On goal programming formulations of the reference point method

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Abstract

This paper discusses connections between the multi-criteria techniques of goal programming (GP) and the reference point method (RPM). Both approaches use a certain target point in the criterion (outcome) space as a key element to model decision maker preferences. Therefore, RPM can be expressed, similarly to GP, in the modelling framework of deviational variables. The paper gives a systematic survey and analysis of the lexicographic GP models of RPM. The corresponding preference models are formalised and analysed with respect to target values interpretations as well as the Pareto-efficiency of their solutions. The properties of equity among the individual achievements of solutions are also analysed with respect to the Rawlsian principle of justice.

Keywords: Multi-objective; goal programming; reference point method; efficiency; equity

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Introduction

The techniques of Goal Programming (GP),[1] and the Reference Point Method (RPM)[2,3] form the well-known multi-criteria decision making (MCDM) methodologies. The approaches have a clear common root: they use a certain target point in the criterion (outcome) space to model decision maker (DM) preferences. This point within GP is a vector of aspiration levels which represents the most desired values for the several criteria (outcomes). Within RPM the target point is a vector of reference levels to be used in an interactive way by the DM. Moreover, both approaches assume that a solution with all individual outcomes equal to the corresponding target values is preferred to any solution with at least one outcome worse than the corresponding target value. Thus each of the approaches may be considered as some specification of the satisficing model[4] for the decision process. Typical GP approaches aim to minimise the distance between the target point and the actual outcome values thus implementing the strict satisficing model where no solution can be considered to be better than that generating the target values. This requires a carefully selected target point to guarantee Pareto-efficiency of solutions. RPM implements the so-called quasi-satisficing decision model[5] where the target points are interpreted consistently with basic concepts of Pareto-efficiency in the sense that the optimisation is continued even when the target point has been reached already. This allows DMs to use targets as mobile reference points since the generated solutions are always efficient.

Despite the similarities in the approaches of GP and RPM, they are usually introduced in the literature as completely independent methodologies. Moreover, different tools are usually employed to present the methodology. In particular, GP approaches focus on modelling the preferences via the deviational variables and they widely use the lexicographic order. RPM is usually introduced via the scalarising achievement function which may be directly interpreted as a utility function (temporal in the interactive process). RPM can be expressed in the GP modelling framework of deviational variables. However, the corresponding RPM models[6,7,8] differ from typical GP formulations since RPM always guarantees the efficiency of solutions.

This paper is organised as follows. In the next section we systematically summarise the lexicographic GP models of RPM. There are covered all the models present in the literature as well as introducing a new parametric model allowing the consideration of some level of compensation between individual achievements. The main results of the paper are related to the analysis of the preference models underlying the approaches of GP and RPM. To the extent of our knowledge, it is the first strictly formalised comparison of the corresponding preference relations. First, the preference relations are analysed with respect to the target value interpretations as well as the Pareto efficiency of their solutions. Next, the property of equity among the individual achievements in the RPM solutions is examined. There it is shown that the RPM models generate efficient solutions satisfying the perfect equity of individual achievements, whenever such an efficient solution exists. When there does not exist an efficient solution with perfectly equal individual achievements, then RPM generates another efficient solution but still provides some equitability of individual achievements by the implementation of an approximation to the Rawlsian principle of justice.

Lexicographic models

In this paper, without loss of generality, it is assumed that all the criteria are maximised (that is, for each outcome 'more is better'). Hence, we consider the following multi-criteria problem:

$$\max \{ \ [f_1(x), f_2(x), \ldots, f_q(x)] \ : \ x \in Q \ \} \tag{1}$$

where x denotes a vector of decision variables to be selected within the feasible set Q, and $f_i(x)$ is a mathematical expression for the ith outcome (the ith criterion). Note that neither any specific form of the feasible set Q is assumed nor any special form of criteria $f_i(x)$ is required. The following notation is further used for the techniques under examination:

$y_i =$ the ith outcome, value of the ith criterion ($y_i = f_i(x)$)

$b_i =$ aspiration or reference level attached to the ith outcome

$b_i^* =$ ideal value corresponding to the ith outcome ($b_i^* = \max\{f_i(x) : x \in Q\}$)

$n_i =$ negative deviational variable ($n_i = \max\{b_i - f_i(x), 0\}$)

$p_i =$ positive deviational variable ($p_i = \max\{f_i(x) - b_i, 0\}$)

RPM is an interactive technique where the DM specifies preferences in terms of reference levels. Depending on the specified reference levels a scalarising achievement function is built which, when optimised, generates an efficient solution to the problem. The scalarising achievement function may be directly interpreted as expressing utility to be maximised. However, to keep the discussion consistent with GP models we will assume that the scalarising achievement function is minimised (thus representing dis-utility). The generic scalarising achievement function takes then the following form:[3]

$$\max_{1 \le i \le q} \{s_i(b_i, f_i(x))\} + \varepsilon \sum_{i=1}^{q} s_i(b_i, f_i(x)) \tag{2}$$

where ε is an arbitrary small positive number and $s_i : R^2 \rightarrow R$, for $i = 1, 2, \ldots, q$, are the individual achievement functions measuring actual achievement of the ith outcome with respect to the corresponding reference levels b_i. For any reference value b_i, function $s_i(b_i, y_i)$ must be strictly decreasing with respect to y_i (the ith outcome) and it has to take value 0 for $y_i = b_i$.

It has been indicated by Steuer[9] that, when the regularisation constant ε in (2) is used as a parameter, one may generate efficient solutions compatible with different DM's structure of preferences. The consequences of a direct use of the analytic scalarising achievement function with not necessarily arbitrarily small value of ε were widely studied in the RPM literature.[10,11] It was shown that any given value of ε restricts the search for efficient solutions to a subset of properly efficient solutions[12] with actually bounded trade-off coefficients, where the bound is given with a simple function of ε.

The standard RPM methodology[2,3] assumes the parameter ε in formula (2) to be arbitrarily small. Thus, when accepting the loss of a direct utility interpretation, one may consider a limiting case with $\varepsilon \rightarrow 0_+$ which results in the lexicographic order being applied to two separate terms of function (2). Therefore, RPM may be also considered as the following lexicographic problem:[6,7,13]

$$\text{lex min} \left\{ \left[\max_{1 \leq i \leq q} \{s_i(b_i, f_i(x))\}, \sum_{i=1}^{q} s_i(b_i, f_i(x)) \right] \ : \ x \in Q \right\} \tag{3}$$

The advantage of the above lexicographic model is that it allows the DM to generate all efficient solutions whereas only properly efficient solutions can be obtained with the minimisation of (2).[5]

Problem (3) is a lexicographic regularisation of the minmax aggregation:

$$\min \left\{ \max_{1 \leq i \leq q} \{s_i(b_i, f_i(x))\} \ : \ x \in Q \right\} \tag{4}$$

The latter was widely studied in the multi-criteria optimisation methodology.[9] The optimal set of the minmax aggregation always contains an efficient solution. Thus the unique optimal solution of (4) is efficient. In the case of multiple optimal solutions, one of them is efficient but also some of them may be inefficient. It is a serious flaw since practical large problems usually have multiple optimal solutions and typical optimisation solvers generate one of them (essentially at random). Therefore, to overcome this flaw of the minmax aggregation in the RPM, problem (3) is additionally regularised with the weighted additive aggregation which guarantees the efficiency of solutions.

Various functions s_i provide a wide modelling environment for measuring individual achievements.[14,10,15] For the sake of computational robustness, the piecewise linear functions

s_i are usually employed. In the simplest models, they take a form of two segment piecewise linear functions:

$$s_i(b_i, y_i) = \begin{cases} v_i^n(b_i - y_i), & \text{for} \quad y_i \leq b_i \\ v_i^p(b_i - y_i), & \text{for} \quad y_i > b_i \end{cases} \tag{5}$$

where v_i^n and v_i^p are positive weights corresponding to underachievements and overachievements, respectively, for the ith outcome. It is usually assumed that v_i^n is much larger than v_i^p. Moreover, RPM is frequently implemented in the form of so-called aspiration/reservation based decision support (ARBDS)[14] which in addition to the main target (aspiration) levels b_i employs also reservation levels r_i, so that the DM can specify desired as well as required values for given outcomes. This allows implicit definition of weights as $v_i^n = 1/(b_i - r_i)$ and $v_i^p = \beta v_i^n$ where β is a small positive parameter ($0 < \beta \ll 1$).

Under the assumption that $v_i^n \geq v_i^p > 0$ for $i = 1, 2, \ldots, q$, the lexicographic RPM model (3) with piecewise linear individual achievement functions (5) can be expressed in terms of the GP implementation environment as the following RGP model:[7]

$$\text{lex min} \quad \left[\max_{1 \leq i \leq q} \{v_i^n n_i - v_i^p p_i\}, \sum_{i=1}^{q} (v_i^n n_i - v_i^p p_i) \right] \tag{6}$$

subject to

$$f_i(x) + n_i - p_i = b_i; \;\; n_i, p_i \geq 0 \quad i = 1, 2, \ldots, q \tag{7}$$

$$x \in Q \tag{8}$$

As shown by Ogryczak,[7] the RGP model always generates an efficient solution to the original multi-criteria problem satisfying simultaneously the RPM rules. It provides a complete parameterisation of the efficient set, in the sense that for any efficient solution $\bar{x} \in Q$ there exist the reference levels allowing to find $\bar{x}$ as an optimal solution of the corresponding RGP problem. Moreover, a solution with all individual outcomes equal to the corresponding reference values is preferred to any solution with at least one outcome worse than its reference level.

The RGP model (6)–(8) is similar to the standard Minmax (fuzzy) GP model:[16]

$$\min \quad \left[\max_{1 \leq i \leq q} \{v_i^n n_i\} \right] \tag{9}$$

subject to (7) and (8)

However, the RGP model differs from (9) due to the use of negative weights ($-v_i^p$) attached to positive deviational variables and the additional regularisation term $\sum_{i=1}^{q}(v_i^n n_i - v_i^p p_i)$ to be minimised lexicographically. The Minmax GP itself does not guarantee the efficiency of solutions, even when reference levels are fixed at their ideal values.[17,18] Thus, for any reference levels, the use of the additional regularisation term is important to guarantee the efficiency of solutions generated by the minmax aggregations.

Recall that under the assumption $v_i^n \geq v_i^p > 0$ for $i = 1, 2, \ldots, q$, the lexicographic RPM model (3) with piecewise linear individual achievement functions (5) can be expressed as the RGP model (6)–(8). It is usually assumed[5] that minimisation of any negative deviation n_i is preferred to maximisation of each positive deviation p_i for $i = 1, 2, \ldots, q$. This can be modelled with v_i^n much larger than v_i^p or, as a limiting case, with the (four level) lexicographic problem:[8]

$$\begin{array}{ll} \text{lex min} & \left[\max_{1 \leq i \leq q} \{v_i^n n_i\}, \sum_{i=1}^{q} v_i^n n_i, \max_{1 \leq i \leq q} \{-v_i^p p_i\}, -\sum_{i=1}^{q} v_i^p p_i \right] \\ & \text{subject to (7) and (8)} \end{array} \tag{10}$$

called hereafter lexicographic RGP (LRGP) model. As shown by Ogryczak,[8] the above model for any reference values b_i and for any positive weights ($v_i^n > 0$ and $v_i^p > 0$ for $i = 1, 2, \ldots, q$) always generates an efficient solution to the original multi-criteria problem, satisfying simultaneously the RPM rules. Note that v_i^n and v_i^p represent here a freely selected preferential weights attached to the ith outcome for minimisation its underachievements or maximisation its overachievements, respectively, when comparing to the target b_i. As usual in the lexicographic optimisation, the objective terms placed in lower priorities become redundant if the optimisation problem corresponding to a higher priority level has no alternative solutions. Nevertheless, all four objective terms (priority levels) are important in a general case. Arbitrary skipping of any objective[17] may result in inefficient solutions.[18]

The LRGP model (10) excludes any compensation among individual achievements (deviations from the targets) whereas a specific DM structure of preferences may be considered additive.[17] One can introduce some level of compensation, and therefore additive preferences, by the use of the regularisation constant ε (instead of lexicographic optimisation) applied to the objective terms with weighted deviations (ie to the second and the fourth terms in (10)). Avoiding impractically large values of ε, it can be formulated as the following parameterisation of LRGP:

$$\begin{array}{ll} \text{lex min} & \left[(1-\varepsilon) \max_{1 \leq i \leq q} \{v_i^n n_i\} + \varepsilon \sum_{i=1}^{q} v_i^n n_i, (1-\varepsilon) \max_{1 \leq i \leq q} \{-v_i^p p_i\} - \varepsilon \sum_{i=1}^{q} v_i^p p_i \right] \\ & \text{subject to (7) and (8)} \end{array} \tag{11}$$

where the parameter ε by taking values between zero and one ($0 < \varepsilon \leq 1$) defines the (relative) level of compensation. This is a new parametric model covering the LRGP model (10) as a limiting case when $\varepsilon \to 0_+$. For $\varepsilon = 1$, model (11) becomes the lexicographic additive model introduced by Romero et al:[17]

$$\text{lex min} \quad \left[\sum_{i=1}^{q} v_i^n n_i \, , - \sum_{i=1}^{q} v_i^p p_i \right] \qquad (12)$$
$$\text{subject to (7) and (8)}$$

Note that, opposite to (2), the use of the regularisation constant ε in (10) does not restrict the search to a subset of properly efficient solutions. In the Appendix we show that model (11) guarantees the efficiency of solutions (Theorem 2) and that it is possible to generate all efficient solutions using model (11) by appropriately choosing the reference point (Theorem 3). Thus model (11) and additive model (12) as its special case ($\varepsilon = 1$) satisfy the basic requirements for the RPM approaches. The reasons why the additive model is not considered in the RPM methodology are related to the interactive nature of RPM. Namely, worse controllability of the interactive analysis based on the weighted additive aggregation of individual achievements, when comparing to the minmax aggregation, is usually noticed.[7,19] Theorem 3, although formally justifying the controllability, has limited practical importance, since in order to generate certain efficient solution the reference point has to be the efficient solution itself.

Preference relations

Every optimisation model, either scalar or lexicographic, defines a corresponding preference model which ranks the outcome vectors with a complete preorder. The preference model is completely characterized by the relation of weak preference,[20] denoted hereafter with $\succeq$. Namely, we say that outcome vector y' is (strictly) preferred to y'' ($y' \succ y''$) iff $y' \succeq y''$ and $y'' \not\succeq y'$. Similarly, we say that outcome vector y' is indifferent or equally preferred to y'' ($y' \cong y''$) iff $y' \succeq y''$ and $y'' \succeq y'$. In this section we use the preference relation to introduce a formal comparison of the GP and RPM models with respect to their specifications of the satisficing approach[4] to the decision process. Although the specifications have been widely discussed[10] 'in words', to extent of our knowledge, it is the first analysis based on strictly formalised properties of the preference relation.

The presented RPM models show that this approach can be expressed in the GP modelling framework of deviational variables and lexicographic optimisation. However, RPM independently from the target specification always generates an efficient solution. To meet this requirement all valid RPM models need to define preference models satisfying the principle of (strict) monotonicity which can be written as:

$$(y' \geq y'' \text{ and } y' \neq y'') \quad \Rightarrow \quad y' \succ y'' \tag{13}$$

or, taking advantages of transitivity, as:

$$y + \varepsilon e_i \succ y \quad \forall\ i = 1, \ldots, q;\ \varepsilon > 0 \tag{14}$$

where e_i denotes the ith unit vector. Therefore, the RPM models differ from typical GP formulations due to the use of negative weights and additional regularisation of the minmax aggregation. Both these elements are important to guarantee the strict monotonicity and thereby the efficiency of solutions. When the negative and positive deviational variables are optimised hierarchically, then each separate minmax aggregation requires the corresponding regularisation to guarantee the efficiency of solutions.

Both GP and RPM may be considered as some specification of the satisficing approach[4] to the decision process. In this approach, depending on recurrent observation, it is assumed that people tend to summarize their learning of the state of the world by forming aspirations of desirable outcomes for their decisions. When the outcomes fail to satisfy their aspirations, people tend to seek ways to improve the outcomes. When their aspirations are satisfied (or outcomes cannot be improved), however, their attention turns to other outcomes. The quasi-satisficing approach[5] extends the decision process assuming that, due to the learning factor, having achieved all aspirations (or having achieved some aspirations and having no possibility to improve other outcomes) one further tries to improve the outcomes (advances the aspirations when attainable).

Classical GP models (including model (9)) specify the satisficing approach with the assumption that no outcome vector can be preferred to the aspiration vector b.[16] In terms of the preference relation, this can be formalised as the following specification:

$$b \succeq y \quad \forall y \tag{15}$$

Specification (15), in general, contradicts the monotonicity principle (13). Therefore, the classical GP models require a carefully selected target point to guarantee the efficiency of solutions. Within RPM the target points are interpreted consistently with the monotonicity principle. It is assumed that a solution with all individual outcomes equal to the corresponding reference values is preferred to any solution with at least one outcome worse than the corresponding reference level.[5] This leads us to the following formal specification:

$$y \not\geq b \quad \Rightarrow \quad b \succ y \tag{16}$$

This specification of the quasi-satisficing behaviour is satisfied by the lexicographic RPM model (3) as well as by all the GP formulations of RPM, ie models (6), (10), (11) and (12).

Both specifications (15) and (16) focus on the case when all the target values b_i are achieved while leaving unspecified the situations when only some target values can be achieved. One may notice that having given an aspiration vector b, according to the (quasi-)satisficing model one should prefer outcome vector $(b_1 - 1, b_2, b_3)$ to outcome vector $(b_1 - 1, b_2 - 1, b_3 + 10)$. This is, in general, not guaranteed by generic RPM. In the RGP model this preference depends on a specific setting of the weights. On the other hand, the LRGP model (10) and the additive model (12) both guarantee that $(b_1 - 1, b_2, b_3)$ is always preferred to $(b_1 - 1, b_2 - 1, b_3 + 10)$. Namely, the lexicographic models with negative deviations minimised prior to the maximisation of positive deviations specify better the quasi-satisficing behaviour. They implement an additional assumption that any (small) improvement of an outcome not reaching its aspiration is preferred to any (large) improvement of an outcome already satisfying its aspiration. Formally, this can be written as:

$$y_i < b_i \text{ and } y_j > b_j \quad \Rightarrow \quad y + \varepsilon' e_i \succ y + \varepsilon'' e_j \quad \forall\, 0 < \varepsilon' \leq b_i - y_i;\; 0 < \varepsilon'' \leq y_j - b_j \qquad (17)$$

Equity of individual achievements

Every RPM (or GP) technique builds the individual achievement functions which measure actual achievement of each outcome with respect to the corresponding reference level. Thus, all the outcomes are transformed into a uniform scale of individual achievements. Romero et al[17] have raised an important issue of equity among individual achievements in the context of the RPM and GP approaches. Unfortunately, the issue is not quite resolved in that paper. Romero et al[17] (p 987) claim that when reference levels are fixed at their ideal values, then any solution of the corresponding Minmax GP model (9) is perfectly equilibrated; that is, the weighted gaps between the ideal values and the actual achievement of goals are equal:

$$v_1^n(b_1^* - f_1(x)) = \cdots = v_i^n(b_i^* - f_i(x)) = \cdots = v_q^n(b_q^* - f_q(x)) \qquad (18)$$

This assertion is quite intuitive and valid in the case of bi-criteria convex problems.[21] However, in general, it is too strong and not true. One can easily build examples that no solution of the corresponding Minmax GP problems satisfies (18).[18] Actually, the feasible set may do not contain any solution satisfying the requirement of perfect equilibration (18).

On the other hand, for any reference levels b_i and any strictly decreasing individual achievement functions $s_i(b_i, y_i)$, RPM generates an efficient solution providing the perfect equity of the individual achievements:

$$s_1(b_1, f_1(x)) = \cdots = s_i(b_i, f_i(x)) = \cdots = s_q(b_q, f_q(x)) \tag{19}$$

if such an efficient solution exists. More precisely, the following assertion is valid.

Theorem 1 *For any reference levels b_i and any individual achievement functions $s_i(b_i, y_i)$ strictly decreasing with respect to y_i, if there exists an efficient solution $\bar{x} \in Q$ satisfying the equilibration requirement (19), then each optimal solution of the minmax problem (4) is efficient and perfectly equilibrated.*

Proof. Let $\bar{x} \in Q$ be an efficient solution satisfying the equilibration requirement (19) with some reference levels b_i and strictly decreasing individual achievement functions $s_i(b_i, y_i)$, $i = 1, 2, \ldots, q$. This means, there exists a number α such that $s_i(b_i, f_i(\bar{x})) = \alpha$ for $i = 1, 2, \ldots, q$.

Let $x \in Q$ be an optimal solution of the minmax problem (4). If $f_i(x) = f_i(\bar{x})$ for $i = 1, 2, \ldots, q$, then x is, obviously, efficient and perfectly equilibrated. Suppose, there exists some index i_0 such that $f_{i_0}(x) \neq f_{i_0}(\bar{x})$. Due to the optimality of x, we have:

$$s_i(b_i, f_i(x)) \leq \max_{1 \leq j \leq q} \{s_j(b_j, f_j(x))\} \leq \max_{1 \leq j \leq q} \{s_j(b_j, f_j(\bar{x}))\} = \alpha = s_i(b_i, f_i(\bar{x})) \quad \forall\, i = 1, \ldots, q$$

Thus, due to strictly decreasing individual achievement functions $s_i(b_i, y_i)$, we get $f_i(x) \geq f_i(\bar{x})$ for all $i = 1, \ldots, q$ and $f_{i_0}(x) \neq f_{i_0}(\bar{x})$ which contradicts the assumption that $\bar{x}$ is efficient. □

Note that if every solution of the minmax aggregations (4) satisfies the condition of perfect equity (19), then all the solutions generate the same values of the outcomes (they are equal in the criterion space). Therefore, the property of perfect equity is not disturbed by any additional regularisation. Hence, the lexicographic RPM model (3) and its specifications in the form of RGP and LRGP models, for any reference levels b_i, generate efficient solutions satisfying the perfect equity of individual achievements, whenever it is possible (whenever such an efficient solution exists). Certainly, this does not apply to the additive model (12). When there does not exist an efficient solution with perfectly equal individual achievements, then RPM generates another efficient solution but still providing some equitability of individual achievements by implementation of an approximation to the Rawls[22] principle of justice. We take an opportunity to explain this approximation[23] as it has not been discussed in the international literature.

The Rawls[22] principle of justice defines a ranking for different 'social states' by the rule that any two states should be ranked according to the least well-off individuals in these societies; if the comparison yields a tie, one should consider the next-least well-off individuals, and so on. Thus the principle of justice defines in words the mathematical concept of the lexicographic minmax optimisation[24] known also as the nucleolus in the game theory.

Let a_i denote the individual achievement for the ith outcome ($a_i = s_i(b_i, f_i(x))$ for $i = 1, 2, \ldots, q$) and $a = (a_1, a_2, \ldots, a_q)$ represent the achievement vector. The lexicographic minmax approach can be mathematically formalised as follows. Within the space of achievement vectors we introduce map $\Theta = (\theta_1, \theta_2, \ldots, \theta_q)$ which orders the coordinates of achievements vectors in a nonincreasing order, i.e., $\Theta(a_1, a_2, \ldots, a_q) = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_q)$ iff there exists a permutation τ such that $\bar{a}_i = a_{\tau(i)}$ for all i and $\bar{a}_1 \geq \bar{a}_2 \geq \ldots \geq \bar{a}_q$. Note that the standard minmax approach (4) depends on minimisation of $\bar{a}_1 = \theta_1(a)$ and it ignores values of $\bar{a}_i$ for $i \geq 2$. In order to take into account all the achievement values, we look for a lexicographic minimum among the ordered achievement vectors. The lexicographic minmax solution is an optimal solution of the following lexicographic problem:

$$\text{lex min } \{ \ [\theta_1(a), \theta_2(a), \ldots, \theta_q(a)] \ : \ a_i = s_i(b_i, f_i(x)), \ i = 1, \ldots, q; \ x \in Q \ \} \qquad (20)$$

The concept of lexicographic minmax solution is a consequent regularisation of the minmax aggregation according to the Rawlsian principle of justice. It is the only one regularisation of the minmax approach satisfying the reduction (addition/deleting) principle.[25] Namely, if the individual achievement of an outcome does not distinguish two solutions, then it does not affect the preference relation:

$$(a'_1, \ldots, a'_i, a^*, a'_{i+1}, \ldots, a'_q) \succeq (a''_1, \ldots, a''_i, a^*, a''_{i+1}, \ldots, a''_q) \quad \Leftrightarrow \quad a' \succeq a'' \qquad (21)$$

Moreover, due to strictly monotonic individual achievement functions, the reduction principle is also satisfied in the original outcome space.

Every optimal solution of the lexicographic minmax model (20) is an efficient solution to the original multi-criteria optimisation problem.[24] Note that every lexicographic minmax solution is also an optimal solution to the standard minmax problem (4). Hence, by virtue of Theorem 1, the lexicographic minmax model (20), for any reference levels b_i, generates efficient solutions satisfying the perfect equity of individual achievements (19), whenever such an efficient solution exists. When there does not exist any efficient solution with perfectly equal individual achievements, then the lexicographic minmax model generates another efficient solution but still providing equitability of individual achievements with respect to the Pigou-Dalton principle of transfers.[26] The principle of transfers states, in the context considered here, that a transfer of small amount from an individual achievement to any relatively worse-off individual achievement results in a more preferred achievement vector, ie:

$$a_i < a_j \quad \Rightarrow \quad a + \varepsilon e_i - \varepsilon e_j \succ a \quad \forall\, 0 < \varepsilon \leq a_j - a_i \tag{22}$$

Recall that this property applies to uniformly measured individual achievements and it does not enforce any equitability of the original outcomes.

One may consider the lexicographic minmax problem (20) as a basis for a corresponding nucleolar RPM model. In the case of multi-criteria linear programming the lexicographic minmax optimisation can be solved, like the standard lexicographic optimisation problems, by sequential optimisation with elimination of the dominating functions.[24] Thus the nucleolar RPM model (with piecewise linear individual achievement functions) may be considered implementable in the case of linear problems with a limited number of criteria. However, in general, the lexicographic minmax problem (20) is too complex to be solved during an interactive analysis.

Note that the lexicographic RPM model (3) can be expressed as the following problem:

$$\text{lex}\ \min \left\{ \left[\theta_1(a), \sum_{i=2}^{q} \theta_i(a)\right] \ :\ a_i = s_i(b_i, f_i(x)),\ i = 1, \ldots, q;\ x \in Q \right\}$$

thus representing exactly the lexicographic minmax (20) in the case of two criteria ($q = 2$). For larger number of criteria ($q > 2$) model (3) only approximates the lexicographic minmax (20) as all the lower priority objective terms are aggregated at the second priority level. Hence, the lexicographic RPM model (3) fulfills the principle of transfers only in the case of an improvement of the worst individual achievement, ie:

$$a_i < a_j\ \forall\, i : i \neq j \quad \Rightarrow \quad a + \varepsilon e_i - \varepsilon e_j \succ a \quad \forall\, 0 < \varepsilon \leq a_j - a_i \tag{23}$$

The same applies to the RGP model (6) with respect to individual achievements defined by the piecewise linear functions (5).

The additive aggregation, in general, does not maintain any equitable preferences. However, the additive model (12), due to minimisation of all negative deviations prior to the maximisation of any positive deviation, supports some equitability of individual achievements. It follows from (17) that, for individual achievements a_i defined by the piecewise linear function (5), the additive model (12) fulfills the principle of transfers restricted to the case of improvement of the underachievements and worsening of the overachievements, ie:

$$a_i < 0 < a_j \quad \Rightarrow \quad a + \varepsilon e_i - \varepsilon e_j \succ a \quad \forall\, 0 < \varepsilon \leq \min\{a_j, |a_i|\} \tag{24}$$

The four level LRGP model (10), being based on the minmax aggregation and minimising negative deviations prior to the maximisation of positive deviations, combines the equitability properties of the standard RGP model (6) and that of the additive model (12). Namely, with respect to individual achievements a_i defined by the piecewise linear function (5), the LRGP model fulfills both (23) and (24).

Conclusions

The RPM approach can be expressed in the GP modelling framework of deviational variables and lexicographic optimisation. However, the preference models of RPM and those of typical GP represent different specifications of the satisficing approach to the decision process. Therefore, the corresponding RGP model differs from typical GP formulations. It makes use of negative weights assigned to the positive deviational variables and of the additional regularisation of the minmax aggregation. Both these elements are important to guarantee the efficiency of solutions. For better modelling of the satisficing behaviour, one may consider the lexicographic RGP model where the negative and positive deviational variables are optimised hierarchically. However, each separate minmax aggregation requires then the corresponding regularisation to guarantee the efficiency of solutions.

The RPM models generate efficient solutions satisfying the perfect equity of individual achievements, whenever such an efficient solution exists. When there does not exist an efficient solution with perfectly equal individual achievements, then RPM generates another efficient solution but still providing some equitability of individual achievements by implementation of an approximation to the Rawlsian principle of justice. The lexicographic RGP model, where the negative and positive deviational variables are optimised hierarchically, enhances the equitability properties of the standard RGP model.

We hope that the analysis presented in this paper gives a clear overview of GP models of the RPM approaches and their properties. Real-life applications of the RPM methodology usually deal with more complex individual achievement functions. They are defined with multiple reference points[14,10,15] which enriches the preference models and simplifies the interactive analysis. Nevertheless, the main properties of the RPM models remain the same as discussed in this paper.

Appendix

In this Appendix we formally prove the relations between the efficient solutions of the multi-criteria optimisation problem (1) and the optimal solutions of the corresponding parametric model (11). Note that the following theorems cover also to the case of $\varepsilon = 1$ thus justifying the lexicographic additive RGP model (12) as a limiting case.

Theorem 2 *For any reference levels b_i, any positive weight coefficients $v_i^n, v_i^p > 0$ and $0 < \varepsilon \leq 1$, if $(\bar{x}, \bar{n}, \bar{p})$ is an optimal solution of the problem (11), then $\bar{x}$ is an efficient solution of the corresponding multi-criteria problem (1).*

Proof. Let $(\bar{x}, \bar{n}, \bar{p})$ be an optimal solution of the problem (11) with some $v_i^n, v_i^p > 0$ and $0 < \varepsilon \leq 1$. Note that, due to $v_i^n, v_i^p > 0$ and $0 < \varepsilon \leq 1$, the deviations $\bar{n}$ and $\bar{p}$ satisfy

$$\bar{n}_i = \max\{b_i - f_i(\bar{x}), 0\} \quad \text{and} \quad \bar{p}_i = \max\{f_i(\bar{x}) - b_i, 0\} \quad \text{for } i = 1, 2, \ldots, q \tag{25}$$

thus $\bar{n}_i \bar{p}_i = 0$ for all i.

Suppose that $\bar{x}$ is not efficient to the multi-criteria problem (1). This means, there exists a decision vector $x \in Q$ such that

$$f_i(x) \geq f_i(\bar{x}) \text{ for } i = 1, 2, \ldots, q \quad \text{and} \quad f_{i_o}(x) > f_{i_o}(\bar{x}) \tag{26}$$

where i_o is some outcome index $(0 \leq i_o \leq q)$. Let us define:

$$n_i = \max\{b_i - f_i(x), 0\} \quad \text{and} \quad p_i = \max\{f_i(x) - b_i, 0\} \quad \text{for } i = 1, 2, \ldots, q \tag{27}$$

The triple (x, n, p) is then a feasible solution of problem (11). Moreover, from (25), (26) and (27) it follows that $n_i \leq \bar{n}_i$ and $p_i \geq \bar{p}_i$ for $i = 1, 2, \ldots, q$ where either strict inequality $n_{i_o} < \bar{n}_{i_o}$ or strict inequality $p_{i_o} > \bar{p}_{i_o}$ holds. Due to $v_i^n, v_i^p > 0$ and $0 < \varepsilon \leq 1$, the latest assertion contradicts the optimality of $(\bar{x}, \bar{n}, \bar{p})$ for problem (11), which completes the proof. □

Theorem 3 *If $\bar{x}$ is an efficient solution of the multi-criteria problem (1), then there exist reference levels b_i such that the triple $\bar{x}$, $\bar{n} = (\max\{b_i - f_i(\bar{x}), 0\})_{i=1,\ldots,q}$ and $\bar{p} = (\max\{f_i(\bar{x}) - b_i, 0\})_{i=1,\ldots,q}$ is an optimal solution of the corresponding problem (11), for any positive weight coefficients $v_i^n, v_i^p > 0$ and $0 < \varepsilon \leq 1$.*

Proof. Let us set the reference levels as $b_i = f_i(\bar{x})$, for $i = 1, 2, \ldots, q$. One can easily verify that the triple: $\bar{x}$, $\bar{n} = (\max\{b_i - f_i(\bar{x}), 0\})_{i=1,\ldots,q} = 0$ and $\bar{p} = (\max\{f_i(\bar{x}) - b_i, 0\})_{i=1,\ldots,q} = 0$ is a feasible solution of the corresponding problem (11), for any positive weight coefficients $v_i^n, v_i^p > 0$ and $0 < \varepsilon \leq 1$. Moreover,

$$(1 - \varepsilon) \max_{1 \leq i \leq q} \{v_i^n \bar{n}_i\} + \varepsilon \sum_{i=1}^{q} v_i^n \bar{n}_i = 0 \quad \text{and} \quad (1 - \varepsilon) \max_{1 \leq i \leq q} \{-v_i^p \bar{p}_i\} - \varepsilon \sum_{i=1}^{q} v_i^p \bar{p}_i = 0$$

Suppose that for some $v_i^n, v_i^p > 0$ and $0 < \varepsilon \leq 1$ the triple $(\bar{x}, 0, 0)$ is not an optimal solution of the corresponding problem (11). This means, there exists a feasible triple (x, n, p) such that

$$\left[(1-\varepsilon) \max_{1 \leq i \leq q} \{v_i^n n_i\} + \varepsilon \sum_{i=1}^{q} v_i^n n_i \,, (1-\varepsilon) \max_{1 \leq i \leq q} \{-v_i^p p_i\} - \varepsilon \sum_{i=1}^{q} v_i^p p_i \right] <_{lex} [0, 0]$$

Hence, $n_i = 0$ for all $i = 1, 2, \ldots, q$ and there exists i_o $(1 \leq i_o \leq q)$ such that $p_{i_o} > 0$. Recall that $b_i = f_i(\bar{x})$. This implies $f_i(x) = b_i + p_i \geq f_i(\bar{x})$ for all $i = 1, 2, \ldots, q$ and $f_{i_o}(x) > f_{i_o}(\bar{x})$, which contradicts the efficiency of $\bar{x}$. □

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References

1 Charnes A and Cooper WW (1961). *Management Models and Industrial Applications of Linear Programming*. Wiley: New York.

2 Wierzbicki AP (1977). Basic properties of scalarizing functionals for multiobjective optimization. *Optimization* **8**: 55–60.

3 Wierzbicki AP (1982). A mathematical basis for satisficing decision making. *Math Modelling* **3**: 391–405.

4 Simon HA (1957). *Models of Man*. MacMillan: New York.

5 Wierzbicki AP (1986). On completeness and constructiveness of parametric characterizations to vector optimization problems. *OR Spektrum* **8**: 73–87.

6 Ogryczak W and Lahoda S (1992). Aspiration/reservation decision support — A step beyond goal programming. *J Multi–Criteria Dec Anal* **1**: 101–117.

7 Ogryczak W (1994). A goal programming model of the reference point method. *Annals of Opns Res* **51**: 33–44.

8 Ogryczak W (1997). Preemptive reference point method. In: Climaco J (ed). *Multicriteria Analysis*. Springer: Berlin, pp 156–167.

9 Steuer RE (1986). *Multiple Criteria Optimization — Theory, Computation and Applications*. Wiley: New York.

10 Wierzbicki AP, Makowski M and Wessels J (2000). *Model Based Decision Support Methodology with Environmental Applications*. Kluwer: Dodrecht.

11 Kaliszewski I (1994). *Quantitative Pareto Analysis by Cone Separation Technique*. Kluwer: Dodrecht.

12 Geoffrion AM (1968). Proper efficiency and the theory of vector maximization, *J Math Anal Appls* **22**: 618–630.

13 Korhonen P and Halme M (1996). Using lexicographic parametric programming for searching a non-dominated set in multiple-objective linear programming. *J Multi–Criteria Dec Anal* **5**: 291–300.

14 Lewandowski A and Wierzbicki AP (ed) (1989). *Aspiration Based Decision Support Systems — Theory, Software and Applications*. Springer: Berlin.

15 Granat J and Makowski M (2000). ISAAP — Interactive specification and analysis of aspiration-based preferences. *Eur J Opl Res* **122**: 469–485.

16 Ignizio JP (1982). *Linear Programming in Single and Multiple Objective Systems*. Prentice-Hall: Englewood Cliffs, NJ.

17 Romero C, Tamiz M and Jones DF (1998). Goal programming, compromise programming and reference point method formulations: linkages and utility interpretations. *J Opl Res Soc* **49**: 986–991.

18 Ogryczak W (2001). Viewpoint on: 'Goal programming, compromise programming and reference point method formulations: linkages and utility interpretations, *J Opl Res Soc* **49**: 986–991'. *J Opl Res Soc*, submitted.

19 Nakayama H (1997). Some remarks on trade-off analysis in multiobjective programming. In: Climaco J (ed). *Multicriteria Analysis*. Springer: Berlin, pp 179–190.

20 Vincke Ph (1992). *Multicriteria Decision–Aid*. Wiley: New York.

21 Ballestero E and Romero C (1991). A theorem connecting utility function optimization and compromise programming. *Opns Res Lett* **10**: 421–427.

22 Rawls J (1971). *The Theory of Justice*. Harvard Univ Press: Cambridge.

23 Ogryczak W (1997). *Linear and Discrete Optimization with Multiple Criteria: Preference Models and Applications to Decision Support* (in Polish). Warsaw Univ Press: Warsaw.

24 Marchi E and Oviedo JA (1992). Lexicographic optimality in the multiple objective linear programming: the nucleolar solution. *Eur J Opl Res* **57**: 355–359.

25 Ehrgott M (1997). *Multiple Criteria Optimization: Classification and Methodology*. Shaker: Aachen.

26 Kostreva MM and Ogryczak W (1999). Linear optimization with multiple equitable criteria. *RAIRO Oper Res* **33**: 275–297.