Equitable Approaches to Location Problems

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Abstract

Location problems can be considered as multicriteria models where for each client (spatial unit) there is defined an individual objective function, which measures the effect of a location pattern with respect to the client satisfaction (e.g. it expresses the distance or travel time between the client and the assigned facility). In this approach the geographical space, essentially, represents both the decision space and the criterion space. Thus, the approach is well suited for a GIS environment. The individual objective functions are usually conflicting when optimized and the decision–maker or planner needs to select some compromise solution for implementation. Moreover, while locating public facilities, the distribution of effects (distances) among the clients is a crucial issue and the preference model should take into account some equity aspects. In this chapter various equitable multicriteria solution concepts are analyzed. The analysis provides a theoretical basis for development of solution procedures.

Key Words: Location, Multiple Criteria, Efficiency, Equity.

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Introduction

Public goods and services are typically provided and managed by governments in response to perceived and expressed need. The spatial distribution of public goods and services is influenced by facility location decisions. A host of operational models has been developed to deal with facility location optimization (c.f., Love et al., 1988; Francis et al., 1992; Current et al., 1990). Most classical location studies focus on some aspects of two major approaches: the minimax (center) or the minisum (median) solution concepts. Both concepts minimize only simple scalar characteristics of the distribution: the maximal distance and the average distance, respectively. In this chapter all the distances for the individual clients are considered as the set of multiple uniform criteria to be minimized. This results in a multiple criteria model taking into account the entire distribution of distances. Moreover, the model enables us to link location problems with theories of inequality measurement (in particular the Pigou–Dalton approach) (Sen, 1973).

The generic location problem that we consider may be stated as follows. There is given a set of m clients (service recipients). Each client is represented by a specific point in the geographical space. There is also given a set of n potential locations for the facilities. It may be, in particular, a subset (or the entire set) of points representing the clients. Further, the number (or the maximal number) p of facilities to be located is given ($p \leq n$). Thus, we limit our discussion to discrete location problems (Mirchandani and Francis, 1990). They can be viewed, however, as network location problems with possible locations restricted to some subset of the network vertices (Labbé et al., 1996).

The main decisions to be made in the location problem can be described with the binary variables:

x_j — equal to 1 if location j is to be used and equal to 0 otherwise

$(j = 1, 2, \ldots, n)$.

To meet the problem requirements, the decision variables x_j have to satisfy the following constraints:

$$\sum_{j=1}^{n} x_j = p \ , \quad x_j \in \{0,1\} \ , \quad \text{for } j = 1, 2, \ldots, n. \tag{1}$$

Where the equation is replaced with the inequality ($\leq$) if p specifies the maximal number of facilities to be located. Note that constraints (1) take a very simple form of the binary knapsack problem with all the constraint coefficients equal to 1. However, for most location problems the feasible set has a more complex structure due to explicit consideration of allocation decisions. These decisions are usually modeled with the additional allocation variables:

x'_{ij} — equal to 1 if location j is used to service client i and equal to 0 otherwise $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$.

The allocation variables have to satisfy the following constraints:

$$\sum_{j=1}^{n} x'_{ij} = 1 \ , \qquad \text{for } i = 1, 2, \ldots, m. \tag{2}$$

$$x'_{ij} \leq x_j \ , \qquad \text{for } i = 1, 2, \ldots, m \quad \text{and} \quad j = 1, 2, \ldots, n. \tag{3}$$

$$x'_{ij} \in \{0,1\} \ , \qquad \text{for } i = 1, 2, \ldots, m \quad \text{and} \quad j = 1, 2, \ldots, n. \tag{4}$$

In the capacitated location problem the capacities of the potential facilities are given as q_j (for $j = 1, 2, \ldots, n$). This implies the additional constraints:

$$\sum_{i=1}^{m} x'_{ij} \leq q_j \ , \quad \text{for} \quad j = 1, 2, \ldots, n.$$

Let us assume that for each client i $(i = 1, 2, \ldots, m)$ a function $f_i(\mathbf{x})$ of the location pattern $\mathbf{x}$ has been defined. This function, called the individual objective function, measures the outcome (effect) of the location pattern for client i (Marsh and Schilling, 1994). Individual objective functions f_i depend on effect of several allocation decisions. Thus they depend on allocation effect coefficients $d_{ij} \geq 0$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. Hereafter d_{ij} are called simply distance coefficients or distances as in the simplest problems they usually express the distance between client i and location j. However, we emphasize to the reader that we do not restrict our considerations to the case of outcomes measured as distances. They can be measured (modeled) as travel time, travel costs as well as in a more subjective way as relative travel costs (e.g., travel costs by clients incomes) or ultimately as the levels of clients dissatisfaction (individual disutility) of allocations.

For the standard uncapacitated location problem it is assumed that all the potential facilities provide the same type of service and each client is serviced by the nearest located facility. The individual objective functions then take the following form:

$$f_i(\mathbf{x}) = \min_{j=1,\dots,n} \{d_{ij} \ : \quad x_j = 1\} \ , \quad \text{for } i = 1, 2, \dots, m.$$

With the explicit use of the allocation variables and the corresponding constraints (2)–(3) the individual objective functions f_i can be written in the linear form:

$$f_i(\mathbf{x}) = \sum_{j=1}^{n} d_{ij} x'_{ij} \ , \quad \text{for } i = 1, 2, \dots, m. \tag{5}$$

These linear functions of the allocation variables are applicable for the uncapacitated as well as for the capacitated facility location problems.

In typical formulations of location problems related to desirable facilities a smaller value of the individual objective function means a better effect (higher service quality or client satisfaction). This remains valid for location of obnoxious facilities if the distance coefficients are replaced with their complements to some large number: $d'_{ij} = d - d_{ij}$, where $d > d_{ij}$ for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. Therefore, without loss of generality, we can assume that each function f_i is to be minimized. Hence, the generic location problem can be viewed as the following multiple criteria minimization problem:

$$\min \{\mathbf{f}(\mathbf{x}) \ : \quad \mathbf{x} \in Q\}. \tag{6}$$

Where:

$\mathbf{f} = (f_1, \dots, f_m)$	is a vector–function that maps the decision space $X = R^n$ into the criterion space $Y = R^m$,
$Q \subset X$	denotes the feasible set of location patterns,

$\mathbf{x} \in X$ denotes the vector of decision variables (the location pattern).

The individual objective functions f_i are usually conflicting when minimized. Therefore, the location problem (6) is a multiple criteria decision problem and the decision–maker or planner needs to select some compromise solution for implementation. An integration of multiple criteria decision approaches with geographical information system (GIS) capabilities has recently been recognized as one of the most important areas for future developments in decision support for spatial planning (Carver, 1991; Pereira and Duckstein, 1993). GIS usually focuses on the capture, storage, manipulation, analysis and display of geographically referenced data and only implicitly assumes a support of spatial decision–making through analytical modeling operations (Densham and Goodchild, 1989). The display capabilities of GIS typically provide the user with a number of techniques that can be used to visualize the problem and the solution in geographical space. Note that in our multiple criteria location problem (6) the geographical space, essentially, covers both the decision space and the criterion space. Therefore, multiple criteria approach to location problems based on model (6) seems to be well suited for the development of interactive solution procedures to be used within the GIS environment. The analysis presented in this chapter may provide a theoretical basis for such developments. As a source of additional motivation we remark that a similar model was considered for the purpose of management of ecological resources (Kostreva et al., 1998), and that model is another candidate for the equity approach described here.

We do not assume any special form of the feasible set while analyzing properties of the solution concepts. We rather allow the feasible set to be a general discrete (nonconvex) set. Therefore, the results of our analysis apply to various discrete location problems. Similarly, we do not assume any special form of the individual objective functions nor their special properties (like convexity) while analyzing properties of the solution concepts. However, computational procedures for many solution concepts may assume that the individual objective functions are defined in terms of formula (5).

One may be interested in putting into location model (1)–(6) some additional client weights $v_i > 0$ to represent the service demand. In typical applications such a weight represents the number of clients located at the same geographical point. Integer weights can be interpreted as numbers of unweighted vertices located at exactly the same place (with distances 0 among them). For theoretical considerations we will assume that the problem is transformed (disaggregated) to the unweighted one (that means all the client weights are equal to 1). However, such a disaggregation usually dramatically increases the problem size. Therefore, while discussing specific solution concepts, we will analyze how they can be applied directly to the weighted problem.

Model (6) specifies that we are interested in the minimization of all objective functions f_i for $i \in I = \{1, 2, \dots, m\}$. There is, however, a specificity of problem (6) related to the location decision circumstances. In typical multiple criteria problems values of the individual objective functions are assumed to be incomparable (Steuer, 1986). The individual objective functions in our multiple criteria location model express the same quantity (usually the distance) for various clients. Thus the functions are uniform in the sense of the scale used and their values are directly comparable. This is ultimately true for all location models as long as the modeler is capable to express the individual outcomes (and the outcome coefficients d_{ij}) in the unique scale of clients dissatisfaction (disutility). Moreover, especially when locating public facilities, the clients should be considered impartially and equally. Thus the distribution of distances (outcomes) among the clients is more important than the assignment of several distances (outcomes) to the specific clients. In other words, a location pattern generating individual distances: 4, 2 and 0 for clients 1, 2 and 3, respectively, should be considered equally good as a solution generating distances 0, 2 and 4. Moreover, according to the requirement of equal treatment of all clients a location pattern generating all distances equal to 2 should be considered better than both the above solutions. Our approach will take into account this specificity of the multiple criteria location model (6).

Preference model

Model (6) only says that we are interested in the minimization of all objective functions f_i for $i \in I = \{1, 2, \dots, m\}$. In order to make it operational, one needs to assume some solution concept specifying what it means to minimize multiple objective functions. Vector–function $\mathbf{f}$ maps the feasible set Q (as a subset of the decision space) into the criterion space Y. The elements of the criterion space we refer to as achievement vectors. An achievement vector $\mathbf{y} \in Y$ is attainable if it expresses outcomes of a feasible solution $\mathbf{x} \in Q$ ($\mathbf{y} = \mathbf{f}(\mathbf{x})$). The set of all the attainable achievement vectors is denoted by Y_a, i.e. $Y_a = \{\mathbf{y} \in Y \; : \; \mathbf{y} = \mathbf{f}(\mathbf{x}), \; \mathbf{x} \in Q\}$.

We say that achievement vector $\mathbf{y}' \in Y$ *dominates* $\mathbf{y}'' \in Y$ if at least one individual achievement is better whereas no other one is worse (if $y'_i \leq y''_i$ for all $i \in I$ where at least one strict inequality holds). It is clear, or rather commonly accepted, that achievement vector $\mathbf{y}'$ is better than $\mathbf{y}''$ if $\mathbf{y}'$ dominates $\mathbf{y}''$. In fact, it is the most general assumption about the preference model underlying the multiple criteria optimization. This assumption is called the Pareto–optimality (or efficiency) principle. In accordance with the Pareto–optimality principle, we treat all the objective functions, and thereby all the clients, in the same way. We do not make any specific assumption about the decision–maker preference model except for the general assumption that for each individual objective function less means better (minimization), i.e. in terms of location problems, for each spatial unit, closer to the service means better.

Each feasible solution (location pattern) for which one cannot improve any individual achievement without worsening another one is a *Pareto–optimal* (efficient) solution. We say that a solution concept for problem (6) complies with the Pareto–optimality principle if it always generates a Pareto–optimal solution. Ususally there exist many Pareto–optimal solutions and they are different not only in the decision space but also in the criterion space. Therefore, there may exist many quite different solution concepts complying with the Pareto–optimality principle.

Typical solution concepts for the location problems are based on some scalar measures of the achievement vectors. However, there are some concepts, like the lexicographic minimax (Ogryczak, 1997), which do not introduce directly any scalar measure, even though they rank the achievement vectors with a complete preorder. Therefore, we prefer to focus our analysis of solution concepts on the properties of the corresponding preference model. We assume that solution concepts depend only on evaluation of the achievement vectors and they do not take into account other solution properties not represented within achievement vectors. In fact, to the extent of our knowledge, all the solution concepts for location problems present in the literature satisfy this assumption. Thus, we can limit our considerations to the preference model in the criterion space Y.

The preference model is completely characterized by the relation of weak preference (Vincke, 1992), denoted hereafter with $\preceq$. Namely, we say that achievement vector $\mathbf{y}' \in Y$ is (strictly) preferred to $\mathbf{y}'' \in Y$ ($\mathbf{y}' \prec \mathbf{y}''$) iff $\mathbf{y}' \preceq \mathbf{y}''$ and $\mathbf{y}'' \not\preceq \mathbf{y}'$. Similarly, we say that achievement vector $\mathbf{y}' \in Y$ is indifferent or equally preferred to $\mathbf{y}'' \in Y$ ($\mathbf{y}' \cong \mathbf{y}''$) iff $\mathbf{y}' \preceq \mathbf{y}''$ and $\mathbf{y}'' \preceq \mathbf{y}'$. If a solution concept is defined by the minimization of some scalar function $g(\mathbf{y})$, then the corresponding preference model is defined by the relation

$$\mathbf{y}' \preceq \mathbf{y}'' \quad \text{iff} \quad g(\mathbf{y}') \leq g(\mathbf{y}'').$$

All the scalar solution concepts, as well as all the solutions concepts considered in this chapter, generate complete preorders in the criterion space. That means the corresponding preference relation $\preceq$ is *complete*

$$\text{for any} \quad \mathbf{y}', \mathbf{y}'' \in Y, \qquad \mathbf{y}' \preceq \mathbf{y}'' \quad \text{or} \quad \mathbf{y}'' \preceq \mathbf{y}', \tag{7}$$

reflexive

$$\mathbf{y} \preceq \mathbf{y} \tag{8}$$

and *transitive*

$$(\mathbf{y}' \preceq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \preceq \mathbf{y}''') \quad \Rightarrow \quad \mathbf{y}' \preceq \mathbf{y}'''. \tag{9}$$

A solution concept defined by the preference relation $\preceq$ depends on finding $\mathbf{y}^0 \in Y_a$ such that $\mathbf{y}^0 \preceq \mathbf{y}$ for all $\mathbf{y} \in Y_a$. To the extent of our knowledge, all the solution concepts for location problems present in the literature satisfy these properties. Under the assumption of transitivity of the preference relation, the Pareto–optimality principle may be expressed as a property of the preference relation, called strict monotonicity. We say that preference relation $\preceq$ is *strictly monotonic* if for any achievement vector $\mathbf{y}$ and for any $i \in I$

$$\mathbf{y} - \varepsilon \mathbf{e}_i \prec \mathbf{y} \quad \text{for} \quad \varepsilon > 0 \tag{10}$$

where $\mathbf{e}_i$ denotes the i–th unit vector in the criterion space. A solution concept which preference relations satisfies (7)–(10) we call hereafter an *efficient* solution concept.

Recall that in the multiple criteria location problem (6) all the individual objective functions are uniform and equally important. Moreover, we want to consider all the clients, and thereby all the individual objective functions, impartially. Thus we are interested in comparison of distributions of outcomes. Note that having two possible location patterns generating achievement vectors $\mathbf{y}' = (5, 0, 5)$ and $\mathbf{y}'' = (0, 1, 0)$, respectively, we recognize both the location patterns as efficient. In fact, neither $\mathbf{y}'$ dominates $\mathbf{y}''$ nor $\mathbf{y}''$ dominates $\mathbf{y}'$. However, the first location pattern generates two outcomes (distances) equal to 5 and one outcome equal to 0, whereas the second pattern generates one outcome equal to 1 and two outcomes equal to 0. Thus, the second location pattern is clearly better.

For multiple criteria problems with uniform and equally important objective functions we introduce an efficiency concept based rather on the distribution of outcomes than on the achievement vectors themselves. For this purpose, we assume that the preference model satisfies the principle of impartiality (anonimity)

$$(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}) \cong (y_1, y_2, \ldots, y_m) \;, \quad \text{for any } \tau \in \Pi(I) \qquad (11)$$

where $\Pi(I)$ is the set of all permutations of the set I. Condition (11) means that any permutation of the achievement vector is equally good (indifferent) as the original achievement vector. Adding the principle of impartiality to the domination relation leads us to the concept of symmetric domination which is not affected by any permutation of the achievement vector coefficients (Ogryczak, 1998).

While locating public facilities, the preference model should take into account equity of the effects (distances). Equity is, essentially, an abstract socio–political concept that implies fairness and justice (Young, 1994). Nevertheless, equity is usually quantified with the so–called inequality measures to be minimized. Inequality measures were primarily studied in economics (Sen, 1973). However, Marsh and Schilling (1994) describe twenty different measures proposed in the literature to gauge the level of equity in facility location alternatives. Among many inequality measures perhaps the most commonly accepted by economists is the Gini coefficient, which has been recently also analyzed in the location context (Mulligan, 1991; Erkut, 1993). The *Gini coefficient* is one half the relative mean absolute difference (Kendall and Stuart, 1958). It can be relatively easily introduced into the location models with tools of linear programming (Mandell, 1991). When applied to the multiple criteria problem, direct minimization of typical inequality measures contradicts the strict monotonicity axiom (10) in the multiple criteria optimization preference model. As noticed by Erkut (1993), it is rather a common flaw of all the relative inequality measures that while moving away from the spatial units to be serviced one gets better values of the measure as the relative distances become closer to one–another. As an extreme, one may consider an unconstrained continuous (single–facility) location problem and find that the facility located at (or near) infinity will provide (almost) perfectly equal service (in fact, rather lack of service) to all the spatial units.

According to the theory of equity measurement (Sen, 1973; Allison, 1978), the preference model should satisfy the (Pigou–Dalton) principle of transfers. The principle of transfers states that a transfer of small amount from an outcome to any relatively worse–off outcome results in a more preferred achievement vector. As a property of the preference relation, the principle of transfers takes the form of the following axiom

$$y_{i'} > y_{i''} \quad \Rightarrow \quad \mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''} \prec \mathbf{y} \quad \text{for } 0 < \varepsilon < y_{i'} - y_{i''}. \tag{12}$$

Requirement of impartiality (11) and the principle of transfers (12), i.e. two crucial axioms of inequality measures, do not contradict the multiple criteria optimization axioms (8)–(10). Therefore, we can consider solution concepts based on the preference model defined by axioms (7)–(12). A solution concept satisfying all the properties (7)–(12) we call hereafter an *equitably efficient (E–E) solution concept* and the location pattern generated by this concept we call the *equitably efficient solution*. In the next section we develop the basic theory and methodology for the E–E solution concepts for location problems.

Scale invariance is widely considered an additional axiom for equity measures. We say that preference relation $\preceq$ is *scale invariant* (satisfies the principle of scale invariance) if for any achievement vectors $\mathbf{y}', \mathbf{y}'' \in Y$ and for any positive constant c

$$\mathbf{y}' \preceq \mathbf{y}'' \quad \Rightarrow \quad c\mathbf{y}' \preceq c\mathbf{y}''. \tag{13}$$

We do not assume the principle of scale invariance as an axiom for E–E solution concept. Nevertheless, we pay attention if solution concepts comply with it as such a principle is important for maintaining stability of the solution, and for creating well–defined models. In fact, all the concepts discussed here comply with the principle of scale invariance.

Equitable dominance

Consider the multiple criteria problem (6) with the preference model defined by axioms (8)–(12). Recall that in the standard Pareto–optimality preference model based on (8)–(10), achievement vector $\mathbf{y}' \in Y$ dominates $\mathbf{y}'' \in Y$ if at least one individual outcome is better whereas no other one is worse (if $y'_i \leq y''_i$ for all $i \in I$ where at least one strict inequality holds). While introducing the principle of transfers (12) we enforce the dominance relation by the requirement that a transfer of small amount from an outcome to any relatively worse–off outcome results in a more preferred achievement vector. The principle of impartiality further enforces the dominance relation as any achievement vector $\mathbf{y}$ (due to transitivity) dominates all the vectors dominated by any permutation of $\mathbf{y}$. Thus, finally, we say that achievement vector $\mathbf{y}' \in Y$ *equitably dominates* $\mathbf{y}'' \in Y$, or $\mathbf{y}''$ is equitably dominated by $\mathbf{y}'$, iff there exists a finite sequence of vectors $\mathbf{y}^0, \mathbf{y}^1, \ldots, \mathbf{y}^t$ such that $\mathbf{y}^0 = \mathbf{y}''$, $\mathbf{y}^t = (y'_{\tau(1)}, y'_{\tau(2)}, \ldots, y'_{\tau(m)})$ for some permutation τ of I and for each $k = 1, 2, \ldots, t$ either $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'} + \varepsilon_k \mathbf{e}_{i''}$ with $0 < \varepsilon_k < y^{k-1}_{i'} - y^{k-1}_{i''}$ or $\mathbf{y}^k$ dominates $\mathbf{y}^{k-1}$. Figure 1 shows the achievement vectors equitably dominated by $\mathbf{y} \in R^2$ (i.e., in the case $m = 2$).

The relation of equitable dominance can be expressed as a vector inequality on the cumulative ordered achievement vectors. This can be mathematically formalized as follows. First, we introduce the ordering map $\Theta : R^m \rightarrow R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \geq \theta_2(\mathbf{y}) \geq \cdots \geq \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, 2, \ldots, m$. This allows us to focus on distributions of outcomes impartially. Next, we apply to ordered achievement vectors $\Theta(\mathbf{y})$, a linear cumulative map to get the *cumulative ordering map* $\bar{\Theta} = (\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_m)$ defined as

$$\bar{\theta}_i(\mathbf{y}) = \sum_{j=1}^{i} \theta_j(\mathbf{y}) \quad \text{for } i = 1, 2, \ldots, m. \tag{14}$$

The coefficients of vector $\bar{\Theta}(\mathbf{y})$ express, respectively: the largest out-

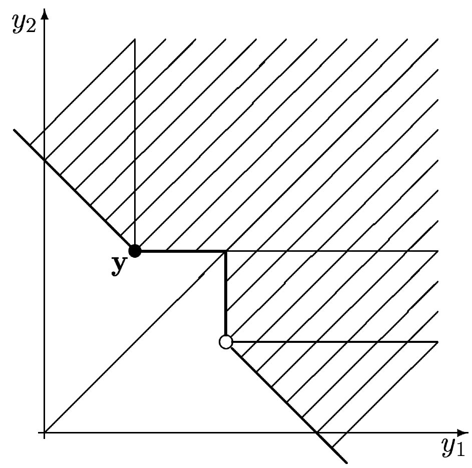

Figure 1: Achievement vectors equitably dominated by $\mathbf{y} \in R^2$

come, the total of the two largest outcomes, the total of the three largest outcomes, etc.

Directly from the definition of the map $\bar{\Theta}$, it follows that for any two achievement vectors $\mathbf{y}', \mathbf{y}'' \in Y$ equation $\bar{\Theta}(\mathbf{y}') = \bar{\Theta}(\mathbf{y}'')$ holds if and only if $\mathbf{y}'$ and $\mathbf{y}''$ have the same distribution of outcomes (i.e., $\Theta(\mathbf{y}') = \Theta(\mathbf{y}'')$). Similarly, inequality $\Theta(\mathbf{y}') \leq \Theta(\mathbf{y}'')$ implies $\bar{\Theta}(\mathbf{y}') \leq \bar{\Theta}(\mathbf{y}'')$ but the reverse implication is not valid. For instance, $\bar{\Theta}(2,2,2) = (2,4,6) \leq (3,5,6) = \bar{\Theta}(3,2,1)$ and simultaneously $\Theta(2,2,2) \not\leq \Theta(3,2,1)$.

The relation $\bar{\Theta}(\mathbf{y}') \leq \bar{\Theta}(\mathbf{y}'')$ was extensively analyzed within the theory of majorization (Marshall and Olkin, 1979), where it is called the relation of weak submajorization. The theory of majorization includes the results which allow us to derive the following proposition (Kostreva and Ogryczak, 1998).

Proposition 1 *Achievement vector* $\mathbf{y}' \in Y$ *equitably dominates* $\mathbf{y}'' \in Y$*, if and only if* $\bar{\theta}_i(\mathbf{y}') \leq \bar{\theta}_i(\mathbf{y}'')$ *for all* $i \in I$ *where at least one strict inequality holds.*

In income economics the Lorenz curve is a popular tool to explain inequalities (Young, 1994). In the context of income distribution, the Lorenz curve is a cumulative population versus income curve. First, all individuals are ranked by income, from poorest to richest. For each rank, we compute the proportion of the income earned by all individuals at this rank and all ranks below this rank. The relationship between the proportions of population and income defines the Lorenz curve. A perfectly equal distribution of income has the diagonal line as the Lorenz curve. All other distributions generate convex Lorenz curves below the diagonal line.

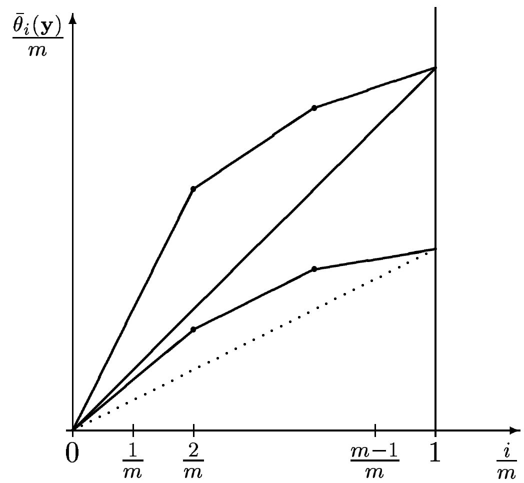

Figure 2: $\bar{\Theta}(\mathbf{y})$ as Lorenz–type curves

Note that the definition of values $\bar{\theta}_i(\mathbf{y})$ for $i = 1, 2, \ldots, m$ is similar to the construction of the Lorenz curve for the population of m clients (outcomes). The main difference depends on inverse ordering, from the largest to the smallest value. It is due to minimization problem (6) opposite to the incomes. If considered in connection with some obnoxious quantity, we get the upper Lorenz curves which are concave and fall above the diagonal equity line. If the curve corresponding to distribution A falls below the curve corresponding to distribution B, then distribution A is considered as less unequal than the latter one.

Vector $\bar{\Theta}(\mathbf{y})$ can be viewed graphically with the Lorenz–type curve connecting point (0,0) and points $(i/m, \bar{\theta}_i(\mathbf{y})/m)$ for $i = 1, 2, \ldots, m$. In the case of two achievement vectors $\mathbf{y}', \mathbf{y}'' \in Y$ with the same total of outcomes ($\bar{\theta}_m(\mathbf{y}') = \bar{\theta}_m(\mathbf{y}'')$), the inequality $\bar{\Theta}(\mathbf{y}') \leq \bar{\Theta}(\mathbf{y}'')$ is equivalent to the dominance $\mathbf{y}'$ over $\mathbf{y}''$ in the sense of upper Lorenz curves. In the general case, the upper Lorenz curves may be considered the graphs of vectors $\bar{\Theta}(\mathbf{y})/\bar{\theta}_m(\mathbf{y})$. Graphs of vectors $\bar{\Theta}(\mathbf{y})$ take the form of unnormalized concave curves (Fig. 2), similar to the upper Lorenz curves.

Note that in terms of the Lorenz curves no achievement vector can be better than the vector of equal outcomes. Equitable dominance takes into account also values of outcomes. Vectors of equal outcomes are distinguished according to the value of outcomes. They are graphically represented with various ascent lines in Fig. 2. With the relation of equitable dominance an achievement vector of small unequal outcomes may be preferred to an achievement vector with large equal outcomes.

Example 1 In order to illustrate the concept of equitable dominance, let us consider an example (Ogryczak, 1997) of location two facilities among ten spatial units, where each spatial unit can be considered as a potential location. We assume that the facilities have unlimited capacities and each spatial unit is served by the nearest facility. Thus the problem takes the form (1)–(6) with $m = n = 10$ and $p = 2$. To make possible an easy analysis of the problem without complex computations, we consider several units U1, U2,...,U10 as points on a line, say the X-axis, with coordinates: 0, 4, 5, 6, 8, 17, 18, 19, 20 and 28, respectively.

Solution		Outcomes y_i									
U2	U9	4	0	1	2	4	3	2	1	0	8
U1	U9	0	4	5	6	8	3	2	1	0	8
U3	U8	5	1	0	1	3	2	1	0	1	9
U1	U10	0	4	5	6	8	11	10	9	8	0

Table 1: Outcomes of location solutions in Example 1

Table 1 contains (four) various solutions to the location problem. The first one corresponds to the lexicographic minimax solution (Ogryczak, 1997), where in addition to the largest distance we minimize also the second largest distance, the third largest and so on. This solution depends on location facilities in spatial units U2 and U9. In the second row of Table 1 there are presented distances for another, in our opinion the worst, minimax solution. It is based on location facilities in spatial units U1 and U9. Further, we have included the minisum (median) solution and the solution minimizing the Gini coefficient. The minisum solution is based on locations in units U3 and U8, whereas the Gini solution uses locations U1 and U10. Note that among four solutions (achievement vectors) presented in Table 1 no one is dominated by any other. In fact, all these solution are efficient as, due to the problem specificity, each feasible solution is efficient.

Solution		Cumulative ordered outcomes $\bar{\theta}_i(\mathbf{y})$									
U2	U9	8	12	16	19	21	23	24	25	25	25
U1	U9	8	16	22	27	31	34	36	37	37	37
U3	U8	9	14	17	19	20	21	22	23	23	23
U1	U10	11	21	30	38	46	52	57	61	61	61

Table 2: Cumulative ordered outcomes in Example 1

Comparing cumulative ordered outcomes $\bar{\Theta}(\mathbf{y})$ given in Table 2, one can see, however, that cumulative ordered achievement vector of the second solution is dominated by that of the first one. The cumulative ordered achievement vector of the fourth solution is dominated by each of other three vectors. Thus, both the second and the fourth solutions are not equitably efficient. □

Note that Proposition 1 permits one to express the relationship between equitable efficiency for problem (6) and the Pareto–optimality for the multiple criteria problem with objectives $\bar{\Theta}(\mathbf{f}(\mathbf{x}))$:

$$\min \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) \;:\; \mathbf{x} \in Q\}. \tag{15}$$

Corollary 1 *A location pattern $\mathbf{x} \in Q$ is an equitably efficient solution of the multiple criteria problem (6), if and only if it is an efficient solution of the multiple criteria problem (15).*

We emphasize to the reader the importance of this result, as it allows us to derive E–E solution concepts for problem (6) from standard efficient solution concepts for the modified problem (15). In what follows we will use Corollary 1 to introduce and analyze E–E solution concepts.

Equitably efficient solution concepts

Efficient solutions of the multiple criteria problem (6) can be generated with simple scalarizations of the problem. Most of them are based on the minisum approach:

$$\min \{\sum_{i=1}^{m} f_i(\mathbf{x}) \;:\; \mathbf{x} \in Q\}, \tag{16}$$

or on the minimax approach:

$$\min \{ \max_{i=1,\ldots,m} f_i(\mathbf{x}) \ : \quad \mathbf{x} \in Q \}. \tag{17}$$

However, the latter generates an efficient solution only in the case of a unique optimal solution. In the general case, the optimal set of (17) includes an efficient solution and some additional refinement (regularization) is necessary to select the optimal solution which is efficient. For location problems, the minisum and the minimax approaches represent the *median* and the *center* solution concepts, respectively. Most classical location studies focus on the minimization of the mean (or total) distance (the median concept) or the minimization of the maximum distance (the center concept) to the service facilities (Morrill and Symons, 1977).

Both the median and the center solution concepts are well defined for aggregated location models using client weights $v_i > 0$ to represent several clients (service demand) at the same location. Exactly, for the weighted location problem, the center solution concept (17) is not affected by the client weights whereas the median problem takes then the following form

$$\min \{ \sum_{i=1}^{m} v_i f_i(\mathbf{x}) \ : \quad \mathbf{x} \in Q \}. \tag{18}$$

Unfortunately, neither center nor median solution concept complies with the principle of transfers. Thus they are not E–E solution concepts.

Note that Corollary 1 allows one to generate equitably efficient solutions of (6) as efficient solutions of problem (15). The median solution concept, minimizing the sum of outcomes (16), corresponds to minimization of the last (m–th) objective in problem (15). Similar, the center solution concept, based on the minimax scalarization (17), corresponds to minimization of the first objective in (15). Thus both the concepts use only one objective in the multiple criteria problem (15).

In the case of efficiency one may use the weighted sum of objective functions to generate various efficient solutions (Steuer, 1986). In the case of equitable multiple criteria programming one cannot assign various weights to individual objective functions, as that violates the requirement of impartiality (11). However, due to Corollary 1, the weighting approach can be applied to problem (15) resulting in the scalarization

$$\min \{\sum_{i=1}^{m} w_i \bar{\theta}_i(\mathbf{f}(\mathbf{x})) \;\; : \quad \mathbf{x} \in Q\}. \tag{19}$$

Note that, due to the definition of map $\bar{\Theta}$ with (14), the above problem can be expressed in the form with weights $\bar{w}_i = \sum_{j=i}^{m} w_j$ $(i = 1, 2, \ldots, m)$ allocated to coefficients of the ordered achievement vector. Such an approach to multiple criteria optimization was introduced by Yager (1988) as the so–called *Ordered Weighted Averaging (OWA)*. When applying OWA to problem (6) we get

$$\min \{\sum_{i=1}^{m} w_i \theta_i(\mathbf{f}(\mathbf{x})) \;\; : \quad \mathbf{x} \in Q\}. \tag{20}$$

If weights w_i are strictly decreasing and positive, i.e.

$$w_1 > w_2 > \cdots > w_{m-1} > w_m > 0, \tag{21}$$

then each optimal solution of the OWA problem (20) is an equitably efficient solution of (6). Thus the OWA approach defines a parametric family of E–E solution concepts for location problem (6).

As the limiting case of the OWA problem (20), when the differences among weights w_i tend to infinity, we get the lexicographic problem

$$\text{lexmin} \{(\theta_1(\mathbf{f}(\mathbf{x})), \theta_2(\mathbf{f}(\mathbf{x})), \ldots, \theta_m(\mathbf{f}(\mathbf{x}))) \;\; : \quad \mathbf{x} \in Q\}, \tag{22}$$

where first $\theta_1(\mathbf{f}(\mathbf{x}))$ is minimized, next $\theta_2(\mathbf{f}(\mathbf{x}))$ and so on. Problem (22) represents the lexicographic minimax approach to the original multiple criteria problem (6). In the location context this solution concept is called the *lexicographic center* (Ogryczak, 1997). The lexicographic center is indeed a refinement (regularization) of the center solution concept (17), but in the former, in addition to the largest outcome, we minimize also the second largest outcome (provided that the largest one remains as small as possible), minimize the third largest (provided that the two largest remain as small as possible), and so on. The lexicographic minimax solution of location problem (6) can be found by sequential optimization as shown by Ogryczak (1997).

Due to (14), problem (22) is equivalent to the problem

$$\text{lexmin} \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) \;\; : \quad \mathbf{x} \in Q\}$$

which can be considered the standard lexicographic optimization applied to problem (15). As the lexicographic optimization generates efficient solutions, thus due to Corollary 1, we get the following result.

Corollary 2 *The optimal solution of the lexicographic minimax problem (22) is an equitably efficient solution of the multiple criteria problem (6).*

Corollary 2 shows that the lexicographic center (22) is an E–E solution concept which refines the standard center solution concept.

The lexicographic center is unique with respect to the ordered achievement vectors $\Theta(\mathbf{f}(\mathbf{x}))$. It can be considered in some sense the "most equitable solution". Note that one may wish to consider the multiple criteria problem (15) as an equitable problem (with an equitable rational preference relation). In such a situation we should apply Corollary 1 to problem (15). It results in the problem with doubly cumulative ordered criteria which again may be considered as equitable. As the limit of such an approach we get the lexicographic minimax problem (22). One may wish to look for the "least equitable solution" (or "the most efficient equitable solution") applying reverse lexicographic minimization to the problem (15), i.e. solving the lexicographic problem

$$\text{lexmin}\ \{(\bar{\theta}_m(\mathbf{f}(\mathbf{x})), \bar{\theta}_{m-1}(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_1(\mathbf{f}(\mathbf{x})))\ :\quad \mathbf{x} \in Q\}, \tag{23}$$

where first $\bar{\theta}_m(\mathbf{f}(\mathbf{x}))$ is minimized, next $\bar{\theta}_{m-1}(\mathbf{f}(\mathbf{x}))$ and so on. While the lexicographic center (22) is a refinement of the standard center solution concept (17), the problem (23) is a lexicographic refinement of the median approach (16). Therefore, we refer to it as the *lexicographic median* problem. As the lexicographic optimization generates efficient solutions, from Corollary 1, we get the following corollary.

Corollary 3 *The optimal solution of the lexicographic median problem (23) is an equitably efficient solution of the location problem (6).*

According to Corollary 3 the lexicographic median (23) is an E–E solution concept which shows that the standard median solution concept may be, similar to the center, refined to an E–E solution concepts. In other words, the median solution is equtably efficient if it is unique and otherwise (23) allows us to find among many median solutions the equitably efficient one. Note that multiple solutions to the median problem are not so common as to the center problem but still for many location problems it may happen quite frequently (e.g., multiple medians of networks; Labbé et al., 1996).

The OWA model (20) defines the multidimensional continuum of E–E location concepts spanning the space between the center and the median. Although rich with E–E solutions concepts, some caution is required. The OWA approach, in general, requires the disaggregation of location problem with the client weights v_i. This may restrict computation due to the size of the problem required. However, for some special sequences of the OWA weights w_i this solution concept can be easily defined for the weighted location problem without disaggregation and the solution procedure may be quite simple. For instance

$$\sum_{i=1}^{m} [2(m-i)+1]\theta_i(\mathbf{y}) = \sum_{i=1}^{m} \sum_{k=1}^{m} \max\{y_i, y_k\}.$$

Hence, the OWA problem given by the weights with equal differences $w_i - w_{i+1}$ depends on minimization of a piecewise linear function which can be directly defined for the weighted location problem as

$$M(\mathbf{x}) = \sum_{i=1}^{m} \sum_{k=1}^{m} v_i v_k \max\{f_i(\mathbf{x}), f_k(\mathbf{x})\}. \tag{24}$$

Further research is necessary to identify wider class of easily solvable OWA solution concepts.

As a simplified approach one may consider bicriteria center/median solution concepts. Recall that the median solution concept corresponds to minimization of the last (m–th) objective in problem (15) and the center solution concept corresponds to minimization of the first objective in (15). Thus, in the case of bicriteria problems ($m = 2$), the set of equitably efficient solutions is equal to the set of efficient solutions of the bicriteria problem with objectives defined as the maximum and the sum of the original two objectives. In general the following corollary is valid.

Corollary 4 *Except for location patterns with identical mean and worst outcome, every efficient solution to the bicriteria problem*

$$\min \{(\max_{i=1,\dots,m} f_i(\mathbf{x}), \sum_{i=1}^{m} f_i(\mathbf{x})) \ : \quad \mathbf{x} \in Q\} \tag{25}$$

is an equitably efficient solution of the problem (6).

The median solution concept based on the minimization of aggregate distance (16) is primarily considered as concerned with spatial efficiency. The center solution concept based on the minimax objective (17) addresses the geographical equity issues. It is of particular importance in spatial organization of emergency service systems, such as fire, police, medical ambulance services, civil defense and accident rescue. On the other hand, locating a facility at the center may cause a large increase in the total distance thus generating a substantial loss in spatial efficiency. This has led to a search for some compromise solution concept to reduce as much as possible discrepancies in accessibility among clients. Most of them are based on the bicriteria center/median model (25) thus defining the so-called cent–dians. The *convex cent–dian* solution concept (Halpern, 1978) for the weighted location problem depends on minimization of the function

$$H_\lambda(\mathbf{x}) = \lambda \max_{i=1,\ldots,m} f_i(\mathbf{x}) + (1-\lambda) \sum_{i=1}^{m} v_i f_i(\mathbf{x}).$$

The convex cent–dian is a parametric solution concept which covers as a special case the center ($\lambda = 1$) and the median ($\lambda = 0$). For $0 < \lambda < 1$, it minimizes a convex combination of the average and maximum distance, thus taking into account both the efficiency and equity criteria. In the case of discrete location problems, we consider, not all efficient solutions of the center/median problem (25) can be identified with the convex cent–dians. This can be achieved with *Chebyschev cent–dians* (Ogryczak, 1997a) based on the minimization of

$$\bar{H}_\lambda(\mathbf{x}) = \max\{\lambda \max_{i=1,\ldots,m} f_i(\mathbf{x}), (1-\lambda) \sum_{i=1}^{m} v_i f_i(\mathbf{x})\}.$$

Corollary 4 partially justifies cent–dian approaches as E–E solution concepts provided that the corresponding solutions are unique. To transform cent–dians into true E–E solution concept one may regularize them with additional objective (24). This allows us to define the *regularized convex cent–dian*

$$\operatorname{lex}\min \{(H_\lambda(\mathbf{x}), M(\mathbf{x})) \;:\; \mathbf{x} \in Q\} \tag{26}$$

and the *regularized Chebyschev cent–dian*

$$\operatorname{lex\,min}\ \{(\bar{H}_\lambda(\mathbf{x}), M(\mathbf{x})) \ : \quad \mathbf{x} \in Q\}. \tag{27}$$

The lexicographic minimization in (26) and (27) means that first we minimize $H_\lambda(\mathbf{x})$ ($\bar{H}_\lambda(\mathbf{x})$) on $\mathbf{x} \in Q$, and next we minimize $M(\mathbf{x})$ on the optimal set of $H_\lambda(\mathbf{x})$ ($\bar{H}_\lambda(\mathbf{x})$). The second minimization is only needed when the optimal solution of $H_\lambda(\mathbf{x})$ ($\bar{H}_\lambda(\mathbf{x})$) is not unique. Corollary 1 implies the following result.

Corollary 5 *For any $0 \leq \lambda \leq 1$, optimal solutions to the regularized cent-dian problems (26) and (27) are equitably efficient solutions of the problem (6).*

Corollary 5 shows that the regularized cent–dian models (26) and (27), similar to the OWA model, define the continuum of E–E location concepts spanning the space between the center and the median. Certainly, cent–dians do not allow us to define the entire richness of the OWA solution concepts. On the other hand, the regularized cent–dians can be applied directly to the weighted location problem without the necessity of disaggregation.

Concluding remarks

In modern society there are many policies and safeguards designed to promote equity and to ensure fairness for all citizens. The need for such policies is clear considering the limited shared public service resources provided by the government. When equity and/or fairness is violated, in this context litigation and difficult negotiations may arise, and society at-large suffers until the violation is removed.

This problem of lack of equitable treatment of all citizens may be due to a lack of foundations in the theory of management of public resources. Our contribution to this volume presents a theory of equitable efficiency in location analysis, which must be considered as foundational. It is shown how to construct the theory from well established socio-economic and political theories and other more mathematical first principles. Once the theory is presented, mathematical models of multiple criteria optimization and appropriate solution methods are described. These computational devices allow the theory to be fully realized, and eventually applied to enhance decision–making for policy refinement. Specially, we have made equitable solutions available through the solution concepts of ordered weighted averaging, lexicographic center, lexicographic median, together with the regularized convex cent–dian and the regularized Chebyschev cent–dian. A simple bicriteria center/median model which "almost always" works is also discussed. Depending on the problem data, several of these E–E solution concepts may serve the decision–maker well.

Our discussion includes remarks telling why certain existing theoretical constructs are inadequate, and how they may be modified and enhanced to obtain a consistent equitable theory. Such information will reinforce the need for a new theory, especially for experienced analysts, who have likely observed these difficulties first-hand.

Finally, we refer the interested reader to a paper published recently elsewhere (Kostreva and Ogryczak, 1998) which contains the relevant mathematical proofs and we suggest that there are many opportunities for further research in this interesting, applicable subject.

References

Allison, P.D. (1978), 'Measures of Inequality', *American Sociological Review*, vol. 43, pp. 865–880.

Carver, S.J. (1991), 'Integrating Multi–Criteria Evaluation with Geographical Information Systems', *International Journal of Geographical Information Systems*, vol. 5, pp. 321–339.

Current, J., Min, H. and Schilling, D. (1990), 'Multiobjective Analysis of Facility Location Decisions', *European Journal of Operational Research*, vol. 49, pp. 295–307.

Densham, P.J. and Goodchild, M.F. (1989), 'Spatial Decision Support Systems: A Research Agenda', in *GIS/LIS'89 Proceedings*, Orlando, Florida, pp. 707–716.

Erkut, E. (1993), 'Inequality Measures for Location Problems', *Location Science*, vol. 1, pp. 199–217.

Francis, R.L., McGinnis, L.F. and White, J.A. (1992), *Facility Layout and Location: An Analytical Approach*, Prentice–Hall, Englewood Cliffs, MA.

Halpern, J. (1978), 'Finding Minimal Center–Median Convex Combination (Cent–Dian) of a Graph', *Management Science*, vol. 24, pp. 534–544.

Kendall, M.G. and Stuart, A. (1958), *The Advanced Theory of Statistics, Volume 1: Distribution Theory*, Griffin, London.

Kostreva, M.M. and Ogryczak, W. (1998), 'Linear Optimization with Multiple Equitable Criteria' *RAIRO Recherche Opérationnelle*, in print.

Kostreva, M.M., Ogryczak, W. and Tonkyn, D.W. (1998), 'Relocation Problems Arising in Conservation Biology', *Computers & Mathematics*, in print.

Labbé, M., Peeters, D. and Thisse, J.-F. (1996), 'Location on Networks', in M.O. Ball, T.L. Magnanti, C.L. Monma and G.L. Nemhauser (Eds.), *Handbook in Operations Research and Management Science: Network Routing*, North–Holland, Amsterdam, The Netherlands, pp. 551–624.

Love, R.F., Morris, J.G. and Wesolowsky, G.O. (1988), *Facilities Location: Models & Methods*, North–Holland, New York, NY.

Mandell, M.B. (1991), 'Modelling Effectiveness–Equity Trade–offs in Public Service Delivery Systems', *Management Science*, vol. 37, pp. 467–482.

Marsh, M.T. and Schilling, D.A. (1994), 'Equity Measurement in Facility Location Analysis: A Review and Framework', *European Journal of Operational Research*, vol. 74, pp. 1–17.

Marshall, A.W. and Olkin, I. (1979), *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York.

Mirchandani, P.B. and Francis, R.L., (Eds.). (1990), *Discrete Location Theory*, John Wiley and Sons, New York, NY.

Morrill, R.L. and Symons, J. (1977), 'Efficiency and Equity Aspects of Optimum Location', *Geographical Analysis*, vol. 9, pp. 215–225.

Mulligan, G.F. (1991), 'Equity Measures and Facility Location', *Papers in Regional Science*, vol. 70, pp. 345–365.

Ogryczak, W. (1997), 'On the Lexicographic Minimax Approach to Location Problems', *European Journal of Operational Research*, vol. 100, pp. 566–585.

Ogryczak, W. (1997a), 'On Cent–Dians of General Networks', *Location Science*, vol. 5, pp. 15–28.

Ogryczak, W. (1998), 'On the Distribution Approach to Location Problems', *Computers & Industrial Engineering*, in print.

Pereira, J.M. and Duckstein, L. (1993), 'A Multiple Criteria Decision-Making Approach to GIS-based Land Suitability Evaluation', *International Journal of Geographical Information Systems*, vol. 7, pp. 407–424.

Sen, A. (1973), *On Economic Inequality*, Clarendon Press, Oxford.

Steuer, R.E. (1986), *Multiple Criteria Optimization — Theory, Computation & Applications*, John Wiley and Sons, New York, NY.

Vincke, Ph. (1992), *Multicriteria Decision–Aid*, John Wiley and Sons, New York, NY.

Yager, R.R. (1988), 'On Ordered Weighted Averaging Aggregation Operators in Multicriteria Decision Making', *IEEE Transactions on Systems, Man and Cybernetics*, vol. 18, pp. 183–190.

Young, H.P. (1994), *Equity in Theory and Practice*, Princeton University Press, Princeton.