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REFERENCE POINT METHOD WITH LEXICOGRAPHIC MIN-ORDERING OF INDIVIDUAL ACHIEVEMENTS*

Abstract

The reference point method (RPM) is a very effective technic for interactive analysis of the multiple criteria optimization problems. It provides the DM with a tool for an open analysis of efficient frontier either connected or disconnected. The interactive analysis is navigated by the commonly accepted control parameters expressing reference levels for the individual objective functions. The individual achievement functions quantify the DM's satisfaction from the individual outcomes with respect to the given reference levels. The final scalarizing function is built as the augmented max-min aggregation of individual achievements which means that the worst individual achievement is essentially maximized, but the optimization process is additionally regularized with the term representing the average achievement. The regularization by the average achievement is easily implementable, but it may disturb the basic max-min aggregation. In order to avoid inconsistencies caused by the regularization, the max-min solution may be regularized according to the lexicographic min-order, thus leading to the nucleolar RPM model. The nucleolar RPM implements a consequent max-min aggregation taking into account also the second-worst achievement, the third-worse, and so on, thus resulting in much better modeling of the reference levels concept. The lexicographic min-ordering regularization is more complicated in implementation due to the requirement of pointwise ordering of partial achievements. Nevertheless, by taking advantage of piecewise linear expression of the cumulated ordered achievements, the nucleolar RPM can be formulated as a standard lexicographic optimization. Actually, in the case of concave piecewise linear partial achievement functions (typically used in the RPM), the resulting formulation extends the original constraints and criteria with simple linear inequalities, thus allowing for a quite efficient implementation. It can be also approximated with the analytic form using the ordered weighted averages. The paper analyzes both the theoretical and practical issues of the nucleolar RPM.

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Keywords

Multicriteria optimization, reference point method (RPM), lex-min order.

INTRODUCTION

Typical multiple criteria optimization methods aggregate the individual outcomes with some scalarizing functions to generate a satisfactory efficient solution. The scalarizing functions may have various constructions and properties depending on the specific approach to preference modeling applied in several methods. Nevertheless, most scalarizing functions can be viewed as two-stage transformation of the original outcomes. First, the individual outcomes are rescaled to some uniform measures of achievements with respect to several criteria and preference parameters. Thus, the individual achievement functions are built to measure actual achievement of each outcome with respect to the corresponding preference parameters. In particular, in the reference point method (RPM) the strictly monotonic partial achievement functions are built to measure individual performance with respect to given reference levels. Similar constructions appear in fuzzy approaches where the membership functions for various fuzzy targets are such individual achievement measures scaled to the unit interval or in goal programming where scaled deviations from targets may be considered individual achievements.

Having all the outcomes transformed into a uniform scale of individual achievements they are aggregated at the second stage to form a unique scalarization. The aggregation usually measures the total (the average) or the worst individual achievement. While several technic and tools for better modeling of preferences with partial achievement functions are developed [3], the aggregation itself is much less studied. The RPM is based on the so-called augmented (or regularized) max-min aggregation. Thus, the worst individual achievement is essentially maximized, but the optimization process is additionally regularized with the term representing the average achievement. The max-min aggregation guarantees fair treatment of all individual achievements by implementing an approximation to the Rawlsian principle of justice.

The max-min aggregation is crucial for allowing the RPM to generate all efficient solutions even for nonconvex (and particularly discrete) problems. On the other hand, the regularization is necessary to guarantee that only efficient solution are generated. The regularization by the average achievement is easily implementable, but it may disturb the basic max-min model. Actually, the only consequent regularization of the max-min aggregation is the lexicographic max-min (nucleolar) solution concept where in addition to the worst achievement, the second worst achievement is also optimized (provided that the worst remains on the optimal level), the third worst is optimized (provided that the two worst remain optimal), and so on. Such a nucleolar regularization is the only max-min regularization satisfying the addition/deleting principle, thus making the corresponding nucleolar RPM not affected by any passive criteria. The recent progress in optimization methods of ordered averages allows one to implement the nucleolar RPM quite effectively. The paper analyzes both the theoretical and practical issues of the nucleolar RPM.

1. SCALARIZATIONS OF THE REFERENCE POINT METHOD

In this paper, without loss of generality, it is assumed that all the criteria are maximized (that is, for each outcome "more is better"). Hence, we consider the following multiple criteria optimization problem:

$$\max \{ (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})) \ : \ \mathbf{x} \in Q \} \tag{1}$$

where x denotes a vector of decision variables to be selected within the feasible set $Q \subset R^n$, and $\mathbf{f}(x) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ is a vector function that maps the feasible set Q into the criterion space R^m. Note that neither any specific form of the feasible set Q is assumed nor any special form of criteria $f_i(\mathbf{x})$ is required. We refer to the elements of the criterion space as outcome vectors. An outcome vector $\mathbf{y}$ is attainable if it expresses outcomes of a feasible solution, i.e. $\mathbf{y} = \mathbf{f}(\mathbf{x})$ for some $\mathbf{x} \in Q$.

Model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. Thus, it allows only to identify (to eliminate) obviously inefficient solutions leading to dominated outcome vectors, while still leaving the entire efficient set to look for a satisfactory compromise solution. In order to make the multiple criteria model operational for the decision support process, one needs assume some solution concept well adjusted to the DM preferences. This can be achieved with the so-called quasi-satisficing approach to multiple criteria decision problems. The best formalization of the quasi-satisficing approach to multiple criteria optimization was proposed and developed mainly by Wierzbicki [20] as the reference point method. The RPM was later extended to permit additional information from the DM and, eventually, led to efficient implementations of the so-called aspiration/reservation based decision support (ARBDS) approach with many successful applications [5,21].

The RPM is an interactive technic. The basic concept of the interactive scheme is as follows. The DM specifies requirements in terms of reference levels, i.e. by introducing reference (target) values for several individual outcomes. Depending on the specified reference levels, a special scalarizing achievement function is built which may be directly interpreted as expressing utility to be maximized. Maximization of the scalarizing achievement function generates an efficient solution to the multiple criteria problem. The computed efficient solution is presented to the DM as the current solution in a form that allows comparison with the previous ones and modification of the reference levels if necessary.

While building the scalarizing achievement function the following properties of the preference model are assumed. First of all, for any individual outcome y_i more is preferred to less (maximization). To meet this requirement the function must be strictly increasing with respect to each outcome. Second, a solution with all individual outcomes y_i satisfying the corresponding reference levels is preferred to any solution with at least one individual outcome worse (smaller) than its reference level. That means, the scalarizing achievement function maximization must enforce reaching the reference levels prior to further improving of criteria. Thus, similar to the goal programming approaches, the reference levels are treated as the targets, but following the quasi-satisficing approach they are interpreted consistently with basic concepts of efficiency in the sense that the optimization is continued even when the target point has been reached already.

The generic scalarizing achievement function takes the following form [20]:

$$S(\mathbf{y}) = \min_{1 \leq i \leq m} \{s_i(y_i)\} + \frac{\varepsilon}{m} \sum_{i=1}^{m} s_i(y_i) \tag{2}$$

where ε is an arbitrary small positive number and $s_i : R \to R$, for $i = 1, 2, \ldots, m$ are the partial achievement functions measuring actual achievement of the individual outcomes y_i with respect to the corresponding reference levels. Let a_i denote the partial achievement for the i-th outcome $(a_i = s_i(y_i))$ and $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ represent the achievement vector. The scalarizing achievement function (2) is, essentially, defined by the worst partial (individual) achievement, but additionally regularized with the sum of all partial achievements. The regularization term is introduced only to guarantee the solution efficiency in the case when the maximization of the main term (the worst partial achievement) results in a nonunique optimal solution. Due to combining two terms with arbitrarily small parameter ε, formula (2) is easily implementable and it provides a direct interpretation of the scalarizing achievement function as expressing utility. When accepting the loss of a direct utility interpretation, one may consider a limiting case with $\varepsilon \to 0_+$ which results in lexicographic order applied to two separate terms of function (2). That means, the regularization can be implemented with the second level lexicographic optimization [14]. Therefore, RPM may be also considered as the following lexicographic problem ([13] and references therein):

$$\operatorname{lex\,max} \{ \, (\min_{1 \le i \le m} a_i \, , \sum_{i=1}^{m} a_i) \; : \; a_i = s_i(f_i(\mathbf{x})) \; \forall \, i, \; \mathbf{x} \in Q \, \} \tag{3}$$

The following two properties of the lexicographic model (3) are crucial for the RPM methodology:

P1: The aggregation is strictly monotonic in the sense that increase of any partial achievement a_i leads to a preferred solution.

P2: For any given target value ϱ, the solution generating all partial achievements equal to ϱ ($a_i = \varrho \; \forall i$) is preferred to any solution generating at least one partial achievement worse than ϱ.

Property P1 guarantees that while using strictly increasing partial achievement functions s_i, every generated solution is efficient. Property P2 guarantees that while using partial achievement function allocating the same value on achieving the reference level, the solution reaching all the reference levels is preferred to any solution failing achievement of at least one reference level.

Various functions s_i provide a wide modeling environment for measuring partial achievements [21,8]. To take advantages of properties P1 and P2 they need to be strictly increasing and to allocate the same value on reaching the reference level. The basic RPM model is based on a single vector of the reference levels, the aspiration vector $\mathbf{r}^a$. For the sake of computational simplicity, the piecewise linear functions s_i are usually employed. In the simplest models, they take a form of two segment piecewise linear functions:

$$s_i(y_i) = \begin{cases} \lambda_i^+(y_i - r_i^a), & \text{for} \quad y_i \geq r_i^a \\ \lambda_i^-(y_i - r_i^a), & \text{for} \quad y_i < r_i^a \end{cases} \tag{4}$$

where λ_i^+ and λ_i^- are positive scaling factors corresponding to under-achievements and overachievements, respectively, for the i-th outcome. Note that for any outcome reaching the corresponding aspiration level $y_i = r_i^a$ one gets $s_i(r_i^a) = 0$. Hence, when using the RPM (3) with partial achievement functions (4), the solution reaching all the aspiration levels is preferred to any solution failing achievement of at least one aspiration level. It is usually assumed that λ_i^+ and is much larger than λ_i^- . Actually, even linear functions:

$$s_i(y_i) = \lambda_i(y_i - r_i^a) \tag{5}$$

with positive scaling factors λ_i represent simplified (but still valid) partial achievement functions in the sense that while used in the lexicographic RPM scheme (3) it guarantees the property **P2**. Nevertheless, the differentiation of the scaling factor is important to enforce the preferences of achieving more aspiration levels rather than overstep the others, especially in the analytic RPM(2). Figure 1 depicts how differentiated scaling affects the isoline contours of the analytic scalarizing achievement function. Certainly, introducing lexicographic two-level partial achievements optimization would be a better way to model the aspiration properties [11], but also more complicated.

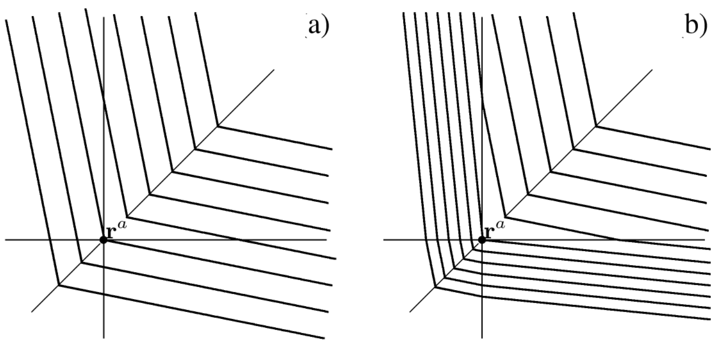

Fig. 1. Isoline contours for the analytic scalarizing achievement function (2): a) with partial achievements (5), (b) with partial achievements (4)

Real-life applications of the RPM methodology usually deal with more complex partial achievement functions defined with more than one reference point [21] which enriches the preference models and simplifies the interactive analysis. In particular, the models taking advantages of two reference vectors: vector of aspiration levels $\mathbf{r}^a$ and vector of reservation levels $\mathbf{r}^r$ [5] are used, thus allowing the DM to specify requirements by introducing acceptable and required values for several outcomes. The partial achievement function s_i can be interpreted then as a measure of the DM's satisfaction with the current value of outcome the i-th criterion. It is a strictly increasing function of outcome y_i with value $a_i = 1$ if $y_i = r_i^a$, and $a_i = 0$ for $y_i = r_i^r$. Thus, the partial achievement functions map the outcomes values onto a normalized scale of the DM's satisfaction. Various functions can be built meeting those requirements. We use the piece-wise linear partial achievement function introduced in an implementation of the ARBDS system for the multiple criteria transshipment problems with facility location [15]:

$$s_i(y_i) = \begin{cases} \gamma \quad (y_i - r_i^r)/(r_i^a - r_i^r), & \text{for} \quad y_i \leq r_i^r \\ \quad\ (y_i - r_i^r)/(r_i^a - r_i^r), & \text{for} \quad r_i^r < y_i < r_i^a \\ \alpha \quad (y_i - r_i^a)/(r_i^a - r_i^r) + 1, & \text{for} \quad y_i \geq r_i^a \end{cases} \tag{6}$$

where α and γ are arbitrarily defined parameters satisfying $0 < \alpha < 1 < \gamma$. Parameter α represents additional increase of the DM's satisfaction over level 1 when a criterion generates outcomes better than the corresponding aspiration level. On the other hand, parameter $\gamma > 1$ represents dissatisfaction connected with outcomes worse than the reservation level.

For outcomes between the reservation and the aspiration levels, the partial achievement function s_i can be interpreted as a membership function μ_i for a fuzzy target. However, such a membership function remains constant with value 1 for all outcomes greater than the corresponding aspiration level and with value 0 for all outcomes below the reservation level (Figure 2).

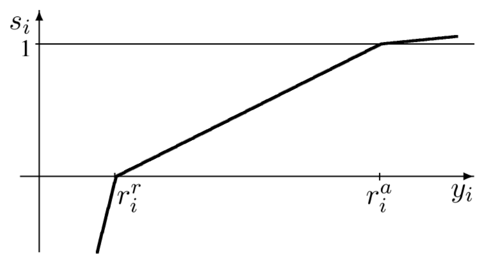

Fig. 2. ARBDS partial achievement function

Hence, the fuzzy membership function is neither strictly monotonic nor concave, thus not representing typical utility for a maximized outcome. The partial achievement function (6) can be viewed as an extension of the fuzzy membership function to a strictly monotonic and concave utility. One may also notice that the aggregation scheme used to build the scalarizing achievement function (2) from the partial ones may also be interpreted as some fuzzy aggregation operator [21]. In other words, maximization of the scalarizing achievement function (2) is consistent with the fuzzy methodology in the case of not attainable aspiration levels and satisfiable all reservation levels while modeling reasonable utility for any values of aspiration and reservation levels.

2. NUCLEOLAR RPM

The crucial properties of the RPM are related to the max-min aggregation of partial achievements while the regularization is only introduced to guarantee the aggregation monotonicity. Unfortunately, the distribution of achievements may make the max-min criterion partially passive when one specific achievement is relatively very small for all the solutions. Maximization of the worst achievement may then leave all other achievements unoptimized. In the lexicographic RPM defined by (3) the regularization term is then optimized on the second level, thus preventing one from selection of any inefficient solution. Nevertheless, the selection is then made according to linear aggregation of the regularization term instead of the max-min aggregation, thus destroying the preference model of the RPM. This can be illustrated with an example of a simple discrete problem of 7 alternative feasible solutions to be selected according to 6 criteria. Table 1 presents six partial achievements for all the solutions where the partial achievements have been defined according to the aspiration/reservation model (6), thus allocating 1 to outcomes reaching the corresponding aspiration level. Solution S7 is the only inefficient alternative. Solution S1 to S5 oversteps the aspiration levels (achievement values 1.2) for four of the first five criteria while failing to reach one of them and the aspiration level for the sixth criterion as well (achievement values 0.3). Solution S6 meets the aspiration levels (achievement values 1.0) for the first five criteria while failing to reach only the aspiration level for the sixth criterion (achievement values 0.3). One may easily notice that the sixth partial achievement (and the corresponding criterion) is constant for the seven alternatives under consideration. Hence, one may expect the same solution selected while taking into account this criterion or not. If focusing on only five first criteria, then the RPM (either lexicographic (3) or analytic (2)) obviously selects solution S6 as reaching all aspiration levels which results in the worst achievement value 1.0. However, while taking into account all six criteria all the solutions generate the same worst achievement value 0.3 and the final selection of the RPM depends on the total achievement (regularization term). Actually, either lexicographic RPM (3) or its analytic version (2) will select then one of solutions S1 to S5 as better than S6.

Table 1

Sample achievements with a passive criterion

Solution	a_1	a_2	a_3	a_4	a_5	a_6	$\min\limits_{i=1,\dots,6}$	$\sum_{i=1}^{6}$	$\min\limits_{i=1,\dots,5}$	$\sum_{i=1}^{5}$
S1	0.3	1.2	1.2	1.2	1.2	0.3	0.3	5.4	0.3	5.1
S2	1.2	0.3	1.2	1.2	1.2	0.3	0.3	5.4	0.3	5.1
S3	1.2	1.2	0.3	1.2	1.2	0.3	0.3	5.4	0.3	5.1
S4	1.2	1.2	1.2	0.3	1.2	0.3	0.3	5.4	0.3	5.1
S5	1.2	1.2	1.2	1.2	1.2	0.3	0.3	5.4	0.3	5.1
S6	1.0	1.0	1.0	1.0	1.0	0.3	0.3	5.3	1.0	5.0
S7	0.3	1.0	0.3	1.0	0.6	0.3	0.3	3.5	0.3	3.2

In order to avoid inconsistencies caused by the regularization, the max-min solution may be regularized according to the Rawlsian principle of justice. Formalization of this concept leads us to the lexicographic max-min ordered or nucleolar solution concept. The approach has been used for general linear programming multiple criteria problems [1,7] as well as for specialized problems related to (multiperiod) resource allocation [6]. In discrete optimization it has been considered for various problems including the location-allocation ones [10]. The lexicographic max-min approach can be mathematically formalized as follows. Within the space of achievement vectors we introduce map $\Theta = (\theta_1, \theta_2, \dots, \theta_m)$ which orders the coordinates of achievements vectors in a nondecreasing order, i.e. $\Theta(a_1, a_2, \dots, a_m) = (\theta_1(\mathbf{a}), \theta_2(\mathbf{a}), \dots, \theta_m(\mathbf{a}))$ if there exists a permutation τ such that $\theta_i(\mathbf{a}) = a_{\tau(i)}$ for all i and $\theta_1(\mathbf{a}) \leq \theta_2(\mathbf{a}) \leq \dots \leq \theta_m(\mathbf{a})$. The standard max-min aggregation depends on maximization of $\theta_1(\mathbf{a})$ and it ignores values of $\bar{a}_i$ for $i \geq 2$. In order to take into account all the achievement values, we look for a lexicographic maximum among the ordered achievement vectors.

Note that the lexicographic RPM model (3) can be expressed as the following problem:

$$\operatorname{lex}\max \{ (\theta_1(\mathbf{a}), \sum_{i=2}^{m} \theta_i(\mathbf{a})) \ : \ a_i = s_i(f_i(\mathbf{x})) \ \forall\, i,\ \mathbf{x} \in Q \}$$

thus, in the case of two criteria $(m = 2)$, representing exactly the lexicographic max-min aggregation. For larger number of criteria $(m > 2)$ model (3) only approximates the lexicographic max-min as all the lower priority objective terms are aggregated at the second priority level. One may consider the lexicographic max-min approach applied to the partial achievement functions (7) as a basis for a corresponding nucleolar RPM model:

$$\operatorname{lex}\max \{ (\theta_1(\mathbf{a}), \theta_2(\mathbf{a}), \ldots, \theta_m(\mathbf{a})) \ : \ a_i = s_i(f_i(\mathbf{x})) \ \forall\, i,\ \mathbf{x} \in Q \} \tag{7}$$

We will use the name nucleolar RPM to avoid any possible misunderstandings when referring to the lexicographic RPM. The nucleolar RPM implements a consequent max-min aggregation, thus resulting in much better modeling of the reference levels concept.

Table 2

Ordered achievements values

Solution	θ_1 (a)	θ_2 (a)	θ_3 (a)	θ_4 (a)	θ_5 (a)	θ_6 (a)
S1	0.3	0.3	1.2	1.2	1.2	1.2
S2	0.3	0.3	1.2	1.2	1.2	1.2
S3	0.3	0.3	1.2	1.2	1.2	1.2
S4	0.3	0.3	1.2	1.2	1.2	1.2
S5	0.3	0.3	1.2	1.2	1.2	1.2
S6	0.3	1.0	1.0	1.0	1.0	1.0
S7	0.3	0.3	0.3	0.6	1.0	1.0

One may easily notice that the nucleolar RPM is not affected by any adding or eliminating passive criterion. While applying the nucleolar RPM the ordered achievement are lexicographically minimized and therefore in our example solution S6 is selected for six criteria as it was selected for five criteria (Table 2). Actually, the lexicographic max-min is the only regularization of the max-min approach satisfying the reduction (addition/deleting) principle [2]. Namely, if the individual achievement of an outcome does not distinguish two solutions, then it does not affect the preference relation:

$$(a'_1, \ldots, a'_i, a^*, a'_{i+1}, \ldots, a'_q) \succeq (a''_1, \ldots, a''_i, a^*, a''_{i+1}, \ldots, a''_q) \quad \Leftrightarrow \quad a' \succeq a'' \qquad (8)$$

Due to strictly monotonic individual achievement functions, the reduction principle is also satisfied in the original outcome space. Moreover, since the aggregation is impartial with respect to partial achievements, it depends only on distribution of achievements independently from their order. Hence, the nucleolar RPM works also properly if the max-min optimization becomes passive despite one cannot identify any passive original criterion. This can be illustrated with data from Table 3 which differ from those of Table 1 only due to permuted achievements of solution S7. This alternative is no longer dominated and the sixth criterion is no longer passive. Nevertheless, as the distributions of achievement values remain the same, the max-min optimization remains passive and the standard forms of the RPM select solution S1 to S5 according to regularization term. Similarly, the ordered values of achievements remain the same as in Table 2, and the nucleolar RPM still selects solution S6 as the best matching the aspiration levels.

Table 3

Sample achievements with passive max-min criterion

Solution	a_1	a_2	a_3	a_4	a_5	a_6
S1	0.3	1.2	1.2	1.2	1.2	0.3
S2	1.2	0.3	1.2	1.2	1.2	0.3
S3	1.2	1.2	0.3	1.2	1.2	0.3
S4	1.2	1.2	1.2	0.3	1.2	0.3
S5	1.2	1.2	1.2	1.2	0.3	0.3
S6	1.0	1.0	1.0	1.0	1.0	0.3
S7	0.3	0.3	0.3	1.0	0.6	1.0

The following assertions shows that the nucleolar RPM model (7) satisfies the basic requirements for the RPM approaches. Namely, model (7) guarantees the efficiency of solutions (Theorem 1) and it is possible to generate all efficient solutions using nucleolar RPM by appropriately choosing the reference vector (Theorem 2).

Theorem 1

For any strictly increasing partial achievement functions $s_i(y_i)$*, if* $\bar{\mathbf{x}}$ *is an optimal solution of the problem (7), then* $\bar{\mathbf{x}}$ *is also an efficient solution of the corresponding multicriteria problem (1).*

Proof

Suppose $\bar{\mathbf{x}}$ optimal to (7) is dominated by some $\mathbf{x}' \in Q$. Thus, due to strictly increasing partial achievement functions one gets $\bar{a}_i = s_i(f_i(\bar{\mathbf{x}})) \leq a'_i = s_i(f_i(\mathbf{x}')) \; \forall \, i$, with at least one inequality strict. Hence, $\Theta(\bar{\mathbf{a}}) <_{lex} \Theta(\mathbf{a}')$ which contradicts optimality $\bar{\mathbf{x}}$ to (7).

■

Theorem 2

For any $\bar{\mathbf{x}}$ *efficient solution, if the reference level are defined as,* $r_i = f_i(\bar{\mathbf{x}})$ *and strictly increasing partial achievement functions* s_i *taking the same value at the reference levels* $(s_i(r_i) = \varrho \; \forall \, i)$ *are used, then* $\bar{\mathbf{x}}$ *is an optimal solution of the corresponding nucleolar RPM problem (7).*

Proof

Note that $\bar{a}_i = s_i(f_i(\bar{\mathbf{x}})) = \varrho \; \forall \, i$. If there exist $a'_i = s_i(f_i(\mathbf{x}'))$ for $\mathbf{x}' \in Q$ such that $\Theta(\bar{\mathbf{a}}) <_{lex} \Theta(\mathbf{a}')$, then $\bar{a}_i =\leq a'_i \; \forall \, i$ with at least one inequality strict. This contradicts the efficiency of $\mathbf{x}$.

■

Note that all typical partial achievement functions, in particular piecewise linear functions of the form (4), (5), or (6) are strictly increasing and they assign the same value at the reference levels. Thus, Theorem 2 justifies the controllability of the nucleolar RPM in the sense that for any $\mathbf{x} \in Q$ efficient solution to multiple criteria problem (1) there exists the reference vector $\mathbf{r}^a$ such that $\mathbf{x}$ is an optimal solution of the corresponding nucleolar RPM problem (7) defined with this reference vector.

3. IMPLEMENTATION ISSUES

An important advantage of the RPM depends on its easy implementation as an expansion of the original multiple criteria model. Actually, even more complicated partial achievement functions of the form (6) are strictly increasing and concave (under the assumption that $0 < \alpha < 1 < \gamma$), thus allowing for implementation of the entire RPM model (2) by an LP expansion [15]. The ordered achievements optimized in the nucleolar RPM (7) are, in general, hard to implement due to the pointwise ordering. Let us consider cumulated achievements $\bar{\theta}_k(\mathbf{a}) = \sum_{i=1}^{k} \theta_i(\mathbf{a})$ expressing, respectively: the worst (smallest) achievement, the total of the two worst achievements, the total of the three worst achievements, etc. Within the lexicographic optimization a cumulation of criteria does not affect the optimal solution. Hence, the nucleolar RPM model (7) can be expressed in terms of the lexicographic maximization of quantities $\bar{\theta}_i(\mathbf{a})$:

$$\text{lex} \ \max \{ \ (\bar{\theta}_1(\mathbf{a}), \bar{\theta}_2(\mathbf{a}), \ldots, \bar{\theta}_m(\mathbf{a})) \ : \ a_i = s_i(f_i(\mathbf{x})) \ \forall \ i, \ \mathbf{x} \in Q \ \} \tag{9}$$

This simplifies dramatically the optimization problem since quantities $\bar{\theta}_k(\mathbf{a})$ can be optimized without use of any integer variables. First, let us notice that for any given vector $\mathbf{a}$, the cumulated ordered value $\bar{\theta}_k(\mathbf{a})$ can be found as the optimal value of the following LP problem:

$$\bar{\theta}_k(\mathbf{a}) = \min_{u_{ik}} \{ \sum_{i=1}^{m} a_i u_{ik} : \sum_{i=1}^{m} u_{ik} = k, \quad 0 \leq u_{ik} \leq 1 \quad \forall i \ \} \tag{10}$$

The above problem is an LP for a given outcome vector a while it becomes nonlinear for a being a vector of variables. This difficulty can be overcome by taking advantage of the LP dual to (10). Introducing dual variable t_k corresponding to the equation $\sum_{i=1}^{m} u_{ik} = k$ and variables d_{ik} corresponding to upper bounds on u_{ik} one gets the following LP dual of problem (10):

$$\bar{\theta}_k(\mathbf{a}) = \max_{t_k, d_{ik}} \{ k t_k - \sum_{i=1}^{m} d_{ik} : a_i \geq t_k - d_{ik}, \ d_{ik} \geq 0 \quad \forall \ i \ \} \tag{11}$$

Due the duality theory, for any given vector a the cumulated ordered coefficient $\bar{\theta}_k(\mathbf{a})$ can be found as the optimal value of the above LP problem. It follows from (11) that $\bar{\theta}_k(\mathbf{a}) = \max \ \{ k t_k - \sum_{i=1}^{m} \ (t_k - a_i)_+ \}$ where $(.)_+$ denotes the nonnegative part of a number and t_k is an auxiliary (unbounded) variable. The latter, with the necessary adaptation to the minimized outcomes in location problems, is equivalent to the computational formulation of the k-centrum model introduced by [17]. Hence, formula (11) provides an alternative proof of that formulation.

Taking advantages of both (9) and (11), the nucleolar RPM can be formulated as a standard lexicographic optimization. Moreover, in the case of concave piecewise linear partial achievement functions (as typically used in the RPM approaches), the resulting formulation extends the original constraints and criteria with linear inequalities. In particular, for strictly increasing and concave partial achievement functions (6), it can be expressed in the form:

$$
\begin{array}{lll}
\text{lex max} & (z_1, z_2, \dots, z_m) & \\
\text{s.t.} & \mathbf{x} \in Q,\ y_i = f_i(\mathbf{x}) & \forall\, i \\
& z_k = k t_k - \sum_{i=1}^{m} d_{ik} & \forall\, k \\
& a_i \geq t_k - d_{ik},\ d_{ik} \geq 0 & \forall\, i, k \\
& a_i \leq \gamma (y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\
& a_i \leq (y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\
& a_i \leq \alpha (y_i - r_i^a)/(r_i^a - r_i^r) + 1 & \forall\, i
\end{array}
\tag{12}
$$

Thus, the nucleolar RPM can be effectively applied to various multiple criteria optimization problems including the discrete ones.

Model (12) provides us with an easily implementable sequential algorithm to generate efficient solutions according to the nucleolar RPM preference specification. However, it does not introduce any explicit scalarizing achievement function which could be directly interpreted as expressing utility to be maximized. In order to get such an analytical form (or rather approximation) of the nucleolar RPM one needs to replace the lexicographic (preemptive) optimization of the ordered achievements in (7) with its weighting approximation. Note that the weights are then assigned to the specific positions within the ordered achievements rather than to the partial achievements themselves, thus representing the so-called Ordered Weighted Averaging (OWA) aggregation. With the OWA aggregation one gets the following RPM model:

$$
\max \{ \sum_{i=1}^{m} v_i \theta_i(\mathbf{a}) \ :\ a_i = s_i(f_i(\mathbf{x}))\ \forall\, i,\ \mathbf{x} \in Q \} \tag{13}
$$

where $v_1 > v_2 > \ldots > v_m > 0$ are positive and strictly decreasing weights. When differences among weights tend to infinity, the OWA aggregation approximates the leximin ranking of the ordered outcome vectors [22]. That means, as the limiting case of (13), we get the nucleolar RPM model (7). Actually, the standard RPM model with the analytic scalarizing achievement function (2) can be expressed as the following OWA model:

$$
\max \{ (1 + \frac{\varepsilon}{m}) \theta_1(\mathbf{a}) + \frac{\varepsilon}{m} \sum_{i=2}^{m} \theta_i(\mathbf{a}) \ :\ a_i = s_i(f_i(\mathbf{x}))\ \forall\, i,\ \mathbf{x} \in Q \}
$$

Hence, the standard RPM model exactly represents the analytic (utility) form of the OWA aggregation (13) with strictly decreasing weights in the case of $m = 2$ $(v_1 = 1 + \varepsilon/2 > v_2 = \varepsilon/2)$. For $m > 2$ it abandons the differences in weighting of the second worst achievement, the third worst one, etc $(v_2 = \ldots = v_m = \varepsilon/m)$.

The OWA aggregation is obviously a piecewise linear function since it remains linear within every area of the fixed order of arguments. Its optimization can be implemented by expressing in terms of the cumulated ordered achievements:

$$\max \{ \sum_{i=1}^{m} w_i \bar{\theta}_i(\mathbf{a}) \ : \ a_i = s_i(f_i(\mathbf{x})) \ \forall\, i;\ \mathbf{x} \in Q \}$$

where $w_i = v_i - v_{i+1}$ for $i = 1, \ldots, m-1$, and taking advantages of the LP expression (11) of $\bar{\theta}_i$ [16]. This leads to a single level computational model similar to (12).

$$\max \sum_{k=1}^{m} w_k z_k \quad \text{s.t.} \quad \begin{array}{ll} z_k = k t_k - \sum_{i=1}^{m} d_{ik} & \forall\, k \\ \mathbf{x} \in Q,\ y_i = f_i(\mathbf{x}) & \forall\, i \\ a_i \geq t_k - d_{ik},\ d_{ik} \geq 0 & \forall\, i,k \\ a_i \leq \gamma(y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\ a_i \leq (y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\ a_i \leq \alpha(y_i - r_i^a)/(r_i^a - r_i^r) + 1 & \forall\, i \end{array} \tag{14}$$

For some special sequences of the OWA weights v_i this solution concept can easily be defined without any need to order outcomes, thus the solution procedure may be quite simple. From the properties of the Gini's mean absolute difference [12] it follows that:

$$\sum_{i=1}^{m}\sum_{k=1}^{m} \min\{a_i, a_k\} = \sum_{i=1}^{m} a_i - \frac{1}{2}\sum_{i=1}^{m}\sum_{k=1}^{m} |a_i - a_k| = \sum_{i=1}^{m} [2(m-i)+1]\theta_i(\mathbf{a})$$

Hence, the OWA aggregation given by the decreasing sequence of weights v_i with a constant step $v_i - v_{i+1} = \Delta$ can be directly expressed as:

$$\sum_{i=1}^{m} v_i \theta_i(\mathbf{a}) = \bar{v}_1 \min_{1 \leq i \leq m} a_i + \frac{\Delta}{2} \sum_{i=1}^{m}\sum_{k=1}^{m} \min\{a_i, a_k\} \tag{15}$$

where $\bar{v}_1 = v_1 - \Delta(2m-1)/2$. Note that formula (15) defines a piecewise linear concave function which guarantees its LP computability when maximized.

The following extension of the analytic RPM model (2)

$$\max \{ \min_{1 \le i \le m} a_i + \frac{\varepsilon}{m^2} \sum_{i=1}^{m} \sum_{k=1}^{m} \min\{a_i, a_k\} \ : \ a_i = s_i(f_i(\mathbf{x})) \ \forall\, i,\ \mathbf{x} \in Q \} \quad (16)$$

due to (15), represents the OWA aggregation given by the decreasing sequence of weights with $v_1 = 1 + (2m-1)\varepsilon$ and the constant step $v_i - v_{i+1} = 2\varepsilon/m^2$. Certainly, such an analytic model is only rough approximation to the nucleolar RPM. Nevertheless, when applying (16) to our sample problem from Table 1, the solution S6 is selected. For strictly increasing and concave partial achievement functions (6) the model can be expressed as:

$$\begin{array}{lll} \max & \underline{a} + \dfrac{\varepsilon}{m^2} \displaystyle\sum_{i=1}^{m} \sum_{k=1}^{m} t_{ik} & \\ \text{s.t.} & \mathbf{x} \in Q,\ y_i = f_i(\mathbf{x}) & \forall\, i \\ & a_i \ge \underline{a} & \forall\, i \\ & a_i \ge t_{ik},\ a_k \ge t_{ik} & \forall\, i,k \\ & a_i \le \gamma(y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\ & a_i \le (y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\ & a_i \le \alpha(y_i - r_i^a)/(r_i^a - r_i^r) + 1 & \forall\, i \end{array} \quad (17)$$

4. ILLUSTRATIVE EXAMPLE

In order to illustrate the nucleolar RPM performances let us analyze the multicriteria problem of information system selection. We consider a billing system selection for a telecommunication company [19]. The decision is based on 7 criteria related to the system functionality, reliability, processing efficiency, investment costs, installation time, operational costs, and warranty period. All these attributes may be viewed as criteria, either maximized or minimized. Table 4 presents all the criteria with their measures units and optimization directions. There are also set the aspiration and reservation levels for each criterion.

Table 4

Criteria and their reference levels for the sample billing system selection

	f_1 Functionality	f_2 Reliability	f_3 Efficiency	f_4 Investment cost	f_5 Installation time	f_6 Operational cost	f_7 Warranty period
Units	# modules	1-10	CAPS	mln PLN	months	mln PLN	years
Optimization	max	max	max	min	min	min	max
Reservation	4	8	50	2	12	1.25	0.5
Aspiration	10	10	200	0	6	0.5	2

Five candidate billing systems have been accepted for the final selection procedure. All they meet the minimal requirements defined by the reservation levels. Table 5 presents for all the systems (columns) their criteria values y_i and the corresponding partial achievement values a_i. The latter are computed according to the piece-wise linear formula (6) with $\alpha = 0.1$.

Table 5

Criteria values y_i and individual achievements a_i for five billing systems

i	System A		System B		System C		System D		System E	
	y_i	a_i	y_i	a_i	y_i	a_i	y_i	a_i	y_i	a_i
1	9	0.83	10	1.00	8	0.67	6	0.33	8	0.67
2	10	1.00	9	0.50	10	1.00	9	0.50	10	1.00
3	200	1.00	100	0.33	170	0.80	90	0.27	150	0.67
4	1	0.50	0.3	0.85	0.8	0.60	0.2	0.90	0.5	0.75
5	10	0.33	3	1.05	6	1.00	8	0.67	5	1.02
6	1	0.33	1	0.33	0.6	0.87	0.2	1.04	1	0.33
7	2	1.00	2	1.00	1	0.33	2	1.00	1.5	0.67

Table 6 presents for all the systems (columns) their partial achievement values ordered from the worst to the best $\theta_i(\mathbf{a})$.. Examining row $\theta_1(\mathbf{a})$ one may notice that except of system D all other systems have the same worst achievement value $\min_i a_i = 0.33$.. Selection among systems A, B, C, and E depends on the achievements aggregation used in the RPM approach. Comparing the second worst achievements (row $\theta_2(\mathbf{a})$) one can see that according to the nucleolar RPM (7) system E is the best selection guaranteeing at least 0:67 achievement levels for six criteria. These selection cannot be done if using the classical RPM with regularization based on the total achievements. Actually, according to row $\sum_i a_i$ either lexicographic RPM (3), or its analytic version (2) will select system C as better than all the others. However, according to row $\sum_i \sum_k \min\{a_i, a_k\}$ even an analytic model a rough approximation to the nucleolar RPM an analytic model (16) turns out to be strong enough to identify system E as the best selection.

Table 6

Ordered achievements for five billing systems

	A	B	C	D	E
θ_1 (**a**)	0.33	0.33	0.33	0.27	0.33
θ_2 (**a**)	0.33	0.33	0.60	0.33	0.67
θ_3 (**a**)	0.50	0.50	0.67	0.50	0.67
θ_4 (**a**)	0.83	0.85	0.80	0.67	0.67
θ_5 (**a**)	1.00	1.00	0.87	0.90	0.75
θ_6 (**a**)	1.00	1.00	1.00	1.00	1.00
θ_7 (**a**)	1.00	1.05	1.00	1.04	1.02
$\sum_i a_i$	4.99	5.06	5.27	4.71	5.11
$\sum_i \sum_k \min\{a_i, a_k\} a_i$	27.33	27.75	29.91	25.10	30.27

CONCLUSIONS

The reference point method is a very convenient technic for interactive analysis of the multiple criteria optimization problems. It provides the DM with a tool for an open analysis of the efficient frontier. The interactive analysis is navigated with the commonly accepted control parameters expressing reference levels for the individual objective functions. The partial achievement functions quantify the DM satisfaction from the individual outcomes with respect to the given reference levels. The final scalarizing function is built as the augmented max-min aggregation of partial achievements, which means that the worst individual achievement is essentially maximized, but the optimization process is additionally regularized with the term representing the average achievement. The regularization by the average achievement is easily implementable, but it may disturb the basic max-min aggregation. In order to avoid inconsistencies caused by the regularization, the max-min solution may be regularized according to the Rawlsian principle of justice leading to the nucleolar RPM model.

The nucleolar RPM implements a consequent max-min aggregation taking into account also the second worst achievement, the third worse, and so on, thus resulting in much better modeling of the reference levels concept. The nucleolar regularization is more complicated in implementation due to the requirement of pointwise ordering of partial achievements. Nevertheless, by taking advantages of piecewise linear expression of the cumulated ordered achievements, the nucleolar RPM can be formulated as a standard lexicographic optimization. Actually, in the case of concave piecewise linear partial achievement functions (typically used in the RPM), the resulting formulation extends the original constraints and criteria with simple linear inequalities, thus allowing for a quite efficient implementation. The nucleolar RPM can be also approximated with the analytic form using the ordered weighted averaging, thus introducing explicit scalarizing achievement function to be interpreted as utility.

The paper is focused on nucleolar refinement of the reference point method. Nevertheless, the same methodology can be easily applied to various multiple criteria approaches requiring some fair (equitable) aggregations. In particular, to the fuzzy goal programming models.

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