographic GP applications but they are the standard GP tool (so-called fuzzy GP, [4]). Moreover, in the case of linear problems, the minmax achievement function does not destroy the linear structure and it can be implemented implicitly in the simplex algorithms [6]. Thus, the achievement function (8) can easily be implemented in the lexicographic GP solution techniques.

As shown by Ogryczak [5], the RGP problem always generates an efficient solution to the original multi-criteria problem (even in the case of nonconvex, e.g. discrete decision problem) satisfying simultaneously the rules of the reference point approach. Namely, whenever a solution with all objectives not worse than the corresponding aspiration levels is attainable, such an efficient solution is provided by optimization of the RGP problem. Moreover, the requirements (6) can be simply omitted in the constraints of the RGP problem provided that the weights satisfy natural for the reference point approach relations

$$w_i^+ > w_i^- > 0 \qquad \text{for } i = 1, 2, \ldots, k \quad (10)$$

Thus the reference point approach may be considered as an extension of goal programming. However, in most real-life applications of goal programming the goals or deviations are grouped according to the predefined priorities (preemptive GP) whereas in the reference point method all the deviations are considered to be equally important. In the next section we show how prioritization of deviations can be incorporated into RGP model of the reference point method. The proposed preemptive reference point method preserves the most important properties of the standard reference point approach.

2 Preemptive Reference Point Method

While incorporating priorities into the reference point method, like in preemptive goal programming, we are interested rather in priorities on the deviation variables than on the original objective functions. That means, we consider the same original multi–criteria optimization problem (1) without any hierarchy of the objective functions. The priorities are considered only as an additional tool for better expression of the DM's preferences during interactive search for an efficient solution.

During the interactive analysis the DM specifies the preferences with vector $\mathbf{a}$ of aspiration levels and two groups of priority sets P_j^+ for $j = 1, 2, \ldots, p_+$ and P_j^- for $j = 1, 2, \ldots, p_-$. The priority sets P_j^+ $(j = 1, 2, \ldots, p_+)$ represent the aspiration levels hierarchy in the sense of importance of achieving outcomes not worse than the corresponding aspiration levels. Similarly, P_j^- $(j = 1, 2, \ldots, p_-)$ represent hierarchy of importance to achieve outcomes better than the corresponding aspiration levels. Both groups of priority sets define partitions of the entire set of objectives. The two partitions may be, in particular, identical. However, for better modeling of real-life preferences it seems to be necessary to allow them to be different, as importance of achieving aspiration levels does not, necessarily, match interests in exceeding the aspiration levels.

Let us summarize the DM's preference model expressed with the aspiration levels and hierarchy sets:

P1. For any individual outcome $F_i(\mathbf{x})$ $(i = 1, 2, \ldots, k)$ less is preferred to more (minimization);

P2. A solution with all individual outcomes $F_i(\mathbf{x})$ equal to the corresponding aspiration levels is preferred to any solution with at least one individual outcome greater than the corresponding aspiration level;

P3. Minimization of any positive deviation d_j^+ is preferred to maximization of each negative deviation d_i^- for $i = 1, 2, \ldots, k$;

P4. If $j < t$ then minimization of positive deviations d_i^+ for $i \in P_j^+$ is preferred to minimization of positive deviations d_i^+ for $i \in P_t^+$;

P5. If $j < t$ then maximization of negative deviations d_i^- for $i \in P_j^-$ is preferred to maximization of negative deviations d_i^- for $i \in P_t^-$;

Property **P1** means that efficient solutions are preferred to nonefficient ones, i.e., the DM's preferences are consistent with the efficiency principle. Property **P2** expresses that the DM prefers outcomes that satisfy all the aspiration levels to any outcome that does not reach one or more of the aspiration levels. In terms of goal deviations (compare (4)–(6)) it means that a solution with positive deviations d_i^+ equal to 0 for all $i = 1, 2, \ldots, k$ is preferred to any solution with positive value of at least one deviation d_i^+. These two properties are crucial for the preference model considered in the reference point approach. Properties **P3**, **P4** and **P5** express the hierarchy of deviations defined with the priority sets.

In order to introduce the hierarchy of deviations into the RGP model we replace the lexicographic achievement function with the following:

$$\text{PRGP:} \quad \text{lexmin } \mathbf{g}(\mathbf{d}^-, \mathbf{d}^+) = [\mathbf{g}^+(\mathbf{d}^+), \mathbf{g}^-(\mathbf{d}^-)] \quad (11)$$
$$\text{subject to (4), (5) and (7)}$$

where

$$\mathbf{g}^+(\mathbf{d}^+) = [g^+_{11}(\mathbf{d}^+), g^+_{12}(\mathbf{d}^+), \ldots, \mathbf{g}^+_{p+1}(\mathbf{d}^+), \mathbf{g}^+_{p+2}(\mathbf{d}^+)] \quad (12)$$
$$\mathbf{g}^-(\mathbf{d}^-) = [g^-_{11}(\mathbf{d}^-), g^-_{12}(\mathbf{d}^-), \ldots, \mathbf{g}^-_{p-1}(\mathbf{d}^-), \mathbf{g}^-_{p-2}(\mathbf{d}^-)] \quad (13)$$
$$g^+_{j1}(\mathbf{d}^+) = \max_{i \in P^+_j}(w^+_i d^+_i) \quad \text{for } j = 1, 2, \ldots, p_+ \quad (14)$$
$$g^+_{j2}(\mathbf{d}^+) = \sum_{i \in P^+_j} w^+_i d^+_i \quad \text{for } j = 1, 2, \ldots, p_+ \quad (15)$$
$$g^-_{j1}(\mathbf{d}^-) = \max_{i \in P^-_j}(-w^-_i d^-_i) \quad \text{for } j = 1, 2, \ldots, p_- \quad (16)$$
$$g^-_{j2}(\mathbf{d}^-) = \sum_{i \in P^-_j} -w^-_i d^-_i \quad \text{for } j = 1, 2, \ldots, p_- \quad (17)$$

Note that constraints of the above PRGP problem do not include the requirements (6) to guarantee proper calculation of goal deviations. The requirements (6) could be simply omitted in the constraints of PRGP problem since all negative deviations d^-_i have assigned lower priorities than any positive deviation d^+_i. This is made precise in Proposition 1.

Proposition 1 *For any aspiration levels a_i and any positive weights w^-_i and w^+_i, any $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ optimal solution to the problem PRGP satisfies requirements (6), i.e., $\bar{d}^-_i \bar{d}^+_i = 0$ for $i = 1, 2, \ldots, k$.*

Proof. Let $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ be an optimal solution for the problem PRGP. Suppose that $\bar{d}^-_{i_0} \bar{d}^+_{i_0} > 0$ for some index $1 \leq i_0 \leq k$. Then we can decrease both $\bar{d}^-_{i_0}$ and $\bar{d}^+_{i_0}$ by the same small positive quantity. That means, for small enough positive δ the vector $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^- - \delta \mathbf{e}_0, \bar{\mathbf{d}}^+ - \delta \mathbf{e}_0)$, where $\mathbf{e}_0$ denotes the unit vector corresponding to index i_0, is feasible to the problem PRGP. Due to positive weights w^-_i and w^+_i, the following inequalities are satisfied

$$g^+_{j1}(\bar{\mathbf{d}}^+ - \delta \mathbf{e}_0) \leq g^+_{j1}(\bar{\mathbf{d}}^+) \text{ and } g^+_{j2}(\bar{\mathbf{d}}^+ - \delta \mathbf{e}_0) \leq g^+_{j2}(\bar{\mathbf{d}}^+) \quad \text{for } j = 1, 2, \ldots, p_+$$

Moreover, there exists j_0 such that $i_0 \in P^+_{j_0}$ and thereby $g^+_{j_02}(\bar{\mathbf{d}}^+ - \delta \mathbf{e}_0) < g^+_{j_02}(\bar{\mathbf{d}}^+)$ which contradicts optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ for the problem PRGP. Thus $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ must satisfy conditions (6). □

By the definition of the achievement functions (12)–(17), it is lucid that the PRGP problem (11)–(17) complies with the properties **P3**, **P4** and **P5**. However, fulfilling of properties **P1** and **P2** (crucial for the reference point approach) is not so clear. We will show (Proposition 2) that the lexicographic problem PRGP always generates an efficient solution to the original multi–criteria optimization problem (property **P1**) complying simultaneously with property **P2** (Proposition 3).

Proposition 2 *For any aspiration levels a_i and any positive weights w_i^- and w_i^+, if $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ is an optimal solution to the problem PRGP, then $\bar{\mathbf{x}}$ is an efficient solution to the multi–criteria optimization problem (1).*

Proof. Let $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ be an optimal solution to the problem PRGP. Suppose that $\bar{\mathbf{x}}$ is not efficient to the problem (1). That means, there exists a vector $\mathbf{x} \in Q$ such that $F_i(\mathbf{x}) \leq F_i(\bar{\mathbf{x}})$ for $i = 1, 2, \ldots, k$ and $F_{i_0}(\mathbf{x}) < F_{i_0}(\bar{\mathbf{x}})$ for some index $1 \leq i_0 \leq k$. The deviations $\bar{d}_i^-$ and $\bar{d}_i^+$ satisfy relations: $\bar{d}_i^+ = (F_i(\bar{\mathbf{x}}) - a_i)_+$ and $\bar{d}_i^- = (a_i - F_i(\bar{\mathbf{x}}))_+$, where $(.)_+$ denotes the nonnegative part of a quantity. Let us define similar deviations for the vector $\mathbf{x}$, i.e., $d_i^+ = (F_i(\mathbf{x}) - a_i)_+$ and $d_i^- = (a_i - F_i(\mathbf{x}))_+$ for $i = 1, 2, \ldots, k$. $(\mathbf{x}, \mathbf{d}^-, \mathbf{d}^+)$ is a feasible solution to the problem PRGP and $d_i^+ \leq \bar{d}_i^+$ and $d_i^- \geq \bar{d}_i^-$ for $i = 1, 2, \ldots, k$. Hence, for any positive weights w_i^- and w_i^+ the following inequalities are satisfied

$$g_{j1}^+(\mathbf{d}^+) \leq g_{j1}^+(\bar{\mathbf{d}}^+) \quad \text{and} \quad g_{j2}^+(\mathbf{d}^+) \leq g_{j2}^+(\bar{\mathbf{d}}^+) \quad \text{for} \quad j = 1, 2, \ldots, p_+$$

$$g_{j1}^-(\mathbf{d}^-) \leq g_{j1}^-(\bar{\mathbf{d}}^-) \quad \text{and} \quad g_{j2}^-(\mathbf{d}^-) \leq g_{j2}^-(\bar{\mathbf{d}}^-) \quad \text{for} \quad j = 1, 2, \ldots, p_-$$

Moreover, there exist j_+ such that $i_0 \in P_{j_+}^+$ and j_- such that $i_0 \in P_{j_-}^-$. Hence, for any positive weights w_i^- and w_i^+, $g_{j_+2}^+(\mathbf{d}^+) < g_{j_+2}^+(\bar{\mathbf{d}}^+)$ or $g_{j_-2}^-(\mathbf{d}^-) < g_{j_-2}^-(\bar{\mathbf{d}}^-)$, which contradicts optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ for the problem PRGP. Thus $\bar{\mathbf{x}}$ must be an efficient solution to the original multi–criteria optimization problem (1). □

Proposition 3 *For any aspiration levels a_i and any positive weights w_i^- and w_i^+, if $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ is an optimal solution to the problem PRGP, then any deviation $\bar{d}_i^+$ is positive only if there does not exist any vector $\mathbf{x} \in Q$ such that $F_i(\mathbf{x}) \leq a_i$ for $i = 1, 2, \ldots, k$.*

Proof. Let $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ be an optimal solution to the problem PRGP. Suppose that $\bar{d}_{i_0}^+ > 0$ (i.e., $F_{i_0}(\bar{\mathbf{x}}) > a_{i_0}$) for some index $1 \leq i_0 \leq k$ and there exists a vector $\mathbf{x} \in Q$ such that $F_i(\mathbf{x}) \leq a_i$ for $i = 1, 2, \ldots, k$. Let us define deviations for the vector $\mathbf{x}$: $d_i^+ = (F_i(\mathbf{x}) - a_i)_+ = 0$ and $d_i^- = (a_i - F_i(\mathbf{x}))_+ \geq 0$ for $i = 1, 2, \ldots, k$. $(\mathbf{x}, \mathbf{d}^-, \mathbf{d}^+)$ is a feasible solution for the problem PRGP and for any positive weights w_i^- and w_i^+ the following inequalities are satisfied

$$g_{j1}^+(\mathbf{d}^+) = 0 \leq g_{j1}^+(\bar{\mathbf{d}}^+) \quad \text{and} \quad g_{j2}^+(\mathbf{d}^+) = 0 \leq g_{j2}^+(\bar{\mathbf{d}}^+) \quad \text{for} \quad j = 1, 2, \ldots, p_+$$

Moreover, there exists j_0 such that $i_0 \in P_{j_0}^+$ and thereby $g_{j_01}^+(\mathbf{d}^+) = 0 < w_{i_0}^+ \bar{d}_{i_0}^+ \leq g_{j_01}^+(\bar{\mathbf{d}}^+)$, which contradicts optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ for the problem PRGP. Thus cannot exist any vector $\mathbf{x} \in Q$ such that $F_i(\mathbf{x}) \leq a_i$ for $i = 1, 2, \ldots, k$. □

In order to show that the PRGP model provides us with a complete parameterization of the efficient set, we show in the following proposition that for each efficient solution $\bar{\mathbf{x}}$ there exists an aspiration vector for which $\bar{\mathbf{x}}$ with the corresponding values of the deviation variables is an optimal solution of the PRGP problem.

Proposition 4 *For any positive weights w_i^- and w_i^+, if $\bar{\mathbf{x}} \in Q$ is an efficient solution of the multi-criteria problem (1), then $(\bar{\mathbf{x}}, \mathbf{0}, \mathbf{0})$ is an optimal solution of the corresponding PRGP problem with aspiration vector $\mathbf{a} = \mathbf{F}(\bar{\mathbf{x}})$.*

Proof. Note that $(\bar{\mathbf{x}}, \mathbf{0}, \mathbf{0})$ is a feasible solution of the PRGP problem with $\mathbf{a} = \mathbf{F}(\bar{\mathbf{x}})$. Suppose that $(\bar{\mathbf{x}}, \mathbf{0}, \mathbf{0})$ is not optimal for PRGP. For any positive weights w_i^- and w_i^+, $\mathbf{g}^+(\mathbf{d}^+) \geqq \mathbf{0}$ and $\mathbf{g}^-(\mathbf{d}^-) \leqq \mathbf{0}$, whereas $\mathbf{g}^+(\mathbf{0}) = \mathbf{0}$ and $\mathbf{g}^-(\mathbf{0}) = \mathbf{0}$. So, there exists a feasible solution $\mathbf{x} \in Q$ such that

$$\mathbf{g}^+((\mathbf{F}(\mathbf{x}) - \mathbf{F}(\bar{\mathbf{x}}))_+) = \mathbf{0} \quad \text{and} \quad \mathbf{g}^-((\mathbf{F}(\bar{\mathbf{x}}) - \mathbf{F}(\mathbf{x}))_+) <_{lex} \mathbf{0}$$

Hence $\mathbf{F}(\mathbf{x}) \leq \mathbf{F}(\bar{\mathbf{x}})$ which contradicts efficiency of $\bar{\mathbf{x}}$ for the multi-criteria problem (1). □

Note that neither proposition assumes convexity of the feasible set Q. Thus the preemptive reference point method can be applied not only for linear problems but also for integer ones where goal programming may fail to generate efficient solutions (see [3]).

The PRGP problem takes a simpler form when all the priority sets are single element sets. Note that in such a case

$$g_{j1}^+(\mathbf{d}^+) = g_{j2}^+(\mathbf{d}^+) = w_j^+ d_j^+ \qquad \text{for } j = 1, 2, \ldots, k$$
$$g_{j1}^-(\mathbf{d}^-) = g_{j2}^-(\mathbf{d}^-) = -w_j^- d_j^- \qquad \text{for } j = 1, 2, \ldots, k$$

Thus vector functions $\mathbf{g}^+(\mathbf{d}^+)$ and $\mathbf{g}^-(\mathbf{d}^-)$ defined by (12)–(17) can be replaced then with the following simpler formulas

$$\begin{aligned}
\mathbf{g}^+(\mathbf{d}^+) &= [g_1^+(\mathbf{d}^+), g_2^+(\mathbf{d}^+), \ldots, g_k^+(\mathbf{d}^+)] \\
\mathbf{g}^-(\mathbf{d}^-) &= [g_1^-(\mathbf{d}^-), g_2^-(\mathbf{d}^-), \ldots, g_k^-(\mathbf{d}^-)] \\
g_j^+(\mathbf{d}^+) &= d_j^+ \qquad \text{for } j = 1, 2, \ldots, k \\
g_j^-(\mathbf{d}^-) &= -d_j^- \qquad \text{for } j = 1, 2, \ldots, k
\end{aligned}$$

As a special case of PRGP one may consider the problem with all deviations d_i^+ belonging to the same priority set $P_1^+ = \{1, 2, \ldots, k\}$ and all deviations d_i^- belonging to the same priority set $P_1^- = \{1, 2, \ldots, k\}$. Achievement functions for such a problem takes the form

$$\mathbf{g}^+(\mathbf{d}^+) = [g_1^+(\mathbf{d}^+), g_2^+(\mathbf{d}^+)] \quad , \quad \mathbf{g}^-(\mathbf{d}^-) = [g_1^-(\mathbf{d}^-), g_2^-(\mathbf{d}^-)] \tag{18}$$

$$g_1^+(\mathbf{d}^+) = \max_{i=1,\ldots,k} (w_i^+ d_i^+) \quad , \quad g_2^+(\mathbf{d}^+) = \sum_{i=1}^{k} w_i^+ d_i^+ \tag{19}$$

$$g_1^-(\mathbf{d}^-) = \max_{i=1,\ldots,k} (-w_i^- d_i^-) \quad , \quad g_2^-(\mathbf{d}^-) = \sum_{i=1}^{k} -w_i^- d_i^- \tag{20}$$

Problem PRGP with achievement functions (18)–(20) defines the reference GP model without priorities for the deviations. However, it differs from the RGP problem due to the form and properties. While in the RGP model weights w_i^- and w_i^+ must satisfy (10), PRGP without priorities requires only positivity of the weights. It is due to property **P3** built in the PRGP model. This property is equivalent to the requirement that minimization of any individual outcome greater than its aspiration level is preferred to minimization of any individual outcome which is less or equal to the corresponding aspiration level. It is a stronger requirement than property **P2** and it is not always satisfied by the standard reference point method as well as by the RGP model (3)–(9). For instance, in the standard reference point method based on the scalarizing achievement function (2) with all $\lambda_i = 1$ or in the RGP problem with $w_1^+ = w_2^+ = w_3^+$ and $0.1w_3^+ < w_3^- < w_3^+$, achievement vector (a_1+1, a_2+1, a_3-10) is preferred to (a_1+1, a_2, a_3). Preferring of vector (a_1+1, a_2, a_3) seems to be better consistent with the basic quasisatisficing rule that the DM concentrates on improvement of these individual outcomes which do not reach their aspiration levels. In the PRGP model (without priorities) vector (a_1+1, a_2, a_3) is preferred to (a_1+1, a_2+1, a_3-10) for any positive weights w_i^- and w_i^+. Thus, with the PRGP model we not only allow the DM to introduce hierarchy of aspiration levels but even without such a hierarchy we refine the standard reference point method with better modeling of the quasisatisficing approach.

3 Illustrative Example

In this section we discuss the PRGP model for a sample decision problem. To keep the problem description short and clear, and the model itself easy solvable by graphical analysis we have decided to use rather a textbook example than a real–life decision problem. Our example is based on the goal programming example from the OR/MS textbook [1].

Consider the media mix problem concerned with allocating the advertising budget among various media. For simplicity, we consider only two media: television and radio. Suppose rated exposures (people per month per advertising outlay) per $ 1 000 of advertising expenditure are 10 000 and 7 500, respectively, for television and radio. Assume the management has set the following goals it wish to achieve, arranged from highest to lowest priority:

1. Avoid expenditures of more than $ 100 000;
2. Reach at least 750 000 exposures;
3. Avoid expenditures of more than $ 70 000 for TV advertisements;
4. Maximize number of exposures;
5. Minimize total expenditures;
6. Minimize expenditures for TV advertisements.

One may easily notice that the baseline problem is three-criteria optimization problem where the objective functions are: number of exposures (maximized), total expenditures (minimized) and expenditures for TV advertisements (minimized). However, the management has specified its preferences with aspiration levels and priorities.

Introducing two decision variables x_1 and x_2, expressing expenditures for several media in thousands of dollars, one gets the following goal constraints:

$$\begin{array}{rrrrr} x_1+ & x_2+ & d_1^- - & d_1^+ = & 100 \\ 10\,000x_1+ & 7\,500x_2+ & d_2^- - & d_2^+ = & 750\,000 \\ x_1+ & & d_3^- - & d_3^+ = & 70 \\ \multicolumn{4}{l}{x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \quad \geq} & 0 \end{array}$$

The management preferences can be expressed with the following PRGP achievement function:

$$\text{lexmin} \;\; [d_1^+, d_2^-, d_3^+, -d_2^+, -d_1^-, -d_3^-]$$

Note that d_2^- has got there higher priority than $-d_2^+$ as the corresponding objective is maximized whereas for theoretical considerations in the previous section we assumed all the objective functions to be minimized.

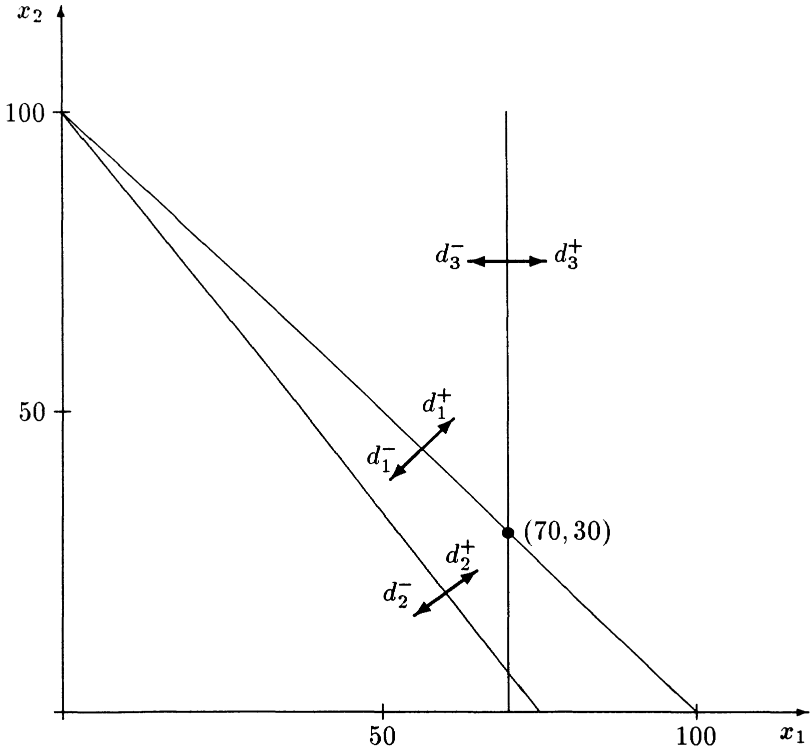

Figure 1: Graphical analysis for the illustrative example

One can easily verify with the graphical analysis (compare Fig. 1) that the above PRGP problem has a unique optimal solution $\mathbf{x} = (70, 30)$ with outcomes: \$ 100 000 of total expenditures including \$ 70 000 expended on TV advertisements, and 925 000 exposures. The graphical analysis of the problem shows that this solution could not be reached for any achievement function using only positive weights. In fact, while solving the problem

$$\text{lexmin} \quad [w_1^+ d_1^+, w_2^- d_2^-, w_3^+ d_3^+, w_2^+ d_2^+, w_1^- d_1^-, w_3^- d_3^-]$$

for any positive weights w_i^- and w_i^+ one gets a unique optimal solution $\mathbf{x} = (0, 100)$ with outcomes: \$ 100 000 of total expenditures all expended on radio advertisements and 750 000 exposures. This solution definitely seems not to be what the management preferred. Thus the usage of negative weights turns out to be important in this example.

4 Concluding Remarks

The most widely used technique for multi–criteria optimization and usually the only one taught in general OR/MS courses is goal programming. Goal programming, however, does not satisfy the efficiency (Pareto-optimality) principle. Simply, the GP approach does not suggest decisions that optimize the objective functions. It only yields decisions that have outcomes closest to the specified aspiration levels. This weakness of goal programming led to the development of the reference point method which though using the same main control parameters as GP always generates an efficient solution.

Ogryczak [5] showed how the reference point method could be modeled within goal programming methodology provided that the nonnegativity restrictions on weights were dropped. It allows us to consider the reference point approach as an extension of goal programming. However, in most of real–life applications of goal programming the goals are grouped according to the predefined priorities whereas in the reference point method all the deviations are considered to be equally important. In this paper we have shown how the priorities can be incorporated into the reference point method. The proposed preemptive reference point method not only preserves the most important properties of the standard reference point method (such as compliance with the efficiency principle and controllability) but even without the use of priorities it refines the standard reference point method with better modeling of the quasisatisficing approach. Moreover, the preemptive reference point method has been developed using GP implementation techniques. So, it can be considered as an extension of preemptive goal programming. It allows us to extend applications of the powerful reference point approach and to build a unique decision support systems providing the DM with both preemptive GP and reference point approaches.

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