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ON CENT-DIANS OF GENERAL NETWORKS

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Abstract—Most classical location studies focus on the minimisation of the average distance or the minimisation of the maximum distance to service facilities. In this paper, we analyse solution concepts related to the bicriteria model, providing some compromise between these two criteria. We show that the classical approaches based on the λ-cent-dian and the generalised centre solution concepts have some flaws, when applied to a general network. In order to avoid these flaws, we propose a new solution concept of the Chebyschev λ-cent-dian. This parametric solution concept allows us to identify all Pareto-optimal compromise locations on any network. We also show how the algorithm for finding λ-cent-dian can be modified to generate Chebyschev λ-cent-dians. © 1997 Elsevier Science Ltd. All rights reserved

Key words: location on networks, centre, median, Pareto-optimality.

1. INTRODUCTION

Public goods and services are typically provided and managed by governments in response to perceived and expressed need. The spatial distribution of public goods and services is strictly related to facility location decisions. A host of operational models has been developed to deal with facility location optimization; see, for example, Francis *et al.*, 1992; Handler and Mirchandani, 1979; Labbé *et al.*, 1995. Most classical location studies focus on the minimization of the average (or total) distance or the minimization of the maximum distance to service facilities (Morrill and Symons, 1977).

Approaches based on the minimization of aggregate or average weighted distance are primarily concerned with spatial efficiency. The corresponding solution concept is called *median*. Since the median approach is based on averaging, it often provides solutions in which remote and low-population density areas are discriminated against in terms of accessibility to public facilities, as compared with centrally situated and high-population density areas. For this reason, an alternative approach, involving the minimization of the maximum distance (or travel time) between any consumer and the closest facility, can be applied. This approach is referred to as the *centre* solution concept (Hakimi, 1965). The minimax objective primarily addresses geographical equity issues. It is of particular importance in spatial organization of emergency service systems, such as fire, police, medical ambulance services, civil defense, and accident rescue.

The centre approach is consistent with the Rawlsian (Rawls, 1971) theory of justice (Harvey, 1972). On the other hand, locating a facility at the centre may cause a large increase in the total distance, thus generating a substantial loss in spatial efficiency. This has led to a search for some compromise solution concept. Halpern (1976, 1978) introduces the *λ-cent-dian* as a parametric solution concept based on the bicriteria centre/median model. He

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modeled the corresponding trade-offs with a convex combination of two objectives. Hansen *et al.* (1991) introduce a solution concept of the *generalised centre*, which minimises the difference between the maximum distance and the average distance.

In this paper, we show some flaws of the λ-cent-dian and the generalised centre solution concepts from the perspective of compromise solutions to the bicriteria centre/median model. We discuss possible modifications to avoid these flaws. As both the solution concepts were introduced for the location of a single facility on a network, we focus on this location model. Nevertheless, the analysis presented can be extended to the multiple facility location problem on a network (Labbé *et al.*, 1995, and references therein) as well as to discrete location problems (Francis *et al.*, 1992, and references therein).

The paper is organised as follows. In Section 2, we review the problem statement and the solution concepts of the λ-cent-dian and the generalised centre. In Section 3, we deal with the generalised centre. We give a paradoxical example, where there exists a location that is simultaneously a centre and a median, but the generalised centre is located at another centre with the worst possible value of the median function. In order to avoid this flaw, the generalised centre solution concept needs to be restricted to locations that are Pareto-optimal for the bicriteria centre/median model. However, with this restriction, the generalised centre turns out to be always a centre, thus not providing us with any compromise location. In Section 4, we show that the solution concept of λ-cent-dian, which allows us to identify all the compromise locations on a tree, may fail to do so in the case of a general network. To overcome this flaw of λ-cent-dians, we introduce the solution concept of the *Chebyschev λ-cent-dian*, which allows us to identify all the Pareto-optimal compromise locations on a general network. Finally, in Section 5, we show how the algorithm for finding λ-cent-dians can be modified to generate the Chebyschev λ-cent-dians.

2. THE MODEL

We assume the usual definition of a network (Labbé *et al.*, 1995), where: $V = \{v_1, v_2, \ldots, v_n\}$ denotes the set of vertices of the network, and N is the union of a finite number of edges $[v_i, v_j]$. For any two points $x_1, x_2 \in N, d(x_1, x_2)$ denotes their distance. For any two points belonging to the same edge $x_1, x_2 \in [v_i, v_j]$, $l\,[x_1, x_2]$ denotes the length of the subedge $[x_1, x_2]$. For each vertex $v_i \in V$ there is given a nonnegative weight w_i.

The average weighted distance between a point $x \in N$ and the vertices of N is given by

$$F(x) = 1/w(V) \sum_{v_i \in V} w_i d(x, v_i)$$

where $w(V) = \Sigma_{v_i \in V} w_i$. A point minimizing F in N is called a *median*.

The maximum distance between a point $x \in N$ and the vertices of N is defined as

$$G(x) = \max \{d(x, v_i) \colon v_i \in V, w_i > 0\}$$

A point minimizing G in N is called a *centre*.

Note that weights w_i do not appear in the formula for the maximum distance $G(x)$. In typical applications, the weight represents the number of clients located at the corresponding vertex. Integer weights can be interpreted as numbers of unweighted vertices located at exactly the same place (with distances 0 among them). The number of identical vertices affects the average distance but it does not affect the maximum distance.

Halpern (1976, 1978) introduces a parametric solution concept of *λ-cent-dian* as a point $x \in N$ which minimises

$$H_\lambda(x) = \lambda G(x) + (1-\lambda)F(x) \tag{1}$$

The λ-cent-dian covers as special cases the centre ($\lambda = 1$) and the median ($\lambda = 0$) solution concepts. For $0 < \lambda < 1$, the λ-cent-dian minimises a convex combination of the average and maximum distance, thus taking into account both spatial efficiency and equity criteria.

The λ-cent-dian solution concept may be viewed as the weighting approach (Steuer, 1986) to the bicriteria centre/median model

$$\min\{[G(x), F(x)]: x \in N\} \tag{2}$$

where both ($G(x)$ and $F(x)$ have to be minimised. A point $x \in N$ is a *Pareto-optimal* solution to the bicriteria centre/median problem (2) if there does not exist another point $y \in N$ for which

$$G(y) \leq G(x) \text{ and } F(y) \leq F(x)$$

where at least one of the inequalities is satisfied as a strict inequality. We denote the set of all Pareto-optimal solutions to problem (2) as PO_2. Note that set PO_2 represents all rational compromises between the values of $G(x)$ and $F(x)$. For any point $y \notin PO_2$ one can find $x \in PO_2$ such that $G(x) \leq G(y)$ and $F(x) \leq F(y)$ where at least one strict inequality holds. It means that, for points not belonging to PO_2, we may improve one criterion ($G(x)$ or $F(x)$) without worsening the other. For points of PO_2 any decrease of $G(x)$ must be offset by some increase of $F(x)$ and vice versa.

Hansen *et al.* (1991) consider the location of a facility to reduce as much as possible discrepancies in accessibility among users. For this purpose, they introduce the solution concept of the *generalised centre*, which minimises the difference between the maximum and the average distances to the vertices. Note that $|G(x) - F(x)| = G(x) - F(x)$ as $G(x) \geq F(x)$ for any $x \in N$. In order to avoid selection of a clearly 'unreasonable' location, which may happen with such a criterion, the selection is restricted to the set PO of points, which are Pareto-optimal (or efficient) with respect to the distances. A point $x \in N$ is *Pareto-optimal with respect to the distances* if there does not exist another point $y \in N$ for which

$$d(v_i, y) \leq d(v_i, x) \text{ for all } v_i \in V \text{ such that } w_i > 0$$

where at least one of the inequalities is satisfied as a strict inequality. Thus, the generalised centre is defined as an optimal solution of the following problem

$$\min\{G(x) - F(x): x \in PO\} \tag{3}$$

Moreover, due to $H_\lambda(x) = F(x) + \lambda(G(x) - F(x))$, Hansen *et al.* (1991) have noticed that the generalised centre can be viewed as a limiting case of λ-cent-dian when $\lambda \to \infty$. Consequently, they consider λ-cent-dians associated with $\lambda > 1$ as solutions to a location problem where both efficiency and equity are important.

3. GENERALISED CENTRE

In this section, we analyze the solution concept of the generalised centre and the λ-cent-dians associated with $\lambda > 1$, when applied to a general network. These solution concepts depend on minimization of the difference $G(x) - F(x)$ which, in general, does not comply with the bicriteria minimization model (2). Note that function $H_\lambda(x)$ can be written as

$$H_\lambda(x) = G(x) + (\lambda - 1)(G(x) - F(x))$$

This form shows that, when minimizing $H_\lambda(x)$ with $\lambda > 1$ (in particular $\lambda \to \infty$), having several possible locations with the same value $G(x)$, the location with the largest (i.e. the worst) value of $F(x)$ will be selected among them. We show a paradoxical example, where there exists a location that is simultaneously a centre and a median, but the generalised centre and all the λ-cent-dians for $\lambda > 1$ are located in another centre with the worst possible value of the median function $F(x)$.

Example 1. Let us consider a simple cyclic network as presented in Fig. 1. Notice that all the points of the network are Pareto-optimal with respect to the distances, provided that all the weights are positive. There are four centres located at points c_1, c_2, c_3 and c_4. Let us consider a set of weights favouring one centre. For instance, let $w_1 = w_2 \gg w_3 = w_4 > 0$, thus generating the set of median solutions on the edge $[v_1, v_2]$. Hence, c_1 is an optimal solution to both the centre and the median problem. Note that, due to symmetry of the network, all the centres generate exactly the same set of distances (two distances 1/2 and two 3/2). Nevertheless, they are quite different with respect to the spatial efficiency. Centre c_1 is the most efficient (minimum of $F(x)$ on the entire network) whereas centre c_3 is the least efficient (maximum of $F(x)$ on the entire network).

Now, let us look for the generalised centre of the network. One may easily find that the generalised centre is located at point c_3, i.e., the centre that is the least efficient. Moreover, all the λ-cent-dians for $\lambda > 1$ are also located at c_3.

Example 1 calls into question the solution concept of the generalised centre as stated by Hansen *et al.* (1991). Let us analyze whether there is a way to modify this solution concept to eliminate paradoxes similar to Example 1. Note that in order to eliminate such paradoxes, we cannot accept worsening of the value of $F(x)$ without simultaneous improvement of the value of $G(x)$ and vice versa. Thus, we should restrict the set of feasible locations to the set PO_2 of points, which are Pareto-optimal for the bicriteria problem (2).

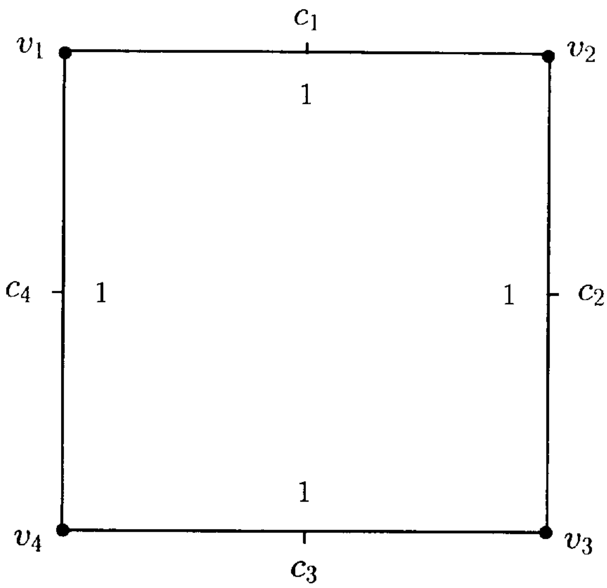

Fig. 1. Sample network for Example 1.

Let us define the *restricted generalised centre* as an optimal solution of the problem

$$\min\{G(x)-F(x):\ x \in PO_2\} \tag{4}$$

Certainly PO_2 is a subset of PO. Thus, we have restricted the feasible set in comparison to the standard generalised centre defined with (3). Similarly, we may introduce the *restricted λ-cent-dian* defined as an optimal solution to the problem

$$\min\{H_\lambda(x):\ x \in PO_2\} \tag{5}$$

For $0<\lambda<1$ the corresponding λ-cent-dian always belongs to the set PO_2. Thus, in this case, the restricted λ-cent-dian is simply a λ-cent-dian. We will show that for all $\lambda \geq 1$, including the limiting case of the restricted generalised centre, the corresponding restricted λ-cent-dian is always a centre.

Note, that the centre of a general network may be nonunique and in such a case not all centres belong to PO_2. For instance, among four centres in Example 1, only c_1 belongs to PO_2. To belong to PO_2, the centre must be unique or it must be a centre with the best value of $F(x)$. Thus, the centre belonging to PO_2 is an optimal solution of the following lexicographic (two-level) problem

$$\text{lex}\min\{[G(x),\ F(x)]:\ x \in N\} \tag{6}$$

The lexicographic minimization in (6) means that first we minimise $G(x)$ on $x \in N$, and next we minimise $F(x)$ on the optimal set of $G(x)$. The second minimization is only needed when the optimal solution of $G(x)$ is not unique. We will call the optimal solution of problem (6) *lexicographic cent-dian*. The lexicographic cent-dian is also a centre and the unique centre is the lexicographic cent-dian.

Proposition 1. On any network N, the restricted generalised centre, as well as any restricted λ-cent-dian for $\lambda \geq 1$, is a lexicographic cent-dian.

Proof. Let x be the lexicographic cent-dian. This means $G(x) \leq G(y)$ for any $y \in N$ and $F(x) \leq F(y)$ for any $y \in N$ such that $G(x)=G(y)$. First, note that $x \in PO_2$. Let $y \in N$ be a point of PO_2. If y is not the lexicographic cent-dian, then $G(x) \neq G(y)$ or $F(x) \neq F(y)$. Hence, being in PO_2, y has to satisfy inequalities

$$G(y)>G(x) \text{ and } F(y)<F(x) \tag{6}$$

Hence, $G(y)-F(y) > G(x)-F(x)$, which proves that the restricted generalised centre is a lexicographic cent-dian. Moreover, for all $\lambda \geq 1$

$$H_\lambda(y)=G(y)+(\lambda-1)(G(y)-F(y))>G(x)+(\lambda-1)(G(x)-F(x))=H_\lambda(x)$$

which proves that for $\lambda \geq 1$ each restricted λ-cent-dian is a lexicographic cent-dian. □

Hansen *et al.* (1991) have proven that, on a tree, the centre is the unique λ-cent-dian for all $\lambda \geq 1$. Note that, on a tree, the centre is unique and therefore it is, in fact, the lexicographic cent-dian. Thus, Proposition 1 may be considered as an analog of that result for the case of a general network.

Proposition 1 can be illustrated in the 2-dimensional criterion space $(G(x),F(x))$, as in Fig. 2. Let $g: N \rightarrow \mathbf{R}^2$ be a mapping from the network N into the plane $\mathbf{R}^2$, defined as $g(x)=(G(x),F(x))$. Further, let m denote the median with the best value of $G(x)$ (among all

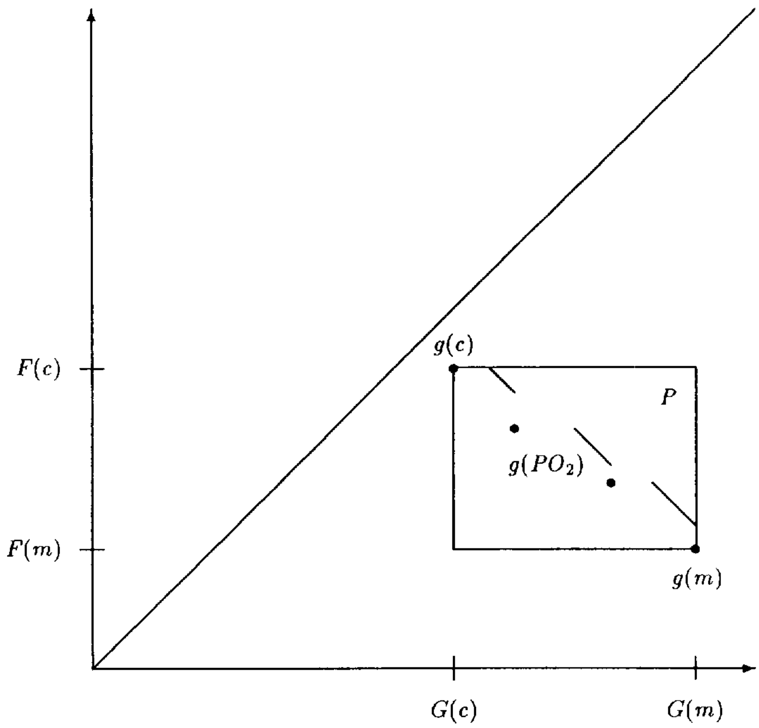

Fig. 2. Image of PO_2 in the criterion space $(G(x), F(x))$.

the medians) and c denote the centre with the best value of $F(x)$ (i.e. a lexicographic cent-dian). The image $g(PO_2)$ is included in rectangle P defined by the vertices $(G(c),F(m))$, $(G(m),F(m))$, $(G(m),F(c))$ and $(G(c),F(c))$, where the last vertex is the image of c. Note that all points $(y_1,y_2) \in P$ satisfy inequality $y_1 \geq y_2$. While looking for a restricted generalised centre, we are interested in a point $(y_1, y_2) \in g(PO_2)$ which is closest to the line $y_1 = y_2$. Point $g(c) \in g(PO_2)$ is the closest point of $P \supset g(PO_2)$. Therefore, c is the restricted generalised centre. Similarly, while considering the restricted λ-cent-dian, for all $\lambda \geq 1$ point $g(c)$ is the best in P.

Proposition 1 provides us with a very simple characteristic of the restricted λ-cent-dian for all $\lambda \geq 1$. **They are simply the centres with the best median values**. That means, finding them is not more difficult than the identification of all the centres. On the other hand, it means that the solution concept of the restricted generalised centre does not provide us with any compromise between the average and the maximum distances.

4. COMPROMISE CENT-DIANS

The solution concept of the λ-cent-dian has been introduced by Halpern (1976, 1978) to provide some compromise between the spatial efficiency (average distance minimization) and the spatial equity (maximum distance minimization). As proven by Halpern (1976), in the case of a tree, the λ-cent-dians with various $0 \leq \lambda \leq 1$ allow us to model various compromises between these two criteria. This compromise can be stated in the form of the following proposition.

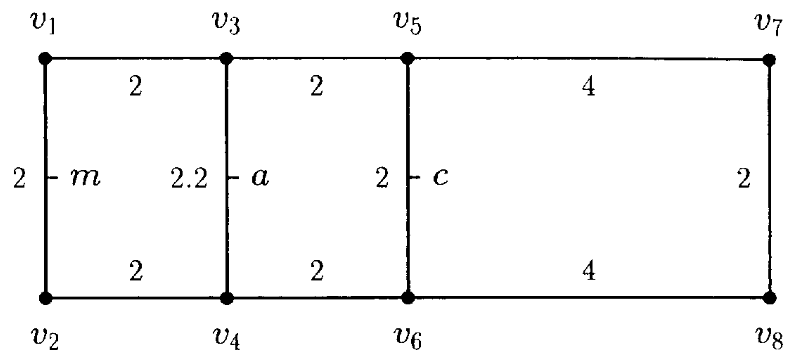

Fig. 3. Sample network for Example 2.

Proposition 2. On a tree, for each $x \in PO_2$ there exists $0 \leq \lambda \leq 1$ such that x is the corresponding λ-cent-dian.

Thus, in the case of a tree, λ-cent-dians with $0 \leq \lambda \leq 1$ provide a complete parameterization of the set PO_2. Unfortunately, due to the lack of convexity (Dearing *et al.*, 1976), this property of λ-cent-dians is not valid in the case of a general network. We illustrate this with Example 2.

Example 2. Let us consider a simple network as presented in Fig. 3. The network has the unique centre at point c (in the middle of edge $[v_5,v_6]$). Let us consider a set of weights favouring locations close to v_1 and v_2. For instance, let $w_1 = w_2 = 47$ and $w_3 = w_4 = w_5 = w_6 = w_7 = w_8 = 1$ thus generating the set of median solutions on the edge $[v_1,v_2]$. Among them, point m (in the middle of edge $[v_1,v_2]$) is the median with the best value of the maximal distance. Both m and c belong to the set PO_2. They generate two corners of rectangle P in Fig. 2.

There are several other points belonging to PO_2. Among them point a (in the middle of edge $[v_3,v_4]$) seems to be a very interesting compromise location to satisfy both criteria. Note that $g(a) = (7.1,3.14)$ whereas $g(c) = (5,4.88)$ and $g(m) = (9,1.28)$. One may easily verify that, for any $0 \leq \lambda \leq 1$, $H_\lambda(m) < H_\lambda(a)$ or $H_\lambda(c) < H_\lambda(a)$. Thus, point a cannot be found as a λ-cent-dian for any $0 \leq \lambda \leq 1$. In fact, for the network under consideration, m and c are the only λ-cent-dians; not other points of the set PO_2 can be generated in that way.

One may suspect that the presented flaw of λ-cent-dians is related to the lack of weights in function $G(x)$. While replacing function $G(x)$ with the maximum weighted distance

$$G_w(x) = \max\{w_i d(x, v_i):\ v_i \in V\} \tag{6}$$

In Example 2, we obtain the weighted centre located at the median point m and the same for all the corresponding λ-cent-dians. Thus, it does not result in any compromise between the centre and the median approaches. For some networks, things may even be worse. Let us consider a simple tree with three vertices, as presented in Fig. 4. Assigning weights $w_1 = 1$,

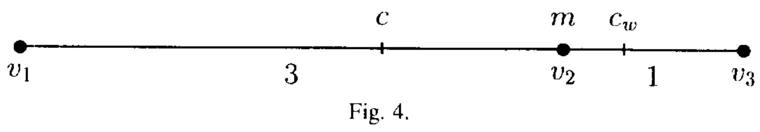

Fig. 4.

$w_2 = 10$ and $w_3 = 5$ to the corresponding vertices, one finds the median m located at the vertex v_2. The centre c is located on the edge $[v_1,v_2]$. While looking for a compromise between the average and the maximum distance, we are interested in the subedge $[c,m]$. However, the weighted centre c_w is located on the edge $[v_2,v_3]$ and it is not an efficient solution for the bicriteria centre/median problem.

In order to identify some compromise cent-dians on a general network, we need a solution concept different from Halpern's λ-cent-dian. According to the theory of multiple objective optimization (Steuer, 1986), in the case of a nonconvex problem, the Pareto-optimal set PO_2 can be completely parameterised with minimization of the weighted Chebyschev norm. Moreover, this optimization should be supported by some regularization (refinement) in the case of nonunique optimal solution. Let us define

$$\bar{H}_\lambda(x) = \max\{\lambda G(x), (1-\lambda)F(x)\} \tag{7}$$

We call point $x \in N$ the *Chebyschev λ-cent-dian* if it is an optimal solution of the following lexicographic (two-level) problem

$$\text{lex min}\ \{[\bar{H}_\lambda(x), H_\lambda(x)]:\ x \in N\} \tag{8}$$

The lexicographic minimization in (8) means that first we minimise $\bar{H}_\lambda(x)$ on $x \in N$, and then we minimise $H_\lambda(x)$ on the optimal set of $\bar{H}_\lambda(x)$. Thus, function $H_\lambda(x)$, defined as the convex linear combination (1), is used in (8) only for regularization purposes, in the case of a nonunique minimum for the main function $\bar{H}_\lambda(x)$ defined with (7). However, this regularization is necessary to guarantee that the Chebyschev λ-cent-dian belongs to PO_2.

Proposition 3. On any network, for each $0<\lambda<1$ the corresponding Chebyschev λ-cent-dian belongs to PO_2.

Proof. Let $x \in N$ be a Chebyschev λ-cent-dian for some $0<\lambda<1$. Suppose that $x \notin PO_2$. This means that there exists $y \in N$ such that

$$G(y) \le G(x) \text{ and } F(y) \le F(x)$$

where at least one of the inequalities is satisfied as a strict inequality. Hence, due to $0<\lambda<1$, we obtain

$$\bar{H}_\lambda(y) \le \bar{H}_\lambda(x) \text{ and } H_\lambda(y) < H_\lambda(x)$$

which contradicts optimality of x for problem (8). Thus, x belongs to PO_2. □

Proposition 4. On any network, for each $x \in PO_2$ there exists $0<\lambda<1$ such that x is the corresponding Chebyschev λ-cent-dian.

Proof. Let x be any point of PO_2. Let us define $\lambda = F(x)/(G(x)+F(x))$. Then $0<\lambda<1$, $1-\lambda = G(x)/(G(x)+F(x))$ and

$$\bar{H}_\lambda(x) = G(x)F(x)/(G(x)+F(x)) = \lambda G(x) = (1-\lambda)F(x) \tag{9}$$

Suppose that x is not the corresponding Chebyschev λ-cent-dian. Thus, there exists $y \in N$ such that

$$\lambda G(y) \le \bar{H}_\lambda(x) \text{ and } (1-\lambda)F(y) \le \bar{H}_\lambda(x)$$

where at least one of the inequalities is satisfied as a strict inequality. Due to (9), it would mean $x \notin PO_2$. Thus, x must be the corresponding Chebyschev λ-cent-dian. □

The Chebyschev λ-cent-dian, similar to the λ-cent-dian, is a parametric solution concept generating various solutions depending on the value of $0<\lambda<1$. Propositions 3 and 4 show that the Chebyschev λ-cent-dian, always belongs to set PO_2 and, vice versa, each point of PO_2 can be found as a Chebyschev λ-cent-dian. Thus, the solution concept of the Chebyschev λ-cent-dian allows us to model all rational compromises between the values of the average and the maximal distance. Selection of λ depends on the type of compromise one seeks. The Chebyschev λ-cent-dian is the centre for $\lambda \geq 1/2$ (Since $G(x) \geq F(x)$), and the median for λ close enough to 0. For λ between 0 and 1/2 one may expect various compromise solutions. One may proceed in the search for a satisfactory compromise in an interactive way.

Recall the network presented in Fig. 3 with weights: $w_1 = w_2 = 47$ and $w_3 = w_4 = w_5 = w_6 = w_7 = w_8 = 1$. In Example 2, we have shown that point a cannot be found as a λ-cent-dian, since for any $0 \leq \lambda \leq 1$, $H_\lambda(m) < H_\lambda(\text{a})$ or $H_\lambda(c) < H_\lambda(a)$. Note that $\bar{H}_\lambda(a) = \max\{7.1\lambda, 3.14(1-\lambda)\}$, $\bar{H}_\lambda(c) = \max\{5\lambda, 4.88(1-\lambda)\}$ and $\bar{H}_\lambda(m) = \max\{9\lambda, 1.28(1-\lambda)\}$. Hence, $\bar{H}_\lambda(a) < \bar{H}_\lambda(c)$ and $\bar{H}_\lambda(a) < \bar{H}_\lambda(m)$ for any $3.14/12.14 < \lambda < 4.88/11.98$. In fact, point a is the Chebyschev λ-cent-dian for all $3.14/11.14 < \lambda < 4.054/11.154$.

Minimization of $\bar{H}_\lambda(x)$ represents, by definition, minimization of the larger value of $\lambda G(x)$ or $(1-\lambda)F(x)$. Note that minimization of $H_\lambda(x)$ on the optimal set of $\bar{H}_\lambda(x)$ is equivalent to minimization of the smaller value of $\lambda G(x)$ or $(1-\lambda)F(x)$, provided that the larger value remains as small as possible. Hence, the lexicographic problem (8) defining the Chebyschev λ-cent-dian may be interpreted as follows. First, by selection of λ, one defines the trade-off for criteria $G(x)$ and $F(x)$. The original criteria are then replaced with the scaled criteria $\lambda G(x)$ and $(1-\lambda)F(x)$, respectively. The scaled criteria are considered as comparable. One seeks the solution which minimises the largest (i.e. worst) value among the criteria. If the solution is not unique, then one selects that which additionally minimises the smaller value. Thus, the Chebyschev λ-cent-dian may be regarded as a result of the Rawlsian (Rawls, 1971) theory of justice applied to the scaled criteria $\lambda G(x)$ and $(1-\lambda)F(x)$.

Recall that in the criterion space (Fig. 2) the entire set $g(PO_2)$ is included in the rectangle P defined by $g(c)$ and $g(m)$. Thus, for a more intuitive understanding of the corresponding trade-offs, one may use the Chebyschev λ-cent-dians for the normalised objective functions

$$\bar{G}(x) = \frac{G(x) - G(c) + \varepsilon}{G(m) - G(c) + \varepsilon} \quad \text{and} \quad \bar{F}(x) = \frac{F(x) - F(m) + \varepsilon}{F(c) - F(m) + \varepsilon}$$

where ε is an arbitrary positive number introduced to guarantee positive values of the functions and ε can be skipped (replaced with 0) if $G(m) > G(c)$ and $F(c) > F(m)$ (the rectangle P is not degenerated). Functions $\bar{G}(x)$ and $\bar{F}(x)$ represent the relative degradations of the corresponding functions $G(x)$ and $F(x)$ to their optimal values $G(c)$ and $F(m)$, respectively. One may easily prove analogs of Proposition 3 and 4 for the Chebyschev λ-cent-dians defined with the use of functions $\bar{G}(x)$ and $\bar{F}(x)$ instead of the original $G(x)$ and $F(x)$. Such a Chebyschev λ-cent-dian solution concept may be considered a special case of the reference point approach in multiple criteria optimization (Wierzbicki, 1982).

5. IDENTIFICATION OF CHEBYSCHEV CENT-DIANS

In this section, we present an algorithm to determine the Chebyschev λ-cent-dian for any $0<\lambda<1$. In fact, we show how the algorithm for finding the λ-cent-dian (Hansen *et al.*, 1991) can be modified to generate the Chebyschev λ-cent-dian. First, we recall the properties of function $H_\lambda(x)$ and prove similar properties of function $\bar{H}_\lambda(x)$.

Point x on edge $[v_i,v_j]$ is called a *bottleneck point* if there exists a vertex v_k with $w_k > 0$ such that

$$d(v_k, x) = d(v_k, v_i) + l[v_i,x] = d(v_k, v_j) + l[v_j, x]$$

Let B_{ij} denote that set of all bottleneck points on edge $[v_i,v_j]$. Along a subedge limited by two successive bottleneck points or vertices (i.e. a subedge not containing any bottleneck point in its interior), the distance from any vertex v_k is either linearly increasing or linearly decreasing. For $x \in [v_i,v_j]$ function $F(x)$ is a piecewise linear concave function with a finite number of breakpoints, all belonging to B_{ij} (Hansen *et al.*, 1987 and references therein).

Consider next the function $G(x)$ on edge $[v_i,v_j]$. It is a piecewise linear continuous function because it is the upper envelope of a family of piecewise linear continuous functions. Furthermore, its breakpoints are either bottleneck points or local minima (Hansen *et al.*, 1987 and references therein). Following Hansen *et al.* (1991), we denote by LM_{ij} the set containing the points of edge $[v_i,v_i]$, which are local minima of $G(x)$ together with the vertices v_i and v_j.

The following proposition by Halpern (1976) describes function $H_\lambda(x)$ and thereby identifies a finite set of points containing all λ-cent-dians for a given $0<\lambda<1$.

Proposition 5. For $x \in [v_i,v_j]$ and a given value $0<\lambda<1$, $H_\lambda(x)$ is a piecewise linear continuous function with a finite number of breakpoints, all belonging to $LM_{ij} \cup B_{ij}$, and a finite number of locally minimal values, all attained at points belonging to LM_{ij}.

Consider now function $\bar{H}_\lambda(x)$ on edge $[v_i,v_j]$. It is a piecewise linear continuous function because it is the upper envelope of two piecewise linear continuous functions. $\bar{H}_\lambda(x)$ may have breakpoints in the set $LM_{ij} \cup B_{ij}$ as well as in some additional points where $\lambda G(x) = (1-\lambda)F(x)$. Let b_k and b_k+1 be two consecutive points of the set $LM_{ij} \cup B_{ij}$. Both functions $G(x)$ and $F(x)$ are linear on subedge $[b_k,b_{k+1}]$. Hence, if

$$[\lambda G(b_k)-(1-\lambda)F(b_k)][\lambda G(b_{k+1})-(1-\lambda)F(b_{k+1})]<0 \tag{10}$$

then there exists a unique interior point $x \in [b_k,b_{k+1}]$ such that $\lambda G(x) = (1-\lambda)F(x)$. This point, called hereafter a *λ-switch*, is determined by the equation

$$|\lambda G(b_k)-(1-\lambda)F(b_k)|\, l[x, b_{k+1}] = |\lambda G(b_{k+1})-(1-\lambda)F(b_{k+1})|\, l[b_k, x] \tag{11}$$

For a given λ, we denote by $S_{ij}(\lambda)$ the set of all λ-switches belonging to edge $[v_i,v_j]$. The following proposition characterises function $\bar{H}_\lambda(x)$ on edge $[v_i,v_j]$.

Proposition 6. For $x \in [v_i,v_j]$ and a given value $0<\lambda<1$, $\bar{H}_\lambda(x)$ is a piecewise linear continuous function with a finite number of breakpoints, all belonging to $LM_{ij} \cup B_{ij} \cup S_{ij}(\lambda)$.

Proof. $\bar{H}_\lambda(x)$ is a piecewise linear continuous function because it is the upper envelope of two piecewise linear continuous functions: $\lambda G(x)$ and $(1-\lambda)F(x)$. Each of these functions has all the breakpoints belonging to $LM_{ij} \cup B_{ij}$. Let b_k and b_{k+1} be two consecutive points in set $LM_{ij} \cup B_{ij}$. Both functions $\lambda G(x)$ and $(1-\lambda)F(x)$ are linear on subedge $([b_k,b_{k+1}]$.
In the case of (10) the corresponding λ-switch defined by (11) is the unique solution to the equation $\lambda G(x) = (1-\lambda)F(x)$, $x \in [b_k,b_{k+1}]$. Thus, it is a unique breakpoint of $\bar{H}_\lambda(x)$ in the interior of subedge $[b_k,b_{k+1}]$. If (10) is not the case, the $\lambda G(x) \geq (1-\lambda)F(x)$ for all

$x \in [b_k,b_{k+1}]$ or $\lambda G(x) \leq (1-\lambda)F(x)$ for all $x \in [b_k,b_{k+1}]$. Thus, $\bar{H}_\lambda(x) = \lambda G(x)$ or $\bar{H}_\lambda(x) = (1-\lambda)F(x)$ for all $x \in [b_k,b_{k+1}]$, which means that $\bar{H}_\lambda(x)$ is linear on $[b_k,b_{k+1}]$. Finally, in both the cases all the breakpoints of function $\bar{H}_\lambda(x)$ belong to the finite set $LM_{ij} \cup B_{ij} \cup S_{ij}(\lambda)$. □

For a given λ, let us define the set $SB_{ij}(\lambda) = \{x \in B_{ij}: \lambda G(x) = (1-\lambda)F(x)\}$ to distinguish bottleneck points that satisfy (11). Next, we extend the set of λ-switches $S_{ij}(\lambda)$ to the set $\bar{S}_{ij}(\lambda) = S_{ij}(\lambda) \cup SB_{ij}(\lambda)$. The following proposition identifies a finite set of points containing all Chebyschev λ-cent-dians for a given $0<\lambda<1$.

Proposition 7. For any given value $0<\lambda<1$, every optimal solution of the problem

$$lex \min \{[\bar{H}_\lambda(x), H_\lambda(x)]: x \in [v_i,v_j]\} \tag{12}$$

belongs to $LM_{ij} \cup \bar{S}_{ij}(\lambda)$.

Proof. An optimal solution of (12) must minimise $\bar{H}_\lambda(x)$ on the edge $[v_i,v_j]$ and in the case of a nonunique minimum it must additionally minimise $H_\lambda(x)$ in the set of minimal points of $\bar{H}_\lambda(x)$.

$\bar{H}_\lambda(x)$ is a piecewise linear continuous function. Therefore, it takes its minimal value in some breakpoint (including the ends of the edge) or on the entire subedge between two consecutive breakpoints. Note that between two consecutive breakpoints b_k and b_{k+1}, $\bar{H}_\lambda(x)$ is linear and equal to $\lambda G(x)$ or $(1-\lambda)F(x)$. As $\lambda G(x)$ is always either increasing or decreasing, $\bar{H}_\lambda(x)$ may be constant on $[b_k,b_{k+1}]$ only if $\bar{H}_\lambda(x) = (1-\lambda)F(x)$ and $F(x)$ is constant. However, in such a case, function $H_\lambda(x)$ is not constant on $[b_k,b_{k+1}]$ and, due to the lexicographic optimization, any interior point of $[b_k,b_{k+1}]$ cannot be an optimal solution to problem (12). Hence, according to Proposition 6, any optimal solution of (12) belongs to $LM_{ij} \cup B_{ij} \cup S_{ij}(\lambda)$.

Furthermore, neither $\lambda G(x)$ nor $(1-\lambda)F(x)$ has local minima in breakpoints belonging to $B_{ij} \backslash LM_{ij}$. Therefore, such points may minimise $\bar{H}_\lambda(x)$ only if they belong to the set $SB_{ij}(\lambda)$. Thus, finally, all the optimal solutions of (12) belong to $LM_{ij} \cup \bar{S}_{ij}(\lambda)$. □

Example 3. Recall the network presented in Fig. 3 with weights: $w_1 = w_2 = 47$ and $w_3 = w_4 = w_5 = w_6 = w_7 = w_8 = 1$. In order to illustrate Propositions 6 and 7, let us analyze edge $[v_3,v_4]$ which includes point a. The set $LM_{3,4}$ consists of two vertices and point a ($LM_{3,4} = \{v_3,a,v_4\}$). There are two bottleneck points ($B_{3,4} = \{b_1,b_2\}$) symmetrically located on edge $[v_3,v_4]$, with distances 0.1 and 2.1 to the vertices. Both $F(x)$ and $G(x)$ are piecewise linear functions completely defined by their values at point of $LM_{3,4} \cup B_{3,4} = \{v_3,b_1,a,b_2,v_4\}$, where $F(v_3) = F(v_4) = 3.042$, $F(b_1) = F(a) = F(b_2) = 3.14$; $G(v_3) = G(v_4) = 8$, $G(b_1) = G(b_2) = 8.1$, $G(a) = 7.1$. The corresponding graphs are presented in Fig. 5.
Figure 6 presents graphs of functions $H_\lambda(x)$ and $\bar{H}_\lambda(x)$ for $\lambda = 0.3$ (similar graphs occur for all $3.14/11.14 < \lambda < 3.14/10.24$). Function $H_{0.3}(x)$ is a piecewise linear function completely defined by values at points of $LM_{3,4} \cup B_{3,4}$. Function $\bar{H}_{0.3}(x)$ is also a piecewise linear function. However, it has two additional breakpoints within subedges $[b_1,a]$ and $[a,b_2]$. They are λ-switches s_1 and s_2 ($S_{3,4}(0.3) = \{s_1,s_2\}$) which are symmetrically located on the edge $[v_3,v_4]$, with the distance 34/135 to point a.

Note that $0.3G(b_1) \neq 0.7F(b_1)$ and $0.3G(b_2) \neq 0.7F(b_2)$. Hence, $\bar{S}_{3,4}(0.3) = S_{3,4}(0.3)$. Following Proposition 7, the optimal solution of the corresponding problem (12) belongs to $LM_{3,4} \cup \bar{S}_{3,4}(0.3) = \{v_3,s_1,a,s_2,v_4\}$. Since $\bar{H}_{0.3}(s_1) = \bar{H}_{0.3}(a) = \bar{H}_{0.3}(s_2) < \bar{H}_{0.3}(v_3) = \bar{H}_{0.3}(v_4)$ and $H_{0.3}(a) < H_{0.3}(s_1) = H_{0.3}(s_2)$, point a is the optimal solution. While solving the corresponding

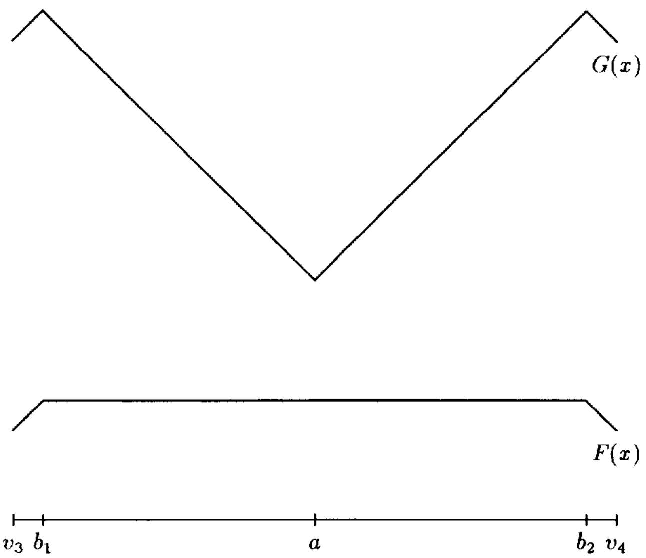

Fig. 5. Graphs of $F(x)$ and $G(x)$ on edge $[V_3, V_4]$.

problem (12) directly, we first minimise function $\bar{H}_{0.3}(x)$ on the edge $[v_3,v_4]$. The optimal solution is not unique. The optimal set consists of the entire subedge $[s_1,s_2]$. Next, we minimise $H_{0.3}(x)$ on the subedge $[s_1,s_2]$. Hence, point a is the optimal solution.

Hansen *et al.* (1991) have proposed algorithm IMA to identify image $g(LM_{ij} \cup B_{ij})$ for edge $[v_i,v_j]$. This algorithm can easily be modified for finding the Chebyschev λ-cent-dians. Algorithm IMA, in at most $O(|V|\,log|V|)$ operations, generates a sorted list of points $LM_{ij} \cup B_{ij}$ and the corresponding values of $G(x)$ and $F(x)$. Without increasing the complexity of the algorithm, additional points of $S_{ij}(\lambda)$ can be identified and the corresponding values of the functions computed. Further, we need to repeat this procedure for every edge and explore all the corresponding sets $LM_{ij} \cup \bar{S}_{ij}(\lambda)$ ($[v_i, v_j] \in E$) to determine the points minimizing the lexicographic objective. Thus the overall complexity of finding the Chebyschev λ-cent-dians (for a given $0<\lambda<1$) is $O(|E|\,|V|\,log|V|)$ operations. Note that in the case of evolving value of parameter λ (in some interactive process) the results of basic computations

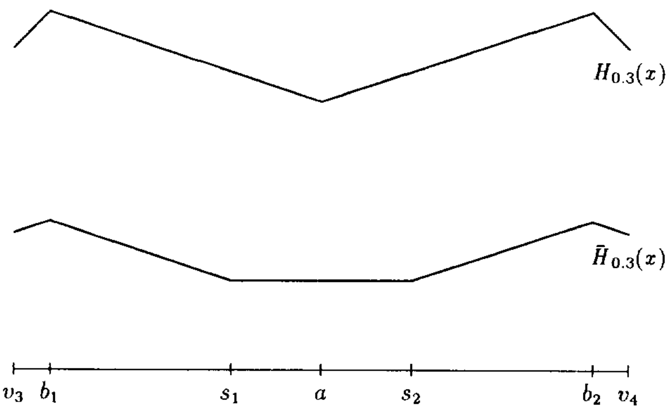

Fig. 6. Graphs of $H_{0.3}(x)$ and $\bar{H}_{0.3}(x)$ on edge $[v_3,v_4]$.

made with IMA for one value of λ (i.e., images $g(LM_{ij} \cup B_{ij})$) remain valid for all other values of λ.

6. CONCLUDING REMARKS

Since the median approach is based on averaging, it often provides solutions where remote and low-population density areas are discriminated against in terms of accessibility to public facilities, as compared with centrally situated and high-population density areas. On the other hand, locating a facility at the centre may cause a large increase in total distance, thus generating a substantial loss in spatial efficiency. This has led to a search for some compromise solution concept. Halpern (1976, 1978) has introduced the λ-cent-dian as a parametric solution concept based on the bicriteria centre/median model. He has modeled the corresponding trade-offs with a convex combination of two objectives. Hansen *et al.* (1991) have introduced a solution concept of the generalised centre which minimises the difference between the maximum distance and the average distance.

We have shown a paradoxical example, where there exists a location being simultaneously a centre and a median, but the generalised centre is located in another centre with the worst possible value of the median function. In order to avoid this flaw of the generalised centre solution concept, one needs to restrict it to locations which are Pareto-optimal for the bicriteria centre/median model. With this restriction the generalised centre turns out to be always a centre, thus not providing us with any compromise location.

The solution concept of λ-cent-dian, in the case of a tree, allows us to identify all the compromise centre/median locations. In the case of a general network it may not, as shown in Example 2. In order to overcome this flaw of λ-cent-dians, we have introduced the solution concept of the Chebyschev λ-cent-dian. This parametric solution concept complies with the theoretical rules as well as the practice of multiple criteria optimization, when applied to the bicriteria centre/median model. The concept of the Chebyschev λ-cent-dian allows us to identify all the compromise (Pareto-optimal) locations on any network. Moreover, the algorithm developed by Hansen *et al.* (1991) for finding λ-cent-dians can easily be modified to generate the Chebyschev λ-cent-dians.

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