On efficient optimisation of the CVaR and related LP computable risk measures for portfolio selection

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Abstract. The portfolio optimisation problem is modelled as a mean-risk bicriteria optimisation problem where the expected return is maximised and some (scalar) risk measure is minimised. In the original Markowitz model the risk is measured by the variance while several polyhedral risk measures have been introduced leading to Linear Programming (LP) computable portfolio optimisation models in the case of discrete random variables represented by their realisations under specified scenarios. Recently, the second order quantile risk measures have been introduced and become popular in finance and banking. The simplest such measure, now commonly called the Conditional Value at Risk (CVaR) or Tail VaR, represents the mean shortfall at a specified confidence level. The corresponding portfolio optimisation models can be solved with general purpose LP solvers. However, in the case of more advanced simulation models employed for scenario generation one may get several thousands of scenarios. This may lead to the LP model with a huge number of variables and constraints, thus decreasing the computational efficiency of the model. We show that the computational efficiency can be then dramatically improved with an alternative model taking advantages of the LP duality. Moreover, similar reformulation can be applied to more complex quantile risk measures like Gini's mean difference as well as to the mean absolute deviation.

Key words: risk measures, portfolio optimisation, computability, linear programming

1 Introduction

In the original Markowitz model [12] the risk is measured by the variance, but several polyhedral risk measures have been introduced leading to Linear Programming (LP) computable portfolio optimisation models in the case of discrete random variables represented by their realisations under specified scenarios. The simplest LP computable risk measures are dispersion measures similar to the variance. Konno and Yamazaki [6] presented the portfolio selection model with the mean absolute deviation (MAD). Yitzhaki [25] introduced the mean-risk model using Gini's mean (absolute) difference as the risk measure. Gini's mean difference turn out to be a special aggregation technique of the multiple criteria LP model [17] based on the pointwise comparison of the absolute Lorenz curves. The latter leads to the quantile

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shortfall risk measures that are more commonly used and accepted. Recently, the second-order quantile risk measures have been introduced in different ways by many authors [2, 5, 15, 16, 22]. The measure, usually called the Conditional Value at Risk (CVaR) or Tail VaR, represents the mean shortfall at a specified confidence level. Maximisation of the CVaR measures is consistent with the second-degree stochastic dominance [19]. Several empirical analyses confirm its applicability to various financial optimisation problems [1, 10]. This paper is focused on computational efficiency of the CVaR and related LP computable portfolio optimisation models.

For returns represented by their realisations under T scenarios, the basic LP model for CVaR portfolio optimisation contains T auxiliary variables as well as T corresponding linear inequalities. Actually, the number of structural constraints in the LP model (matrix rows) is proportional to the number of scenarios T, while the number of variables (matrix columns) is proportional to the total of the number of scenarios and the number of instruments $T + n$. Hence, its dimensionality is proportional to the number of scenarios T. It does not cause any computational difficulties for a few hundred scenarios as in computational analysis based on historical data. However, in the case of more advanced simulation models employed for scenario generation one may get several thousands of scenarios [21]. This may lead to the LP model with a huge number of auxiliary variables and constraints, thus decreasing the computational efficiency of the model. Actually, in the case of fifty thousand scenarios and one hundred instruments the model may require more than half an hour of computation time [8] with the state-of-art LP solver (CPLEX code). We show that the computational efficiency can be then dramatically improved with an alternative model formulation taking advantage of the LP duality. In the introduced model the number of structural constraints is proportional to the number of instruments n, while only the number of variables is proportional to the number of scenarios T, thus not affecting the simplex method efficiency so seriously. Indeed, the computation time is then below 30 seconds. Moreover, similar reformulation can be applied to the classical LP portfolio optimisation model based on the MAD as well as to more complex quantile risk measures including Gini's mean difference [25].

2 Computational LP models

The portfolio optimisation problem considered in this paper follows the original Markowitz' formulation and is based on a single period model of investment. At the beginning of a period, an investor allocates the capital among various securities, thus assigning a nonnegative weight (share of the capital) to each security. Let $J = \{1, 2, \ldots, n\}$ denote a set of securities considered for an investment. For each security $j \in J$, its rate of return is represented by a random variable R_j with a given mean $\mu_j = \mathbb{E}\{R_j\}$. Further, let $x = (x_j)_{j=1,2,\ldots,n}$ denote a vector of decision variables x_j expressing the weights defining a portfolio. The weights must satisfy a set of constraints to represent a portfolio. The simplest way of defining a feasible set $\mathcal{P}$ is by a requirement that the weights must sum to one and they are nonnegative (short sales are not allowed), i.e.,

$$\mathcal{P} = \{x \ : \ \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \dots, n\}. \tag{1}$$

Hereafter, we perform detailed analysis for the set $\mathcal{P}$ given with constraints (1). Nevertheless, the presented results can easily be adapted to a general LP feasible set given as a system of linear equations and inequalities, thus allowing one to include short sales, upper bounds on single shares or portfolio structure restrictions which may be faced by a real-life investor.

Each portfolio $\mathbf{x}$ defines a corresponding random variable $R_{\mathbf{x}} = \sum_{j=1}^{n} R_j x_j$ that represents the portfolio rate of return while the expected value can be computed as $\mu(x) = \sum_{j=1}^{n} \mu_j x_j$. We consider T scenarios with probabilities p_t (where $t = 1, \dots, T$). We assume that for each random variable R_j its realisation r_{jt} under the scenario t is known. Typically, the realisations are derived from historical data treating T historical periods as equally probable scenarios ($p_t = 1/T$). The realisations of the portfolio return R_x are given as $y_t = \sum_{j=1}^{n} r_{jt} x_j$.

Let us consider a portfolio optimisation problem based on the CVaR measure optimisation. With security returns given by discrete random variables with realisations r_{jt}, following [1,9,10], the CVaR portfolio optimisation model can be formulated as the following LP problem:

$$\begin{aligned} &\text{maximise } \eta - \frac{1}{\beta} \sum_{t=1}^{T} p_t d_t \\ &\quad \text{s.t. } \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \dots, n \\ &\qquad d_t - \eta + \sum_{j=1}^{n} r_{jt} x_j \geq 0, \ d_t \geq 0 \quad \text{for } t = 1, \dots, T, \end{aligned} \tag{2}$$

where η is an unbounded variable. Except for the core portfolio constraints (1), model (2) contains T nonnegative variables d_t plus a single η variable and T corresponding linear inequalities. Hence, its dimensionality is proportional to the number of scenarios T. Exactly, the LP model contains $T + n + 1$ variables and $T + 1$ constraints. For a few hundred scenarios, as in typical computational analysis based on historical data [11], such LP models are easily solvable. However, the use of more advanced simulation models for scenario generation may result in several thousands of scenarios. The corresponding LP model (2) contains then a huge number of variables and constraints, thus decreasing its computational efficiency dramatically. If the core portfolio constraints contain only linear relations, like (1), then the computational efficiency can easily be achieved by taking advantage of the LP dual model (2). The

LP dual model takes the following form:

$$\begin{aligned} &\text{minimise } q \\ &\text{s.t. } \; q - \sum_{t=1}^{T} r_{jt} u_t \geq 0 \quad \text{for } j = 1, \ldots, n \\ &\qquad \sum_{t=1}^{T} u_t = 1, \quad 0 \leq u_t \leq \frac{p_t}{\beta} \quad \text{for } t = 1, \ldots, T. \end{aligned} \tag{3}$$

The dual LP model contains T variables u_t, but the T constraints corresponding to variables d_t from (2) take the form of simple upper bounds (SUB) on u_t thus not affecting the problem complexity (c.f., [13]). Actually, the number of constraints in (3) is proportional to the total of portfolio size n, thus it is independent from the number of scenarios. Exactly, there are $T + 1$ variables and $n + 1$ constraints. This guarantees a high computational efficiency of the dual model even for a very large number of scenarios. Note that introducing a lower bound on the required expected return in the primal portfolio optimisation model (2) results only in a single additional variable in the dual model (3). Similarly, other portfolio structure requirements are modelled with a rather small number of constraints, thus generating a small number of additional variables in the dual model.

We have run computational tests on 10 randomly generated test instances developed by Lim et al. [8]. They were originally generated from a multivariate normal distribution for 50 or 100 securities with the number of scenarios of 50,000 just providing an adequate approximation to the underlying unknown continuous price distribution. Scenarios were generated using the Triangular Factorization Method [24] as recommended in [3]. All computations were performed on a PC with a Pentium 4 2.6 GHz processor and 1 GB RAM employing the simplex code of the CPLEX 9.1 package. An attempt to solve the primal model (2) with 50 securities resulted in 2600 seconds of computation (much more than reported in [8]). On the other hand, the dual models (3) were solved in 14.3–27.7 CPU seconds on average, depending on the tolerance level (see Table 1). For 100 securities the optimisation times were longer but still about 1 minute.

Table 1. Computational times (in seconds) for the dual CVaR model (averages of 10 instances with 50,000 scenarios)

Number of securities	$\beta = 0.05$	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.3$	$\beta = 0.4$	$\beta = 0.5$
$n = 50$	14.3	18.7	23.6	26.4	27.4	27.7
$n = 100$	38.1	52.1	67.9	74.8	76.7	76.0

The SSD consistent [14] and coherent [2] MAD model with complementary risk measure ($\mu_\delta(x) = \mathbb{E}\{\min\{\mu(x), R_x\}\}$) leads to the following LP problem [18]:

$$\begin{aligned} \text{maximise } & \sum_{j=1}^{n} \mu_j x_j - \sum_{t=1}^{T} p_t d_t \\ \text{s.t. } & \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \dots, n \\ & d_t - \sum_{j=1}^{n} (\mu_j - r_{jt}) x_j \geq 0, \; d_t \geq 0 \quad \text{for } t = 1, \dots, T. \end{aligned} \tag{4}$$

The above LP formulation uses $T + n$ variables and $T + 1$ constraints while the LP dual model then takes the following form:

$$\begin{aligned} \text{minimise } & q \\ \text{s.t. } & q + \sum_{t=1}^{T} (\mu_j - r_{jt}) u_t \geq \mu_j \quad \text{for } j = 1, \dots, n \\ & 0 \leq u_t \leq p_t \quad \text{for } t = 1, \dots, T, \end{aligned} \tag{5}$$

with dimensionality $n \times (T + 1)$. Hence, there is guaranteed high computational efficiency even for very large numbers of scenarios. Indeed, in the test problems with 50,000 scenarios we were able to solve the dual model (5) in 25.3 seconds on average for 50 securities and in 77.4 seconds for 100 instruments.

For a discrete random variable represented by its realisations y_t, Gini's mean difference measure $\Gamma(x) = \sum_{t'=1}^{T} \sum_{t'' \neq t'-1} \max\{y_{t'} - y_{t''}, 0\} p_{t'} p_{t''}$ is LP computable (when minimised). This leads us to the following GMD portfolio optimisation model [25]:

$$\begin{aligned} \max \; & -\sum_{t=1}^{T} \sum_{t' \neq t} p_t p_{t'} d_{tt'} \\ \text{s.t. } & \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \dots, n \\ & d_{tt'} \geq \sum_{j=1}^{n} r_{jt} x_j - \sum_{j=1}^{n} r_{jt'} x_j, \; d_{tt'} \geq 0 \quad \text{for } t, t' = 1, \dots, T; \; t \neq t', \end{aligned} \tag{6}$$

which contains $T(T - 1)$ nonnegative variables $d_{tt'}$ and $T(T - 1)$ inequalities to define them. This generates a huge LP problem even for the historical data case where the number of scenarios is 100 or 200. Actually, as shown with the earlier experiments [7], the CPU time of 7 seconds on average for $T = 52$ has increased to above 30 s with $T = 104$ and even more than 180 s for $T = 156$. However, similar to the CVaR models, variables $d_{tt'}$ are associated with the singleton coefficient columns. Hence, while solving the dual instead of the original primal, the corresponding dual constraints take the form of simple upper bounds (SUB) which are handled implicitly outside the LP matrix. For the simplest form of the feasible set (1) the dual GMD model takes the following form:

$$\begin{aligned} & \min v \\ & \text{s.t. } v - \sum_{t=1}^{T} \sum_{t' \neq t} (r_{jt} - r_{jt'}) u_{tt'} \geq 0 \quad \text{for } j = 1, \ldots, n \\ & \qquad 0 \leq u_{tt'} \leq p_t p_{t'} \quad \text{for } t, t' = 1, \ldots, T;\ t \neq t', \end{aligned} \tag{7}$$

where original portfolio variables x_j are dual prices to the inequalities. The dual model contains $T(T-1)$ variables $u_{tt'}$ but the number of constraints (excluding the SUB structure) $n+1$ is proportional to the number of securities. The above dual formulation can be further simplified by introducing variables:

$$\bar{u}_{tt'} = u_{tt'} - u_{t't} \quad \text{for } t, t' = 1, \ldots, T;\ t < t', \tag{8}$$

which allows us to reduce the number of variables to $T(T-1)/2$ by replacing (7) with the following:

$$\begin{aligned} & \min v \\ & \text{s.t. } v - \sum_{t=1}^{T} \sum_{t' > t} (r_{jt} - r_{jt'}) \bar{u}_{tt'} \geq 0 \quad \text{for } j = 1, \ldots, n \\ & \qquad -p_t p_{t'} \leq \bar{u}_{tt'} \leq p_t p_{t'} \quad \text{for } t, t' = 1, \ldots, T;\ t < t'. \end{aligned} \tag{9}$$

Such a dual approach may dramatically improve the LP model efficiency in the case of a larger number of scenarios. Actually, as shown with the earlier experiments [7], the above dual formulations let us to reduce the optimisation time to below 10 seconds for $T = 104$ and $T = 156$. Nevertheless, the case of really large numbers of scenarios still may cause computational difficulties, due to the huge number of variables ($T(T-1)/2$). This may require some column generation techniques [4] or nondifferentiable optimisation algorithms [8].

3 Conclusions

The classical Markowitz model uses the variance as the risk measure, thus resulting in a quadratic optimisation problem. Several alternative risk measures were introduced, which are computationally attractive as (for discrete random variables) they result in solving linear programming (LP) problems. The LP solvability is very important for applications to real-life financial decisions where the constructed portfolios have to meet numerous side constraints and take into account transaction costs [10]. The corresponding portfolio optimisation models can be solved with general purpose LP solvers, like ILOG CPLEX providing a set of C++ and Java class libraries allowing the programmer to embed CPLEX optimisers in C++ or Java applications.

Unfortunately, in the case of more advanced simulation models employed for scenario generation one may get several thousands of scenarios. This may lead to the LP model with a huge number of variables and constraints, thus decreasing the computational efficiency of the model. We have shown that the computational efficiency can then be dramatically improved with an alternative model taking advantage of the LP duality. In the introduced model the number of structural constraints (matrix rows) is proportional to the number of instruments thus not seriously affecting the simplex method efficiency by the number of scenarios. For the case of 50,000 scenarios, it has resulted in computation times below 30 seconds for 50 securities or below a minute for 100 instruments. Similar computational times have also been achieved for the dual reformulation of the MAD model. Dual reformulation applied to the GMD portfolio optimisation model results in a dramatic problem size reduction with the number of constraints equal to the number of instruments instead of the square of the number of scenarios. Although, the remaining high number of variables (square of the number of scenarios) still generates a need for further research on column-generation techniques or nondifferentiable optimisation algorithms for the GMD model.

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