Location Problems from the Multiple Criteria Perspective: Efficient Solutions

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Abstract

Location problems can be considered as multiple criteria models where for each client (spatial unit) there is defined an individual objective function, which measures quality of a location pattern with respect to the client satisfaction (e.g. it expresses the distance or travel time between the client and the assigned facility). The individual objective functions are usually conflicting when optimized. Therefore, the decision maker or planner needs to select some compromise solution for implementation. In this paper we analyze various approaches to discrete multiple facility location problems (various solution concepts) from the perspective of the multiple criteria models. We focus our analysis on two aspects of the solution concepts: if a generated solution is an efficient (Pareto-optimal) solution to the multiple criteria problem, and if the solution concept provides some control parameters allowing the decision maker to select every efficient solution of the multiple criteria problem. That means, we analyze if a solution concept complies with the optimality principle for the multiple criteria model as well as if it allows to take into account various preferences of the decision maker.

Key words: Discrete Location Problem, Multiple Criteria, Efficiency.

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1 Introduction

Public goods and services are typically provided and managed by governments in response to perceived and expressed need. The spatial distribution of public goods and services is strictly related to facility location decisions. A host of operational models has been developed to deal with the facility location optimization (c.f., [13,4,10]). Most classical location studies focus on the minimization of the total distance or the minimization of the maximum distance to the service facilities [17]. Even multiple criteria approaches to location problems employ these two types of the objectives [2,8]. In this paper we analyze the location problem from the perspective of multiple criteria minimization of all the distances considered as independent criteria for several users (clients) of the service system.

The generic location problem, we consider, may be stated as follows. There is given a set of n clients (spatial units). In the case of spatial units, each unit can be represented by a specific point (node) situated in this unit. There is also given a set of m potential locations for the facilities. It may be, in particular, a subset (or the entire set) of points representing the clients. Further, the number (or the maximal number) p of facilities to be located is given ($p \le m$). Thus, we limit our discussion to discrete location problems [16]. They can be viewed, however, as network location problems with possible locations restricted to some subset of the network vertices [11].

Further, let us assume that for each client $j = 1, 2, \ldots, n$ there is defined a function $f_j(\mathbf{x})$ of the location pattern $\mathbf{x}$. The function measures quality of the location pattern with respect to the satisfaction of client j. In typical formulations of location problems this function is usually related to the distances and thereby its smaller value means higher service quality and client satisfaction. Therefore, we assume, each function f_j needs to be minimized. Thus, the generic location problem can be viewed as the following multiple criteria minimization problem

$$\min_{\mathbf{x}} \Big\{ \{f_j(\mathbf{x})\}_{j=1,\ldots,n} : \quad \mathbf{x} \in Q \Big\} \tag{1}$$

where Q denotes the feasible set of location patterns.

The main decisions to be made in the location problem can be described with the binary variables:

x_i — equal to 1 if location i is to be used and equal to 0 otherwise ($i = 1, 2, \ldots, m$).

To meet the problem requirements, the decision variables x_i have to satisfy the constraint $\sum_{i=1}^m x_i = p$, where the equation is replaced with the inequality ($\le$) if p specifies the maximal number of facilities to be located. Thus the simplest feasible set for problem (1) takes the form

$$Q = \Big\{ \mathbf{x} = (x_i)_{i=1,\ldots,m} : \quad \sum_{i=1}^{m} x_i = p, \quad x_i \in \{0,1\} \quad \text{for} \quad i = 1, 2, \ldots, m \Big\} \tag{2}$$

Note that the constraints of (2) take a very simple form of the binary knapsack problem with all the constraint coefficients equal to 1. For certain classes of location problems, the feasible set has more complex structure due to explicit consideration of allocation decisions. Such decisions are usually modeled with additional allocation decision variables

x'_{ij} — equal to 1 if location i is used to service client j and equal to 0 otherwise ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$).

The feasible set takes then the following form

$$Q = \Big\{ \mathbf{x} = ((x_i)_{i=1,\ldots,m}, (x'_{ij})_{i=1,\ldots,m;j=1,\ldots,n}) : \quad \sum_{i=1}^{m} x_i = p,$$

$$x_i \in \{0,1\} \quad \text{for} \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x'_{ij} = 1 \quad \text{for} \quad j = 1, 2, \ldots, n$$

$$x'_{ij} \leq x_i, \quad x'_{ij} \in \{0,1\} \quad \text{for} \quad i = 1, 2, \ldots, m \quad \text{and} \quad j = 1, 2, \ldots, n \Big\}$$

We do not assume any special form of the feasible set. We rather consider it as a general discrete (finite) set. Therefore, the results of our analysis apply to various discrete location problems. Whenever we need a specific form of the feasible set (to present some counterexamples or to illustrate some properties) we will use the simplest form of the feasible set, defined by (2).

The objective functions are, in general, nonlinear and they can be very complex. Due to the location problem specificity, we assume that all the objective functions f_j take only nonnegative values. We do not assume any special form of the objective functions nor their special properties (like convexity). Although, we introduce specific functions related to some types of location models to address them (as special cases) in the discussion. Functions f_j depend usually on distance coefficients d_{ij} ($i = 1, \ldots, m; j = 1, \ldots, n$) which express the distance (or travel time) between location i and client j. For the standard uncapacitated facility location problem, it is assumed that all the potential facilities provide the same type of service and each client is serviced by the nearest located facility. The individual objective functions for problem (1)–(2) take then the following form

$$f_j(\mathbf{x}) = \min_{i=1,\ldots,m} \Big\{ d_{ij} : x_i = 1 \Big\} \quad \text{for} \quad j = 1, 2, \ldots, n \tag{3}$$

They can be explicitly written in the form of piecewise linear functions as

$$f_j(\mathbf{x}) = \min_{i=1,\ldots,m} (d_{ij} - d x_i + d) \quad \text{for} \quad j = 1, 2, \ldots, n$$

where d is an arbitrarily large number (greater than all the distance coefficients d_{ij}). With the explicit use of the allocation variables and the corresponding constraints the individual objective functions f_j can be written in the linear form

$$f_j(\mathbf{x}) = \sum_{i=1}^{n} d_{ij} x'_{ij} \quad \text{for} \quad j = 1, 2, \ldots, n$$

However, for our analysis we prefer the formulation (1)–(3), as it makes the individual objective functions explicitly dependent on location patterns $\mathbf{x}$. Our analysis, certainly, covers also the capacitated facility location problem. In that case the allocation decision variables with the corresponding constraints must be used.

In location problems related to desirable facilities a smaller value of the individual objective function means better effect (higher service quality or client satisfaction). This remains valid for location of obnoxious facilities if the distance coefficients are replaced with their complements to some large number: $d'_{ij} = d - d_{ij}$, where $d > d_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Therefore, without loss of generality we can assume that each function f_j needs to be minimized as in model (1).

The individual objective functions f_j are usually conflicting when minimized. Therefore, model (1) is a really multiple criteria decision problem and the decision maker (DM) or planner needs to select some compromise solution for implementation. An integration of multiple criteria decision approaches with geographical information system (GIS) capabilities has recently been recognized as one of the most important areas for future developments in decision support for spatial planning [1,21]. GIS usually focuses on the capture, storage, manipulation, analysis and display of geographically referenced data and only implicitly assumes a support of spatial decision making through analytical modeling operations. The display capabilities of GIS typically provide the user with a number of techniques that can be used to visualize the problem and the solution in geographical space. Note that in our multiple criteria location problem (1) the geographical space, essentially, covers both: the decision space and the criterion space. Therefore, multiple criteria approach to location problems based on model (1) seems to be well suited for the development of interactive solution procedures to be used within the GIS environment. The analysis presented in this paper may provide a theoretical basis for such developments.

Let $\mathbf{F} = (f_1, \ldots, f_n)$ represent a vector of n individual objective functions. Vector–function $\mathbf{F}$ maps the feasible set Q (as a subset of the decision space) into the criterion space of outcomes. The elements of the criterion space we refer to as achievement vectors. An achievement vector $\mathbf{y}$ is attainable if it expresses outcomes of a feasible solution $\mathbf{x} \in Q$ ($\mathbf{y} = \mathbf{F}(\mathbf{x})$). The set of all attainable achievement vectors is denoted by Y, i.e. $Y = \{\mathbf{y} = \mathbf{F}(\mathbf{x}) : \quad \mathbf{x} \in Q\}$. It is clear that an achievement vector is better than another if all of its individual outcomes are better or at least one individual outcome is better whereas no other one is worse. It is mathematically formalized with the domination relation defined on the set Y.

Definition 1
We say that achievement vector $\mathbf{y}' \in Y$ dominates $\mathbf{y}'' \in Y$, or $\mathbf{y}''$ is dominated by $\mathbf{y}'$, if $\mathbf{y}' \neq \mathbf{y}''$ and $y'_j \leq y''_j$ for all $j = 1, 2, \ldots, n$. □

Achievement vector $\mathbf{y}' \in Y$ is considered to be better than $\mathbf{y}'' \in Y$ if $\mathbf{y}''$ is dominated by $\mathbf{y}'$. It means, we treat all the objective functions, and thereby all the clients, in the same way. Thus, we do not make any specific assumption about the DM's preference model except of the general assumption that for each individual objective function less means better (minimization), i.e. in terms of the location problem, for each client closer to the service means better.

Unfortunately, there usually does not exist an achievement vector that dominates all the others with respect to all the criteria. Thus, in terms of strict mathematical relations, we cannot distinguish the best achievement vector. We can only distinguish the achievement vectors which are not dominated by any others.

Definition 2
We say that achievement vector $\mathbf{y} \in Y$ is nondominated if there does not exist a $\mathbf{y}' \in Y$ such that $\mathbf{y}' \neq \mathbf{y}$ and $y'_j \leq y_j$ for all $j = 1, 2, \ldots, n$. □

Definition 3
We say that a feasible solution (location pattern) $\mathbf{x} \in Q$ is an efficient (Pareto–optimal) solution of the multiple criteria problem if $\mathbf{y} = \mathbf{F}(\mathbf{x})$ is a nondominated achievement vector. □

Each feasible solution (location pattern) for which one cannot improve any individual achievement without worsening another one is an efficient solution. There exist usually many efficient solutions and they are different not only in the decision space but also in the criterion space. Hence, there arises a need for further analysis, or rather decision support, to help the DM in selection of one efficient solution for implementation. Certainly, the original objective functions do not allow one to select any efficient solution as better than the others. Therefore, this analysis depends on additional information about the DM's preferences. The DM, working interactively with a decision support system (DSS), specifies the preferences in terms of some control parameters and the DSS provides the DM with an efficient solution which is the best according to the specified control parameters. It is important, however, that the control parameters provide the completeness of the control [28], i.e. that by varying the control parameters, the DM can identify every nondominated achievement vector.

In this paper we analyze various approaches to location problems (solution concepts) from the perspective of the multiple criteria model (1) and their possible use for decision support. We focus our analysis on two aspects of the solution concepts: if a generated solution is an efficient solution to the multiple criteria problem (1), and if the solution concept provides some control parameters allowing the DM to select every efficient solution of (1). The paper is organized as follows. The following section is devoted to the analysis of the classical solution concepts for location problems: the median and the center solution concepts. In both cases we allow to introduce some weights as control parameters. It turns out that the median solution is always an efficient one but there are efficient solutions which cannot be identified by varying weights in the median approach. On the other hand, the weighted center approach allows us to identify each efficient solution to the problem (1) but in the case of nonunique solution it may generate some solutions failing the efficiency requirement. In Section 3 the solution concept of the lexicographic center is introduced and analyzed. It may be considered a refinement (consistent with the center philosophy) of the center solution concept. The solution concept of the lexicographic weighted center seems to be ideal from the perspective of our analysis as it meets our both requirements (efficiency principle and complete parameterization). However, it may require complex computations. Therefore, in Section 4 we introduce another, computationally easier, refinement of the center solution concept. It is derived as a modification of the so–called λ–cent–dian approach [6] which is some form of compromise between the median and center solution concepts. Recall that, despite of not using explicitly the prefix "p-", we consider all the solution concepts as applied to multiple facility location decisions.

2 Median and center approaches

Many location models focus on the minimization of the total distance between clients and the facilities located. The solution to these type of models is called the median solution. Exactly, a feasible decision vector $\bar{\mathbf{x}} \in Q$ is called the median solution of the problem (1) if it is an optimal solution to the single objective problem

$$\min_{\mathbf{x}} \Big\{ \sum_{j=1}^{n} w_j f_j(\mathbf{x}) : \quad \mathbf{x} \in Q \Big\} \tag{4}$$

where w_j $(j = 1, 2, \ldots, n)$ are some positive weights $(w_j > 0)$. In most applications the weights are considered to express the demands for service in the corresponding client points. A median solution may be then interpreted as that solution minimizing the total distance taking into account the clients demands. In our analysis we consider the weights to be parameters modeling the DM's preferences. They can be affected by the service demands as well as by various other factors.

Analyzing the median solution from the perspective of the multiple criteria problem (1) we want to know if the median solution is always an efficient solution. The median solution may be, obviously, interpreted as the weighting approach to the multiple criteria problem, which is known to generate efficient solutions (c.f., [26]). It is made precise in the following proposition.

Proposition 1
For any positive weights $w_j > 0$ $(j = 1, 2, \ldots, n)$, each optimal solution to the median problem (4) is an efficient solution of the multiple criteria location problem (1). □

Unfortunately, the parameters defining the median solution, i.e. the weights w_j $(j = 1, 2, \ldots, n)$, do not provide us with a complete parameterization of the entire efficient set. It is due to the specificity of the weighting approach to multiple criteria. In the case of multiple criteria linear programming it allows to parameterize the entire efficient set [26]. However, in the case of a discrete (and thereby nonconvex) set Y, usually, there exist efficient solutions that cannot be generated as optimal solutions for a single objective problem defined as a convex linear combination of the original objectives. As our multiple criteria location problem is a discrete one, there may exist efficient solutions that cannot be generated as median solutions with any set of positive weights. We illustrate this with a small example.

Example 1
Let us consider a simple single facility location problem with two clients (C1 and C2) and three potential locations (P1, P2 and P3). The distances d_{ij} $(i = 1, 2, 3; j = 1, 2)$ between several potential locations and clients are given as follows: $d_{11} = 1$, $d_{12} = 10$, $d_{21} = 10$, $d_{22} = 1$, $d_{31} = 6$ and $d_{32} = 6$.

The problem can be easily expressed as a planar one with distances according to the Euclidean (l_2) or city-block (l_1) norm. For instance, when the clients have assigned coordinates: C1=(0,1.5) and C2=(9,1.5), and the potential locations have the coordinates: P1=(0,2.5), P2=(9,2.5) and P3=(4.5,0), then our set of distances represents the l_1 distances.

Note that all three feasible solutions are efficient. Despite this they are quite different: location P1 is close to client C1, P2 is close to C2, and P3 is in equal distances from both clients. The problem can be related to the planning situation where two sites to be serviced (clients) are connected via a motorway and some local highway. Along the motorway the facility may be located only on an exit (either close to C1 or close to C2). Along the local highway there exists an opportunity to choose a location on half a way between the clients.

One can easily verify that while dealing with the median approach, location P3 (despite being a very attractive compromise solution) cannot be selected for any set of positive weights assigned to the clients. If C1 has assigned the higher weight than C2 ($w_1 > w_2$), then location P1 is a unique optimal solution to the median problem (4). If C1 has assigned the lower weight than C2 ($w_1 < w_2$), then location P2 is a unique optimal solution to the median problem (4). Finally, if both clients have assigned equal weights ($w_1 = w_2$), then both locations P1 and P2 are optimal. Thus location P3 is never an optimal solution to the median problem (4). □

As values of the objective functions f_j are presumably nonnegative, the median model (4) may be considered a minimization of the weighted l_1 norm of the achievement vector. Similarly, one may consider weighted l_α norms for some $\alpha > 1$ (c.f., [25]). In the case of objective functions (3) it can be easily implemented by replacing the original distance coefficients with its α powers and solving the standard median problem (4) for the modified data. Increasing α usually effects in better parameterization of the efficient set. However, for any fixed α, there may exist efficient solutions not generated as optimal solutions of the corresponding problem (4). Moreover, due to computational reasons (numerical instability) it is usually impossible to deal with α greater than 2 or maybe 3. When increasing α to the infinity we get, as a limiting case, the center solution defined with the Chebyschev norm l_∞.

Since the median approach is based on averaging, it often provides solutions where remote and low-population density areas are discriminated against in terms of accessibility to public facilities, as compared with centrally situated and high-population density areas [3]. For this reason, an alternative approach, involving minimization of the maximum distance (travel time) between any client and the closest facility, may be considered [5]. This class of location problems is referred to as minimax or center problems. The minimax objective primarily addresses the geographical equity issues and it is of particular importance in spatial organization of emergency service systems, such as fire, police, medical ambulance services, civil defense and accident rescue. The minimax location rule is consistent with the Rawls's [23] general difference principle of justice.

The solution concept related to the minimax approach is called center. Exactly, a feasible decision vector $\bar{\mathbf{x}} \in Q$ is called the center solution of the problem (1), if it is an optimal solution to the single objective problem

$$\min_{\mathbf{x}} \Big\{ \max_{j=1,\dots,n} f_j(\mathbf{x}) : \quad \mathbf{x} \in Q \Big\} \tag{5}$$

One can easily find examples of center solutions which are not efficient in terms of the multiple criteria location problem (1). However, the center solution can be dominated only by another center solution. It leads us to the following proposition [26].

Proposition 2
The set of all center solutions, i.e. the optimal set of problem (5), always contains an efficient solution of the multiple criteria location problem (1).
A unique optimal solution of problem (5) is an efficient solution of the multiple criteria location problem (1). □

It often turns out that the distribution of clients (spatial units) in relation to the location of facilities makes the minimax criterion passive in the sense of generating a lot of alternative optimal solutions. Such a situation is caused, for instance, by existence of an isolated client located at a considerable distance from all the locations of facilities. Minimization of the maximum distance is then reduced to minimization of the distance of that single isolated client (e.g., [14]) leaving other location decisions unoptimized. To resolve this problem, the center solution concept needs to be supported by some regularization (refinement) technique to guarantee that only efficient solutions are selected. We discuss the regularization techniques in subsequent sections.

Let us analyze if the center approach can be used to examine any efficient solution. The strict center solution defined with problem (5) has no control parameters at all. However, like in the median approach one may consider positive weights assigned to several clients and look for a weighted center solution. It means, we minimize the weighted Chebyschev norm l_∞ as in the following problem

$$\min_{\mathbf{x}}\Big\{\max_{j=1,\ldots,n} w_j f_j(\mathbf{x}) : \quad \mathbf{x} \in Q\Big\} \tag{6}$$

If the weights are considered to express the demands for service, the weighted center approach expresses minimization of maximal total distance covered by each client.

Similar to the center solution, the weighted center solution can be dominated only by another weighted center solution with the same weights. Thus, the following analogue to Proposition 2 can be easily proven [26].

Proposition 3
For any positive weights w_j the set of all weighted center solutions, i.e. the optimal set of problem (6), contains an efficient solution of the multiple criteria location problem (1).
The unique optimal solution of problem (6) is an efficient solution of the multiple criteria location problem (1). □

It turns out that the solution concept of the weighted center allows us to build a complete parameterization of the entire efficient set to the multiple criteria location problem (1). It is widely known in the multicriteria optimization theory (c.f., [26]) that such an assertion is valid for problems with strictly positive outcomes, which is not a case for typical discrete location problems. However, in the case of discrete location problems, the feasible sets are finite and therefore the assertion is also valid for problems with nonnegative outcomes.

Proposition 4
For any efficient solution $\bar{\mathbf{x}}$ of the multiple criteria location problem (1), there exist positive weights w_j such that $\bar{\mathbf{x}}$ is an optimal solution to the corresponding weighted center problem (6).

Proof
As we deal with discrete location problems, the individual objective functions $f_j(\mathbf{x})$ can take only values from some finite set. As the values of the functions depict some distances we may assume that all the values are nonnegative. Let d_j denote the smallest positive value of the function f_j on the set of feasible locations Q. If such a value does not exist, the corresponding function is constant on Q and it does not affect the efficient set. Let us define weights $\bar{w}_j$ as follows

$$\bar{w}_j = 1/f_j(\bar{\mathbf{x}}) \quad \text{if} \quad f_j(\bar{\mathbf{x}}) > 0 \quad \text{or} \quad \bar{w}_j = 1/d_j \quad \text{if} \quad f_j(\bar{\mathbf{x}}) = 0$$

Note that for the center problem (6) with weights $\bar{w}_j$, vector $\bar{\mathbf{x}}$ has the objective value 1 if at least one individual objective function is positive and 0 otherwise. In the latter case, $\bar{\mathbf{x}}$ is clearly an optimal solution. Let us concentrate on the former case. Suppose that there exists a feasible vector $\mathbf{x}$ with objective value less than 1. Hence

$$f_j(\mathbf{x}) < f_j(\bar{\mathbf{x}}) \quad \text{if} \quad f_j(\bar{\mathbf{x}}) > 0 \quad \text{and} \quad f_j(\mathbf{x}) = 0 \quad \text{if} \quad f_j(\bar{\mathbf{x}}) = 0$$

which contradicts efficiency of the vector $\bar{\mathbf{x}}$. □

3 Lexicographic center

In the traditional center approach to location problem all the functions $f_j(\mathbf{x})$ are equally important and therefore their values are important rather than their assignment to specific spatial units. That means, we compare sets of outcomes $\{f_j(\mathbf{x}) : \quad j \in N\}$ (where $N = \{1, 2, \ldots, n\}$) rather than vectors $(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ with a specified order of coefficients. The center approach minimizes the largest outcomes and it is too crude to guarantee efficiency of the solution in all possible cases. One may consider to minimize also the second largest outcome, the third largest and so on. It leads us to a concept of the lexicographic minimization of vectors $(f_{j_1}(\mathbf{x}), f_{j_2}(\mathbf{x}), \ldots, f_{j_n}(\mathbf{x}))$ with decreasing order of their outcome values, i.e. lexicographic minimax optimization.

This approach can be mathematically formalized as follows. We introduce map $\Theta : R^n \rightarrow R^n$ which orders the coordinates of the achievement vectors in the nonincreasing order, i.e., $\Theta(y_1, y_2, \ldots, y_n) = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)$ iff there exists a permutation τ of N such that $\bar{y}_j = y_{\tau(j)}$ for all $j \in N$ and $\bar{y}_1 \geq \bar{y}_2 \geq \ldots \geq \bar{y}_n$. Further, we introduce the strict lexicographic order $\prec$ by $\mathbf{y} \prec \mathbf{v}$ iff there is an index $k \leq n$ such that $y_j = v_j$ for all $j < k$ and $y_k < v_k$. The weak lexicographic order is defined with the relation $\mathbf{y} \preceq \mathbf{v}$ iff $\mathbf{y} = \mathbf{v}$ or $\mathbf{y} \prec \mathbf{v}$. It is commonly known that the (weak) lexicographic order is complete and therefore one can look for a minimum vector with respect to this relation. We call a location pattern $\mathbf{x}^o \in Q$ the lexicographic center solution if

$$\Theta(f_1(\mathbf{x}^o), f_2(\mathbf{x}^o), \ldots, f_n(\mathbf{x}^o)) \preceq \Theta(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all location patterns $\mathbf{x} \in Q$. That means, the lexicographic center solution is an optimal solution of the following lexicographic problem

$$\operatorname{lex}\min_{\mathbf{x}} \Big\{ \Theta(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})) : \quad \mathbf{x} \in Q \Big\} \tag{7}$$

Example 2
To illustrate the solution concept of the lexicographic center, let us consider a problem of locating two facilities among ten spatial units where each spatial unit can be considered as a potential location. We assume that the facilities have unlimited capacities and each spatial unit is served by the nearest facility. Thus the problem takes the form (1)–(3) with $m = n = 10$ and $p = 2$. To simplify the example we consider several units U1, U2,. . .,U10 as points on one line, say X-axis, with coordinates: 0, 4, 5, 6, 8, 17, 18, 19, 20 and 28, respectively.

Table 1: Achievements vectors for Example 2

			Distances to units									
Solution concept	Locations		1	2	3	4	5	6	7	8	9	10
lexicographic center	U2	U9	4	0	1	2	4	3	2	1	0	8
"worst" center	U1	U9	0	4	5	6	8	3	2	1	0	8
median	U3	U8	5	1	0	1	3	2	1	0	1	9

Table 2: Ordered achievements vectors for Example 2

			Ordered distances									
Solution concept	Locations		1	2	3	4	5	6	7	8	9	10
lexicographic center	U2	U9	8	4	4	3	2	2	1	1	0	0
"worst" center	U1	U9	8	8	6	5	4	3	2	1	0	0
median	U3	U8	9	5	3	2	1	1	1	1	0	0

One can easily verify that the lexicographic center solution is based on locating facilities in spatial units U2 and U9. The distances generated by this location are presented in the first row of Table 1. This solution seems to match very well the geographic equity concept. It is, certainly, also an optimal solution to the classical center problem (5) with a single minimax objective function. However, problem (5) has other optimal solutions which are less desirable from the perspective of minimization of the distances. In the second row of Table 1 there are presented distances for another, in our opinion the worst, optimal solution to the center problem. It is based on locating facilities in spatial units U1 and U9. In this solution two spatial units have the maximal distance 8 to the nearest facility, while in the lexicographic center solution only one does. For easier comparison of the solutions, Table 2 presents for each solution distances ordered in the nonincreasing order. In Tables 1 and 2 we have also included the median solution based on locations in units U3 and U8, which causes that the distance from unit U10 to the nearest facility is equal to 9. □

Proposition 5
The lexicographic center solution is an efficient solution of the multiple criteria location problem (1).
Proof
Let $\bar{\mathbf{x}}$ be the lexicographic center solution. Suppose that $\bar{\mathbf{x}}$ is not an efficient solution for problem (1). Then, there exists a feasible vector $\mathbf{x}$ such that $f_j(\mathbf{x}) \leq f_j(\bar{\mathbf{x}})$ for

$j = 1, 2, \ldots, n$, where for at least one index j_0 strict inequality holds. Thus

$$\Theta(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})) \prec \Theta(f_1(\bar{\mathbf{x}}), f_2(\bar{\mathbf{x}}), \ldots, f_n(\bar{\mathbf{x}}))$$

which contradicts lexicographic optimality of $\bar{\mathbf{x}}$ for problem (7). □

Due to Proposition 5, the lexicographic minimax optimization allows us to select a center solution which is always an efficient solution of the multiple criteria problem (1). The lexicographic center solution concept has no control parameters at all and cannot be directly used to parameterize the entire efficient set. However, similar to the weighted center, we may use for this purpose the lexicographic weighted center solution concept defined as an optimal solution to the problem

$$\operatorname{lex}\min_{\mathbf{x}} \Big\{ \Theta(w_1 f_1(\mathbf{x}), \ldots, w_n f_n(\mathbf{x})) : \quad \mathbf{x} \in Q \Big\} \tag{8}$$

In the two following propositions we show that the solution concept of the lexicographic weighted center complies with the efficiency principle and it parameterizes the entire efficient set of the multiple criteria location problem (1).

Proposition 6
For any positive weights w_j, the lexicographic weighted center solution is an efficient solution of the multiple criteria location problem (1).
Proof
Let $\bar{\mathbf{x}}$ be the lexicographic weighted center solution. Suppose that $\bar{\mathbf{x}}$ is not an efficient solution for problem (1). Then a feasible vector $\mathbf{x}$ must exist such that $f_j(\mathbf{x}) \leq f_j(\bar{\mathbf{x}})$ for $j = 1, 2, \ldots, n$, where for at least one index j_0 strict inequality holds. Thus, due to positive weights w_j

$$w_j f_j(\mathbf{x}) \leq w_j f_j(\bar{\mathbf{x}}) \quad \text{for} \quad j = 1, 2, \ldots, n$$

with at least one strict inequality and therefore

$$\Theta(w_1 f_1(\mathbf{x}), w_2 f_2(\mathbf{x}), \ldots, w_n f_n(\mathbf{x})) \prec \Theta(w_1 f_1(\bar{\mathbf{x}}), w_2 f_2(\bar{\mathbf{x}}), \ldots, w_n f_n(\bar{\mathbf{x}}))$$

which contradicts the lexicographic optimality of $\bar{\mathbf{x}}$ for problem (8). □

Proposition 7
For any efficient solution $\bar{\mathbf{x}}$ of the multiple criteria location problem (1), there exist positive weights w_j such that $\bar{\mathbf{x}}$ is the lexicographic weighted center solution.
Proof
Let, as in Proposition 4, d_j denote the smallest positive value of the function f_j on the set of feasible location patterns Q. If such a value does not exist, the corresponding function is constant on Q and it does not affect the efficient set. Let us define weights $\bar{w}_j$ as follows

$$\bar{w}_j = 1/f_j(\bar{\mathbf{x}}) \quad \text{if} \quad f_j(\bar{\mathbf{x}}) > 0 \quad \text{or} \quad \bar{w}_j = 1 + 1/d_j \quad \text{if} \quad f_j(\bar{\mathbf{x}}) = 0$$

Note that for such defined weights

$$\bar{w}_j f_j(\bar{\mathbf{x}}) = 1 \quad \text{if} \quad f_j(\bar{\mathbf{x}}) > 0 \quad \text{or} \quad \bar{w}_j f_j(\bar{\mathbf{x}}) = 0 \quad \text{if} \quad f_j(\bar{\mathbf{x}}) = 0$$

and for any feasible $\mathbf{x} \in Q$ and $j = 1, 2, \ldots, n$

$$f_j(\mathbf{x}) > f_j(\bar{\mathbf{x}}) \quad \Rightarrow \quad \bar{w}_j f_j(\mathbf{x}) > 1 \geq \max_{k=1,\ldots,n} \bar{w}_k f_k(\bar{\mathbf{x}})$$

Suppose that there exists a feasible vector $\mathbf{x}$ such that

$$\Theta(\bar{w}_1 f_1(\mathbf{x}), \bar{w}_2 f_2(\mathbf{x}), \ldots, \bar{w}_n f_n(\mathbf{x})) \prec \Theta(\bar{w}_1 f_1(\bar{\mathbf{x}}), \bar{w}_2 f_2(\bar{\mathbf{x}}), \ldots, \bar{w}_n f_n(\bar{\mathbf{x}}))$$

Then $\mathbf{F}(\mathbf{x}) \neq \mathbf{F}(\bar{\mathbf{x}})$ and $f_j(\mathbf{x}) \leq f_j(\bar{\mathbf{x}})$ for $j = 1, 2, \ldots, n$, which contradicts the efficiency of vector $\bar{\mathbf{x}}$. Thus, $\bar{\mathbf{x}}$ is the lexicographic weighted center solution with weights $\bar{w}_j$. □

Propositions 6 and 7 show that the lexicographic weighted center is a complete solution concept. It always generates an efficient solution to the multiple criteria location problem (1) and any efficient solution can be found as the lexicographic weighted center solution with appropriate weights. Moreover, it uses the center solution concept on all the optimization levels. Thus the solution concept of weighted lexicographic center seems to be an ideal solution concept from the perspective of our analysis. Figure 1 illustrates the relation between the efficient set and the solution sets for the concepts of weighted center, weighted lexicographic center and (weighted) median.

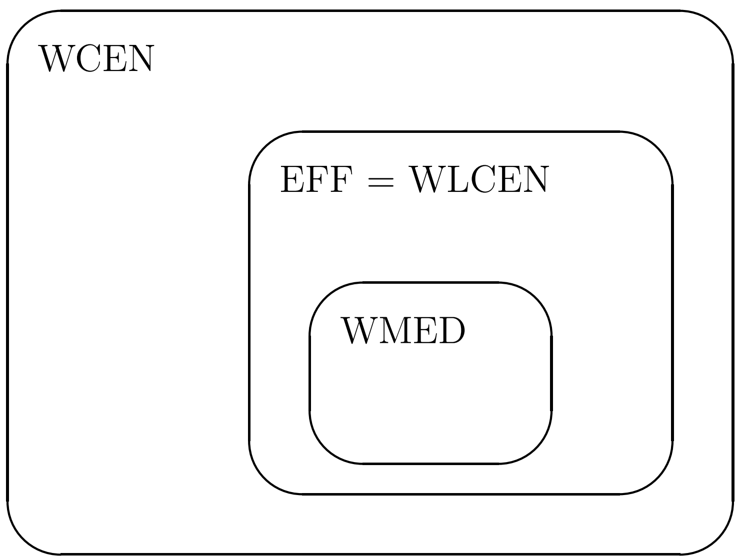

Figure 1: Venn diagram illustrating the relation between the efficient set (EFF) and the solution sets for concepts of the weighted center (WCEN), the weighted lexicographic center (WLCEN) and the weighted median (WMED).

The lexicographic minimax solution is known in the game theory as the nucleolus of a matrix game [24,22]. In matrix games the feasible sets Q are convex polyhedral sets and the functions f_j are linear. In the case of linear objective functions and convex feasible sets, there exists a dominating objective function constant on the entire optimal set of the minimax problem. Therefore, such problems can be easily solved, like the standard lexicographic problems, by sequential optimization with elimination of dominating functions [22]. For the discrete location problems, we consider, there does not exist a dominating objective function which makes the simple sequential optimization procedure inapplicable. As shown by Ogryczak [18] for such problems one can apply the inverse approach taking advantages of the finite number of possible outcomes (values of functions f_j). In this approach the standard lexicographic problem, with apriori defined objective functions and their hierarchy, is solved instead of the lexicographic minimax problem (7). However, the number of the objective functions depends on the number of different possible outcomes (different d_{ij} coefficients). Therefore, the solution concept of the lexicographic weighted center may require complex computations.

4 λ–cent–dian and regularized weighted center

According to Proposition 4, any efficient solution of the multiple criteria location problem (1) is a weighted center solution with some positive weights w_j. Moreover, due to Proposition 3, every unique weighted center solution is always an efficient solution of the multiple criteria location problem (1). In the case of nonunique solutions, the optimal set for the corresponding weighted center problem (6) always contains an efficient solution of problem (1). Thus, to get a complete solution concept, we need only an additional regularization technique that will allow us to select always an efficient weighted center solution. The solution concept of the lexicographic weighted center, discussed in the previous section, is such a regularization of the weighted center problem (6). The solution concept of the lexicographic weighted center seems to be ideal from the perspective of our analysis. However, it may require very complex computations. Therefore, one may consider regularizations based on the median solution concept as easier alternatives.

Halpern [6,7] has introduced a parameterized solution concept based on the bicriterion center/median model

$$\min_{\mathbf{x}}\Big\{[\max_{j=1,\dots,n} f_j(\mathbf{x}), \sum_{j=1}^{n} w_j f_j(\mathbf{x})] : \quad \mathbf{x} \in Q\Big\} \tag{9}$$

Halpern has modeled the corresponding trade-offs with a convex combination of two objectives. He has introduced the λ–cent–dian solution as an optimal solution to the parameterized problem

$$\min_{\mathbf{x}}\Big\{H_\lambda(\mathbf{x}) : \quad \mathbf{x} \in Q\Big\} \tag{10}$$

where

$$H_\lambda(\mathbf{x}) = \lambda \sum_{j=1}^{n} w_j f_j(\mathbf{x}) + (1-\lambda) \max_{j=1,\dots,n} f_j(\mathbf{x}) \tag{11}$$

As suggested by Hansen et al. [9], the median weights w_j should be rather normalized ($\sum_{j=1}^{n} w_j = 1$) in (11) because average and maximum distances are more directly comparable in terms of magnitude. Note that for $\lambda = 0$, H_λ expresses the maximal distance and the problem (10) is then the standard center problem. Similarly, for $\lambda = 1$, H_λ expresses the total (or average if normalized weights are used) distance and the problem (10) is then the standard median problem. Thus, in both these limiting cases the analysis from the previous sections may be applied. Note, however, that the weights w_j are used only for the median objective function whereas the center objective function remains unweighted.

We will further consider a modification of the λ–cent–dian solution concept using the same weights for the center objective function.

In the case of a single facility location on a tree, as originally considered by Halpern [6], the λ–cent–dian solution concept provides the complete modeling of the trade–offs. However, from the perspective of our analysis, it tends to have similar properties as the median solution concept. That means, λ–cent–dian solutions (for $\lambda > 0$) are efficient solutions for the multiple criteria problem (1) but they do not provide us with a capability to parameterize the entire set of efficient solutions.

Proposition 8

For any $0 < \lambda < 1$ and for any positive weights w_j $(j = 1, 2, \ldots, n)$, each optimal solution to the λ–cent–dian problem (10) is an efficient solution of the multiple criteria location problem (1) as well as an efficient solution to the bicriterion problem (9).

Proof

Let $\bar{\mathbf{x}}$ be a λ-cent-dian solution, i.e., an optimal solution to problem (10) with some $0 < \lambda < 1$ and some positive weights w_j $(j = 1, 2, \ldots, n)$. Suppose that $\bar{\mathbf{x}}$ is not an efficient solution for problem (1). Then a feasible vector $\mathbf{x}$ must exist such that $f_j(\mathbf{x}) \leq f_j(\bar{\mathbf{x}})$ for $j = 1, 2, \ldots, n$, where for at least one index j_0 strict inequality holds. Thus

$$\max_{j=1,\ldots,n} f_j(\mathbf{x}) \leq \max_{j=1,\ldots,n} f_j(\bar{\mathbf{x}})$$

and due to positive weights w_j

$$\sum_{j=1}^{n} w_j f_j(\mathbf{x}) < \sum_{j=1}^{n} w_j f_j(\bar{\mathbf{x}})$$

Hence, due to $0 < \lambda < 1$, we get $H_\lambda(\mathbf{x}) < H_\lambda(\bar{\mathbf{x}})$ which contradicts the optimality of $\bar{\mathbf{x}}$ for problem (10).

Suppose now that $\bar{\mathbf{x}}$ is not an efficient solution for the bicriterion problem (9). Then a feasible vector $\mathbf{x}$ must exist such that

$$\max_{j=1,\ldots,n} f_j(\mathbf{x}) \leq \max_{j=1,\ldots,n} f_j(\bar{\mathbf{x}}) \qquad \text{and} \qquad \sum_{j=1}^{n} w_j f_j(\mathbf{x}) \leq \sum_{j=1}^{n} w_j f_j(\bar{\mathbf{x}})$$

where at least one of these two is a strict inequality. Hence, due to $0 < \lambda < 1$, we get $H_\lambda(\mathbf{x}) < H_\lambda(\bar{\mathbf{x}})$ which contradicts the optimality of $\bar{\mathbf{x}}$ for problem (10). □

The λ-cent-dian approach, combining the median and center solution concepts, resolves the problem of inefficient solutions possibly generated by the center approach itself. Unfortunately, similarly to the median approach, the parameters defining the λ-cent-dian solution, i.e., $0 \leq \lambda \leq 1$ and positive weights w_j $(j = 1, 2, \ldots, n)$, do not provide us with a complete parameterization of the entire efficient set of the multiple criteria problem (1) nor of the bicriterion problem (9). We illustrate this with Example 3.

Example 3

Let us consider a simple single facility location problem with two clients (C1 and C2) and three potential locations (P1, P2 and P3). The distances d_{ij} $(i = 1, 2, 3; j = 1, 2)$ between several potential locations and clients are given as follows: $d_{11} = 2$, $d_{12} = 14$, $d_{21} = 10$, $d_{22} = 10$, $d_{31} = 5$ and $d_{32} = 13$.

This problem can be easily expressed as a planar one with distances according to the Euclidean norm (l_2) or city-block norm (l_1). Location P1 is close to client C1 and it is actually the optimal median solution in the case of equal weights. Location P2 is in equal distances from both the clients and it is the optimal center solution. Location P3 is closer to client C1 than to C2, but it is more balanced than location P1 and it is not an optimal solution to the median problem (with equal weights). Location P3 is an efficient solution to the multiple criteria problem (1) as well as to the bicriterion problem (9). In terms of both problems it may be an interesting compromise solution. We will show that P3 cannot be a λ-cent-dian solution for any $0 \leq \lambda \leq 1$ and positive normalized weights w_1, $w_2 > 0$ ($w_1 + w_2 = 1$).

To be a λ-cent-dian solution, location P3 needs to satisfy inequalities: $H_\lambda(P3) \leq H_\lambda(P1)$ and $H_\lambda(P3) \leq H_\lambda(P2)$. Note that $H_\lambda(P1) = 2 + 12(1 - \lambda w_1)$, $H_\lambda(P2) = 10$ and $H_\lambda(P3) = 5 + 8(1 - \lambda w_1)$. Hence we get the inequalities $\lambda w_1 \leq 1/4$ and $\lambda w_1 \geq 3/8$, which are impossible to satisfy. □

The original solution concept of the λ–cent–dian is based on the unweighted center concept. Let us consider the weighted λ–cent–dian solution defined as an optimal solution to the problem

$$\min_{\mathbf{x}} \Big\{ H_{\lambda,\mathbf{w}}(\mathbf{x}) : \quad \mathbf{x} \in Q \Big\} \tag{12}$$

where

$$H_{\lambda,\mathbf{w}}(\mathbf{x}) = \lambda \sum_{j=1}^{n} w_j f_j(\mathbf{x}) + (1 - \lambda) \max_{j=1,\ldots,n} w_j f_j(\mathbf{x}) \tag{13}$$

$$w_j > 0 \quad \text{for} \quad j = 1, 2, \ldots, n \quad \text{and} \quad \sum_{j=1}^{n} w_j = 1 \tag{14}$$

Note that for $\lambda = 0$ problem (12) is the weighted center problem whereas for $\lambda = 1$ it becomes the weighted median problem. Note, moreover, that in both terms of the objective function the same weights w_j are used. Therefore, in this solution concept clearly the individual objective functions f_j are scaled with weights w_j, whereas the parameter λ generates some compromise between the center and median solution concept on the scaled problem. It allows us to take advantages of both approaches: the center and the median.

Proposition 9

For any $0 < \lambda \leq 1$ and for any weights w_j satisfying (14), each optimal solution to the weighted λ–cent–dian problem (12) is an efficient solution of the multiple criteria location problem (1).

Proof

Let $\bar{\mathbf{x}}$ be an optimal solution to problem (12) with some $0 < \lambda \leq 1$ and some weights w_j ($j = 1, 2, \ldots, n$) satisfying (14). Suppose that $\bar{\mathbf{x}}$ is not an efficient solution for problem (1). Then a feasible vector $\mathbf{x}$ must exist such that $f_j(\mathbf{x}) \leq f_j(\bar{\mathbf{x}})$ for $j = 1, 2, \ldots, n$, where for at least one index j_0 strict inequality holds. Thus due to positive weights w_j

$$\max_{j=1,\ldots,n} w_j f_j(\mathbf{x}) \leq \max_{j=1,\ldots,n} w_j f_j(\bar{\mathbf{x}}) \quad \text{and} \quad \sum_{j=1}^{n} w_j f_j(\mathbf{x}) < \sum_{j=1}^{n} w_j f_j(\bar{\mathbf{x}})$$

Hence, due to $0 < \lambda \leq 1$, we get $H_{\lambda,\mathbf{w}}(\mathbf{x}) < H_{\lambda,\mathbf{w}}(\bar{\mathbf{x}})$ which contradicts the optimality of $\bar{\mathbf{x}}$ for problem (12). □

Proposition 10
For any efficient solution $\bar{\mathbf{x}}$ of the multiple criteria location problem (1), there exist weights w_j satisfying (14) and $0 < \lambda < 1$ such that $\bar{\mathbf{x}}$ is an optimal solution to the corresponding weighted λ–cent–dian problem (12).
Proof
Let, as in Proposition 4, d_j denote the smallest positive value of the function f_j on the set of feasible locations Q. If such a value does not exist, the corresponding function is constant on Q and it does not affect the efficient set. Let us define weights $\bar{w}_j$ as follows

$$\bar{w}_j = 1/f_j(\bar{\mathbf{x}}) \quad \text{if} \quad f_j(\bar{\mathbf{x}}) > 0 \quad \text{or} \quad \bar{w}_j = 1/d_j \quad \text{if} \quad f_j(\bar{\mathbf{x}}) = 0$$

Note that for such defined weights and $\lambda = 0$, $H_{0,\bar{\mathbf{w}}}(\bar{\mathbf{x}}) = 1$ if at least one individual objective function is positive and $H_{0,\bar{\mathbf{w}}}(\bar{\mathbf{x}}) = 0$ otherwise. In the latter case $\bar{\mathbf{x}}$ is, clearly, an optimal solution of (12). We will show that it is also valid for the former case. Suppose, there exists a feasible vector $\mathbf{x}$ such that $H_{0,\bar{\mathbf{w}}}(\mathbf{x}) < 1$. Hence

$$f_j(\mathbf{x}) < f_j(\bar{\mathbf{x}}) \quad \text{if} \quad f_j(\bar{\mathbf{x}}) > 0 \quad \text{and} \quad f_j(\mathbf{x}) = 0 \quad \text{if} \quad f_j(\bar{\mathbf{x}}) = 0$$

which contradicts the efficiency of $\bar{\mathbf{x}}$. Thus, vector $\bar{\mathbf{x}}$ is an optimal solution of problem (12) with the weights $\bar{\mathbf{w}}$ and parameter $\lambda = 0$. Moreover, it is a unique optimal solution in terms of the criterion space (achievement vectors). That means, any $\mathbf{x} \in Q$ optimal for the corresponding problem (12) satisfies $\mathbf{F}(\mathbf{x}) = \mathbf{F}(\bar{\mathbf{x}})$.

We will prove that $\bar{\mathbf{x}}$ remains optimal for problem (12) with the weights $\bar{\mathbf{w}}$ and some small positive λ. Due to (13), $H_{\lambda,\mathbf{w}}(\mathbf{x}) = \lambda H_{1,\mathbf{w}}(\mathbf{x}) + (1-\lambda)H_{0,\mathbf{w}}(\mathbf{x})$. Note that, due to discreteness of our location problem, there exist $\varepsilon > 0$ and $E > 0$ such that

$$H_{0,\bar{\mathbf{w}}}(\bar{\mathbf{x}}) \leq H_{0,\bar{\mathbf{w}}}(\mathbf{x}) - \varepsilon \quad \text{for any} \quad \mathbf{x} \in Q, \quad \mathbf{F}(\mathbf{x}) \neq \mathbf{F}(\bar{\mathbf{x}})$$

$$H_{1,\bar{\mathbf{w}}}(\bar{\mathbf{x}}) - E \leq H_{1,\bar{\mathbf{w}}}(\mathbf{x}) \quad \text{for any} \quad \mathbf{x} \in Q$$

Thus, putting $\bar{\lambda} = \varepsilon/(E+\varepsilon)$, we get

$$\bar{\lambda}(H_{1,\bar{\mathbf{w}}}(\bar{\mathbf{x}}) - H_{1,\bar{\mathbf{w}}}(\mathbf{x})) \leq (1-\bar{\lambda})(H_{0,\bar{\mathbf{w}}}(\mathbf{x}) - H_{0,\bar{\mathbf{w}}}(\bar{\mathbf{x}})) \quad \text{for any} \quad \mathbf{x} \in Q$$

Hence, for any $\mathbf{x} \in Q$

$$H_{\bar{\lambda},\bar{\mathbf{w}}}(\bar{\mathbf{x}}) = \bar{\lambda}H_{1,\bar{\mathbf{w}}}(\bar{\mathbf{x}}) + (1-\bar{\lambda})H_{0,\bar{\mathbf{w}}}(\bar{\mathbf{x}}) \leq \bar{\lambda}H_{1,\bar{\mathbf{w}}}(\mathbf{x}) + (1-\bar{\lambda})H_{0,\bar{\mathbf{w}}}(\mathbf{x}) = H_{\bar{\lambda},\bar{\mathbf{w}}}(\mathbf{x})$$

which proves the optimality of vector $\bar{\mathbf{x}}$ for problem (12) with the weights $\bar{\mathbf{w}}$ satisfying (14) and the parameter $0 < \bar{\lambda} < 1$. □

According to Propositions 9 and 10, the weighted λ–cent–dian solution concept satisfies our expectations regarding solution techniques for the multiple criteria location problems (1). That means, each generated (optimal) solution is an efficient solution of problem (1) and for any efficient solution there exists a set of control parameters generating this solution. Note, however, that the value $\bar{\lambda}$ constructed in the proof of Proposition 10 has to be "small enough" and we have used directly the assumption about the finite feasible set (discrete location problem) to define it. Thus, in fact, we use the median term $H_{1,\mathbf{w}}(\mathbf{x})$ of the objective function (13) only as a regularization for the main center term $H_{0,\mathbf{w}}(\mathbf{x})$, to guarantee that in the case of nonunique center solutions that solution with the best median value will be selected. In other words, we use the λ–cent–dian solution concept only to emulate a two level lexicographic optimization with the weighted center and the median objective functions. Thus, we may introduce explicitly the solution concept of the lexicographic cent–dian as an optimal solution to the following lexicographic problem

$$\operatorname{lex}\min_{\mathbf{x}}\Big\{[\max_{j=1,\ldots,n} w_j f_j(\mathbf{x}), \sum_{j=1}^{n} w_j f_j(\mathbf{x})] : \quad \mathbf{x} \in Q\Big\} \tag{15}$$

One can easily adjust the proofs of Propositions 9 and 10 two show similar properties of the lexicographic cent–dian solution concept, as stated in two following propositions.

Proposition 11
For any weights w_j satisfying (14), each optimal solution to the lexicographic problem (15) is an efficient solution of the multiple criteria location problem (1). □

Proposition 12
For any efficient solution $\bar{\mathbf{x}}$ of the multiple criteria location problem (1), there exist weights w_j satisfying (14) such that $\bar{\mathbf{x}}$ is an optimal solution to the corresponding lexicographic problem (15). □

The lexicographic cent–dian solution concept defined as problem (15) seems to be more lucid and better appealing than the equivalent λ–cent–dian model with "small enough" λ. The use of a "small enough" parameter, to combine the objective functions emulating their pre–emptive hierarchy, is usually the simplest implementation technique for the lexicographic optimization. However, especially in the case of objective functions (3), specificity of the location problem allows for easy sequential implementations of problem (15). We need to solve the center problem, and next the median problem with forbidden allocations on (weighted) distances exceeding the optimal value of the center problem. For instance, having solved the center problem, one may replace with infinity (or a very large number) all the distances exceeding the optimal value of the center problem.

The lexicographic cent–dian approach may be considered a special case of the aspiration/reservation based approach [12,19] to multiple criteria optimization which is a modification of goal programming and the reference point method [27]. The aspiration/reservation based approach has been implemented in an experimental DSS for multiple criteria transportation problem with facility location [20] and successfully used for a real-life decision analysis [15]. Efficient solutions generated within the interactive scheme of the aspiration/reservation based approach (in the simplest form of that by Ogryczak and Lahoda [19]) are defined as optimal solutions to the following problem

$$\operatorname{lex}\min_{\mathbf{x}}\Big\{[\max_{j=1,\ldots,n} (f_j(\mathbf{x}) - a_j)/(r_j - a_j), \sum_{j=1}^{n} (f_j(\mathbf{x}) - a_j)/(r_j - a_j)] : \quad \mathbf{x} \in Q\Big\} \tag{16}$$

where the DM's preferences are modeled with the aspiration levels a_j (as the most desired values) and the reservation levels r_j (as the worst acceptable values) for several objective functions. Note, that applying (16) to the multiple criteria location model (1) with $a_j = 0$ and $r_j = 1/w_j$ (for $j = 1, 2, \ldots, n$) we get exactly problem (15). Thus, the lexicographic cent–dian solution concept differs from (16) only due to using always zero as the aspiration levels and the direct use of weights instead of the reservation levels. The latter is only a technical difference as we do not consider how the weights are defined to model the DM's preferences. The former suggests a possibility to introduce various aspiration levels into the lexicographic cent–dian concept. It would improve the controllability of the efficient set parameterization and thereby ease the interactive analysis. One may consider rather impracticable to use various weights in the multiple criteria location model (1). However, the use of aspiration and reservation levels during an interactive analysis within the GIS environment seems to be a reasonable tool to adjust the solution to the DM's preferences.

5 Conclusions

Location problems can be considered as multiple criteria models (1) where for each client there is defined an individual objective function, which measures quality of a location pattern with respect to the client satisfaction. In our multiple criteria location model the geographical space, essentially, covers both: the decision space and the criterion space. Therefore, the multiple criteria approach to location problems based on model (1) seems to be well suited for development of interactive solution procedures to be used within the GIS environment. The analysis presented in this paper may provide a theoretical basis for such developments.

The individual objective functions are usually conflicting when optimized. Therefore, the DM needs to select some compromise solution for implementation. In this paper we have analyzed various approaches to location problems (solution concepts) from the perspective of the multiple criteria model (1). We have focused our analysis on two aspects of the solution concepts: if a generated solution is an efficient (Pareto-optimal) solution to the multiple criteria problem, and if the solution concept provides some control parameters allowing the decision maker to select every efficient solution of the multiple criteria problem. That means, we have analyzed if a solution concept complies with the optimality principle for the multiple criteria model as well as if it allows to take into account various preferences of the DM.

First, we have analyzed the classical solution concepts for location problems: the median and the center. In both the concepts we allow to introduce some weights as control parameters modeling the DM's preferences. It turns out that both the concepts fail to achieve our standards. The median solution is always an efficient one but there are efficient solutions which cannot be identified by varying weights in the median approach. On the other hand, the weighted center approach allows us to identify each efficient solution to the problem (1) but in the case of nonunique solution it may generate some solutions failing the efficiency requirement.

The solution concept of the weighted center needs only an additional regularization in the case of nonunique solutions to meet our requirements. We have introduced the concept of the lexicographic center as a regularization (consistent with the center philosophy) of the center solution concept. The solution concept of the lexicographic weighted center seems to be ideal from the perspective of our analysis as it meets our both requirements (efficiency principle and complete parameterization). However, for most location problems it may require complex computations. Therefore, we have introduced another, computationally easier, regularization of the center solution concept. It has been derived as a modification of the so–called λ–cent–dian approach [6] which is some form of compromise between the median and center solution concepts. The solution concept of the lexicographic cent–dian is computationally robust and meets our standards. Its only weakness depends on some methodological inconsistency as it switches from the center to the median approach. The lexicographic cent–dian approach may be considered a special case of the aspiration/reservation based approach [19] to multiple criteria optimization which is a modification of goal programming and the reference point method [27]. It suggests possible further extensions of this solution concept.

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