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ON DUAL APPROACHES TO EFFICIENT OPTIMIZATION OF LP COMPUTABLE RISK MEASURES FOR PORTFOLIO SELECTION

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In the original Markowitz model for portfolio optimization the risk is measured by the variance. Several polyhedral risk measures have been introduced leading to Linear Programming (LP) computable portfolio optimization models in the case of discrete random variables represented by their realizations under specified scenarios. The LP models typically contain the number of constraints (matrix rows) proportional to the number of scenarios while the number of variables (matrix columns) proportional to the total of the number of scenarios and the number of instruments. They can effectively be solved with general purpose LP solvers provided that the number of scenarios is limited. However, real-life financial decisions are usually based on more advanced simulation models employed for scenario generation where one may get several thousands scenarios. This may lead to the LP models with huge number of variables and constraints thus decreasing their computational efficiency and making them hardly solvable by general LP tools. We show that the computational efficiency can be then dramatically improved by alternative models taking advantages of the LP duality. In the introduced models the number of structural constraints (matrix rows) is proportional to the number of instruments thus not affecting seriously the simplex method efficiency by the number of scenarios and therefore guaranteeing easy solvability.

Keywords: Risk measures; portfolio optimization; computability; linear programming.

1. Introduction

The Asian financial markets have seen strong development during the last decade of the XXth century and the beginning of the XXIth century. Actually, while the largest market in the region, the Tokyo Stock Exchange (TSE) has been more volatile reflecting instability in the Japanese economy over this period, the smaller markets have experienced the growth reaching in total the size comparable to the TSE (Comerton-Forde and Rydge, 2006). Net capital flows to the Asia Pacific region over 1999 to 2003 constituted about 14% of the the world's FDI flows whereas over

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90% of the flows has been in the form of equity and portfolio investments (Ding and Charoenwong, 2006).

Following Markowitz (1952), the portfolio selection problem is modeled as a mean-risk bicriteria optimization problem where the expected return is maximized and some (scalar) risk measure is minimized. In the original Markowitz model the risk is measured by the variance but several other risk measures have been later considered thus creating the entire family of mean-risk (Markowitz-type) models. While the original Markowitz model forms a quadratic programming problem, following Sharpe (1971a), many attempts have been made to linearize the portfolio optimization procedure. Portfolio optimization models for some specific distributions of returns can be transformed into a Linear Programming (LP) problem, like the chance-constrained portfolio selection problem under t-distribution (Wang *et al.*, 2007). More important, several polyhedral risk measures have been introduced leading to LP computable portfolio optimization models in the case of discrete random variables represented by their realizations under specified scenarios (c.f., Speranza (1993) and references therein). The simplest LP computable risk measures are dispersion measures similar to the variance. Konno and Yamazaki (1991) introduced the portfolio selection model with the mean absolute deviation (MAD) and demonstrated its good performance on the TSE. Young (1998) presented the Minimax model while earlier Yitzhaki (1982) introduced the mean-risk model using Gini's mean (absolute) difference as the risk measure. The Gini's mean difference turns out to be a special aggregation technique of the multiple criteria LP model (Ogryczak, 2000a) based on the pointwise comparison of the absolute Lorenz curves. The latter makes the quantile shortfall risk measures directly related to the dual theory of choice under risk (Quiggin, 1982, 1993; Roell, 1987; Yaari, 1987). Recently, the second order quantile risk measures have been introduced in different ways by many authors (Artzner *et al.*, 1999; Embrechts *et al.*, 1997; Ogryczak, 1999, 2000; Rockafellar and Uryasev, 2000). The measure, usually called the Conditional Value at Risk (CVaR) or Tail VaR, represents the mean shortfall at a specified confidence level. The CVaR measures maximization is consistent with the second degree stochastic dominance (Ogryczak and Ruszczyński, 2002). Several empirical analyses confirm its applicability to various financial optimization problems (Andersson *et al.*, 2001; Mansini *et al.*, 2003a). Analyzing the TSE historical data Konno *et al.* (2002) confirmed that while monthly stock returns are almost normally distributed, the daily data exhibits non-symmetric distribution. Similar properties were shown for the Chinese stock markets (Chen and Wang, 2008) as well as for the Asian hedge funds (Hakamada *et al.*, 2007; Wong *et al.*, 2008). This causes a need for the use of LP computable portfolio optimization models capable to deal with non-symmetric distributions and SSD consistent.

This paper is focused on computational efficiency of the LP computable portfolio optimization models. We assume that the instruments returns are represented by their realizations under T scenarios. The basic LP model, for instance for the

CVaR portfolio optimization, contains then T auxiliary variables as well as T corresponding linear inequalities. Actually, the number of structural constraints in the LP model (matrix rows) is proportional to the number of scenarios T, while the number of variables (matrix columns) is proportional to the total of the number of scenarios and the number of instruments $T + n$. Hence, its dimensionality is proportional to the number of scenarios T. It does not cause any computational difficulties for a few hundreds scenarios as in computational analysis based on historical data. However, real-life financial analysis must be usually based on more advanced simulation models employed for scenario generation (Carino *et al.*, 1998). One may get then several thousands scenarios (Pflug, 2001; Guastaroba *et al.*, 2009) thus leading to the LP model with huge number of auxiliary variables and constraints and thereby hardly solvable by general LP tools. Actually, in the case of fifty thousand scenarios and one hundred instruments the model may require more than an hour computation time with the state-of-art LP solver (CPLEX code) or remain unsolved. To overcome this difficulty some alternative solution approaches are searched trying to reformulate the optimization problems as two-stage recourse problems (Künzi-Bay and Mayer, 2006), to employ nondifferential optimization techniques (Lim *et al.*, 2009), cutting planes (Fabian *et al.*, 2009) or to approximate the returns with a factor representation (Konno *et al.*, 2002). We show that the computational efficiency can simply be achieved with an alternative model formulation taking advantages of the LP duality. In the introduced model the number of structural constraints is proportional to the number of instruments n while only the number of variables is proportional to the number of scenarios T thus not affecting so seriously the simplex method efficiency. Therefore, the model can effectively be solved with general LP solvers even for very large numbers of scenarios. Indeed, the computation time for the case of fifty thousand scenarios and one hundred instruments is then below a minute.

Similar reformulations can be applied to the classical LP portfolio optimization models based on the mean absolute deviation as well as to more complex quantile risk measures. The MAD model was introduced by Konno and Yamazaki (1991) with $2T$ auxiliary variables and $2T$ corresponding linear inequalities while reformulated later by Feinstein and Thapa (1993) to the use T auxiliary variables and T corresponding linear inequalities. In our model the number of structural constraints is proportional to the number of instruments n while only the number of variables is proportional to the number of scenarios T. The Tail Gini's measures or the Weighted CVaR measures defined as combinations of CVaR measures for m tolerance levels lead to LP models with the number of structural constraints (matrix rows) proportional to the respectively multiplied number of scenarios mT. In the alternative model taking advantages of the LP duality the number of structural constraints is proportional to the total of the number of instruments and number of tolerance levels $n+m$. This guarantees a high computational efficiency of the dual model even for a very large number of scenarios. The standard LP models for the Gini's mean difference (Yitzhaki, 1982) require T^2 auxiliary constraints which makes them hard already for medium numbers of scenarios, like a few hundred scenarios given by historical data. The models taking advantages of the LP duality allow one to limit the number of structural constraints making it proportional to the number of scenarios T thus increasing dramatically computational performances for medium numbers of scenarios although still remaining hard for very large numbers of scenarios.

The paper is organized as follows. In the next section we introduce briefly basics of the mean-risk portfolio optimization with the LP computable risk measures. In Sec. 3 we develop and test computationally efficient optimization models taking advantages of the LP duality. Further, Sec. 4 is devoted to the similar analysis of more complicated Gini's mean difference LP models and their approximations by the weighted multiple CVaR (WCVaR) measure models.

2. Portfolio Optimization and Risk Measures

The portfolio optimization problem considered in this paper follows the original Markowitz' formulation and is based on a single period model of investment. At the beginning of a period, an investor allocates the capital among various securities, thus assigning a nonnegative weight (share of the capital) to each security. Let $J = \{1, 2, \ldots, n\}$ denote a set of securities considered for an investment. For each security $j \in J$, its rate of return is represented by a random variable R_j with a given mean $\mu_j = \mathbb{E}\{R_j\}$. Further, let $\mathbf{x} = (x_j)_{j=1,2,\ldots,n}$ denote a vector of decision variables x_j expressing the weights defining a portfolio. The weights must satisfy a set of constraints to represent a portfolio. The simplest way of defining a feasible set Q is by a requirement that the weights must sum to one and they are nonnegative (short sales are not allowed), i.e.

$$Q = \left\{ \mathbf{x} : \sum_{j=1}^{n} x_j = 1,\ x_j \geq 0 \text{ for } j = 1, \ldots, n \right\} \tag{2.1}$$

Hereafter, we perform detailed analysis for the set Q given with constraints (2.1). Nevertheless, the presented results can easily be adapted to a general LP feasible set given as a system of linear equations and inequalities, thus allowing one to include short sales, upper bounds on single shares or portfolio structure restrictions which may be faced by a real-life investor.

Each portfolio $\mathbf{x}$ defines a corresponding random variable $R_{\mathbf{x}} = \sum_{j=1}^{n} R_j x_j$ that represents the portfolio rate of return while the expected value can be computed as $\mu(\mathbf{x}) = \sum_{j=1}^{n} \mu_j x_j$. We consider T scenarios with probabilities p_t (where $t = 1, \ldots, T$). We assume that for each random variable R_j its realization r_{jt} under the scenario t is known. Typically, the realizations are derived from historical data treating T historical periods as equally probable scenarios ($p_t = 1/T$). Although the models we analyze do not take advantages of this simplification. The realizations of the portfolio return $R_{\mathbf{x}}$ are given as $y_t = \sum_{j=1}^{n} r_{jt} x_j$.

The portfolio optimization problem is modeled as a mean-risk bicriteria optimization problem where the mean $\mu(\mathbf{x})$ is maximized and the risk measure $\varrho(\mathbf{x})$

is minimized. In the original Markowitz model, the standard deviation was used as the risk measure. Several other risk measures have been later considered thus creating the entire family of mean-risk models (c.f., Mansini *et al.*, 2003, 2003a). These risk measures, similar to the standard deviation, are not affected by any shift of the outcome scale and are equal to 0 in the case of a risk-free portfolio while taking positive values for any risky portfolio. Unfortunately, such risk measures are not consistent with the stochastic dominance order (Müller and Stoyan, 2002) or other axiomatic models of risk-averse preferences (Rotschild and Stiglitz, 1969) and coherent risk measurement (Artzner *et al.*, 1999).

In stochastic dominance, uncertain returns (modeled as random variables) are compared by pointwise comparison of some performance functions constructed from their distribution functions. The first performance function $F^{(1)}_{\mathbf{x}}$ is defined as the right-continuous cumulative distribution function: $F^{(1)}_{\mathbf{x}}(\eta) = F_{\mathbf{x}}(\eta) = \mathbb{P}\{R_{\mathbf{x}} \leq \eta\}$ and it defines the first degree stochastic dominance (FSD). The second function is derived from the first as $F^{(2)}_{\mathbf{x}}(\eta) = \int_{-\infty}^{\eta} F_{\mathbf{x}}(\xi) d\xi$ and it defines the second degree stochastic dominance (SSD). We say that portfolio $\mathbf{x}'$ dominates $\mathbf{x}''$ under the SSD ($R_{\mathbf{x}'} \succ_{_{SSD}} R_{\mathbf{x}''}$), if $F^{(2)}_{\mathbf{x}'}(\eta) \leq F^{(2)}_{\mathbf{x}''}(\eta)$ for all η, with at least one strict inequality. A feasible portfolio $\mathbf{x}^0 \in Q$ is called SSD efficient if there is no $\mathbf{x} \in Q$ such that $R_{\mathbf{x}} \succ_{_{SSD}} R_{\mathbf{x}^0}$. Stochastic dominance relates the notion of risk to a possible failure of achieving some targets. As shown by Ogryczak and Ruszczyński (1999), function $F^{(2)}_{\mathbf{x}}$, used to define the SSD relation, can also be presented as follows: $F^{(2)}_{\mathbf{x}}(\eta) = \mathbb{E}\{\max\{\eta - R_{\mathbf{x}}, 0\}\}$ and thereby its values are LP computable for returns represented by their realizations y_t.

When the mean $\mu(\mathbf{x})$ is used instead of the fixed target the value $F^{(2)}_{\mathbf{x}}$ $(\mu(\mathbf{x}))$ defines the risk measure known as the *downside mean semideviation* from the mean

$$\bar{\delta}(\mathbf{x}) = \mathbb{E}\{\max\{\mu(\mathbf{x}) - R_{\mathbf{x}}, 0\}\} = F^{(2)}_{\mathbf{x}}(\mu(\mathbf{x})). \tag{2.2}$$

The downside mean semideviation is always equal to the upside one and therefore we refer to it hereafter as to the mean semideviation. The mean semideviation is a half of the mean absolute deviation (MAD) from the mean (Ogryczak and Ruszczyński, 1999) $\delta(\mathbf{x}) = \mathbb{E}\{|R_{\mathbf{x}} - \mu(\mathbf{x})|\} = 2\bar{\delta}(\mathbf{x})$. Hence the corresponding portfolio optimization model is equivalent to the MAD. Since $\bar{\delta}(\mathbf{x}) = F^{(2)}_{\mathbf{x}}(\mu(\mathbf{x}))$, the mean semideviation (2.2) is LP computable (when minimized), for a discrete random variable represented by its realizations y_t. Although, due to the use of distribution dependent target value $\mu(\mathbf{x})$, the mean semideviation cannot be directly considered an SSD consistent risk measure. SSD consistency (Ogryczak and Ruszczyński, 1999) and coherency (Mansini *et al.*, 2003a) of the MAD model can be achieved with maximization of for complementary risk measure $\mu_{\delta}(\mathbf{x}) = \mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}) = \mathbb{E}\{\min\{\mu(\mathbf{x}), R_{\mathbf{x}}\}\}$, which also remains LP computable for a discrete random variable represented by its realizations y_t.

An alternative characterization of the SSD relation can be achieved with the so-called Absolute Lorenz Curves (ALC) (Ogryczak, 1999; Shorrocks, 1983) which represent the second quantile functions defined as

$$F_{\mathbf{x}}^{(-2)}(p) = \int_0^p F_{\mathbf{x}}^{(-1)}(\alpha)d\alpha \quad \text{for } 0 < p \leq 1 \quad \text{and} \quad F_{\mathbf{x}}^{(-2)}(0) = 0, \tag{2.3}$$

where $F_{\mathbf{x}}^{(-1)}(p) = \inf\{\eta : F_{\mathbf{x}}(\eta) \geq p\}$ is the left-continuous inverse of the cumulative distribution function $F_{\mathbf{x}}$. The pointwise comparison of ALCs is equivalent to the SSD relation (Ogryczak and Ruszczyński, 2002) in the sense that $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$ if and only if $F_{\mathbf{x}'}^{(-2)}(\beta) \geq F_{\mathbf{x}''}^{(-2)}(\beta)$ for all $0 < \beta \leq 1$. Moreover,

$$F_{\mathbf{x}}^{(-2)}(\beta) = \max_{\eta \in R}[\beta\eta - F_{\mathbf{x}}^{(2)}(\eta)] = \max_{\eta \in R}[\beta\eta - \mathbb{E}\{\max\{\eta - R_{\mathbf{x}}, 0\}\}] \tag{2.4}$$

where η is a real variable taking the value of β-quantile $Q_\beta(\mathbf{x})$ at the optimum. For a discrete random variable represented by its realizations y_t problem (2.4) becomes an LP.

For any real tolerance level $0 < \beta \leq 1$, the normalized value of the ALC defined as

$$M_\beta(\mathbf{x}) = F_{\mathbf{x}}^{(-2)}(\beta)/\beta \tag{2.5}$$

is called the *Conditional Value-at-Risk (CVaR)* or Tail VaR or Average VaR. The CVaR measure is an increasing function of the tolerance level β, with $M_1(\mathbf{x}) = \mu(\mathbf{x})$. For any $0 < \beta < 1$, the CVaR measure is SSD consistent (Ogryczak and Ruszczyński, 2002) and coherent (Pflug, 2000). Opposite to deviation type risk measures, for coherent measures larger values are preferred and therefore the measures are sometimes called safety measures (Mansini *et al.*, 2003a). Due to (2.4), for a discrete random variable represented by its realizations y_t the CVaR measures are LP computable. It is important to notice that although the quantile risk measures (VaR and CVaR) were introduced in banking as extreme risk measures for very small tolerance levels (like $\beta = 0.05$), for the portfolio optimization good results have been provided by rather larger tolerance levels (Mansini *et al.*, 2003a).

For β approaching 0, the CVaR measure tends to the Minimax measure

$$M(\mathbf{x}) = \min_{t=1,\ldots,T} y_t \tag{2.6}$$

introduced to portfolio optimization by Young (1998). Note that the maximum (downside) semideviation

$$\Delta(\mathbf{x}) = \mu(\mathbf{x}) - M(\mathbf{x}) = \max_{t=1,\ldots,T}(\mu(\mathbf{x}) - y_t) \tag{2.7}$$

and the conditional β-deviation

$$\Delta_\beta(\mathbf{x}) = \mu(\mathbf{x}) - M_\beta(\mathbf{x}) \quad \text{for } 0 < \beta \leq 1, \tag{2.8}$$

respectively, represent the corresponding deviation risk measures. They may be interpreted as the drawdown measures (Chekhlov *et al.*, 2002). For $\beta = 0.5$ the measure $\Delta_{0.5}(\mathbf{x})$ represents the mean absolute deviation from the median (Mansini *et al.*, 2003) the risk measure suggested by Sharpe (1971) as the right MAD model.

The commonly accepted approach to implementation of the Markowitz-type mean-risk model (with deviation type risk measures) is based on the use of a specified lower bound μ_0 on expected returns while optimizing the risk measure. This bounding approach provides a clear understanding of investor preferences and a clear definition of optimal portfolio to be sought. For deviation type risk measures ϱ the approach results in the following minimum risk problem:

$$\min\{\varrho(\mathbf{x}) : \mu(\mathbf{x}) \geq \mu_0, \mathbf{x} \in Q\} \tag{2.9}$$

While using the coherent and SSD consistent risk measures μ_ϱ one may focus on the measure maximization without additional constraints

$$\max\{\mu_\varrho(\mathbf{x}) : \mathbf{x} \in Q\} \tag{2.10}$$

or still consider some preferential constraints on the mean expectation

$$\max\{\mu_\varrho(\mathbf{x}) : \mu(\mathbf{x}) \geq \mu_0, \mathbf{x} \in Q\}. \tag{2.11}$$

We demonstrate that both models can be effectively solved for large numbers of scenarios while taking advantages of appropriate dual formulations.

3. Computational LP Models for Basic Risk Measures

3.1. *Coherent measures maximization*

Let us consider portfolio optimization problem with security returns given by discrete random variables with realization r_{jt} thus leading to LP models for coherent risk measures we consider. Let us focus first on measures maximization without additional (preferential) constraints thus considering the optimization models of type 2.10).

Following (2.4) and (2.5), the CVaR portfolio optimization model can be formulated as the following LP problem:

$$\begin{aligned} \max\ & \eta - \frac{1}{\beta}\sum_{t=1}^{T} p_t d_t \\ \text{s.t.}\ & \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1,\ldots,n \\ & d_t - \eta + \sum_{j=1}^{n} r_{jt}x_j \geq 0,\ d_t \geq 0 \quad \text{for } t = 1,\ldots,T \end{aligned} \tag{3.1}$$

where η is unbounded variable. Except from the core portfolio constraints (2.1), model (3.1) contains T nonnegative variables d_t plus single η variable and T corresponding linear inequalities. Hence, its dimensionality is proportional to the number of scenarios T. Exactly, the LP model contains $T+n+1$ variables and $T+1$ constraints. It does not cause any computational difficulties for a few hundreds scenarios as in several computational analysis based on historical data (Mansini *et al.*, 2007). However, in the case of more advanced simulation models employed for scenario generation one may get several thousands scenarios. This may lead to the LP model (3.1) with huge number of variables and constraints thus decreasing the computational efficiency of the model. If the core portfolio constraints contain only linear relations, like (2.1), then the computational efficiency can easily be achieved by taking advantages of the LP dual to model (3.1). The LP dual model takes the following form:

$$
\begin{aligned}
&\min\ q \\
&\text{s.t.}\ \ q - \sum_{t=1}^{T} r_{jt} u_t \geq 0 \quad \text{for } j = 1, \ldots, n \\
&\qquad \sum_{t=1}^{T} u_t = 1 \\
&\qquad 0 \leq u_t \leq \frac{p_t}{\beta} \quad \text{for } t = 1, \ldots, T
\end{aligned}
\tag{3.2}
$$

The dual LP model contains T variables u_t, but the T constraints corresponding to variables d_t from (3.1) take the form of simple upper bounds (SUB) on u_t thus not affecting the problem complexity. Actually, the number of constraints in (3.2) is proportional to the total of portfolio size n, thus it is independent from the number of scenarios. Exactly, there are $T + 1$ variables and $n + 1$ constraints. This guarantees a high computational efficiency of the dual model even for very large number of scenarios. Note that possible additional portfolio structure requirements are usually modeled with rather small number of linear constraints thus generating small number of additional variables in the dual model. Certainly, the optimal portfolio shares x_j are not directly represented within the solution vector of problem (3.2) but they are easily available as the dual variables (shadow prices) for inequalities $q - \sum_{t=1}^{T} r_{jt} u_t \geq 0$. Moreover, the dual model (3.2) may be considered a special case within the general theory of dual representations of coherent measures of risk, following from conjugate duality (Example 4.3 in Ruszczyński and Shapiro, 2006; Section 5 in Miller and Ruszczyński, 2008). This allows variables u_t to have the interpretation of probability distributions, while one looks at the distribution of the portfolio returns with respect to this measure.

The Minimax portfolio optimization model representing a limiting case of the CVaR model for β tending to 0 is even simpler than the general CVaR model. It can be written as the following LP problem:

$$
\begin{aligned}
&\max\ \eta \\
&\text{s.t.}\ \ \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \ldots, n \\
&\qquad -\eta + \sum_{j=1}^{n} r_{jt} x_j \geq 0, \quad \text{for } t = 1, \ldots, T
\end{aligned}
\tag{3.3}
$$

Except from the portfolio weights x_j, the model contains only one additional variable η. Nevertheless, it still contains T linear inequalities in addition to the core constraints (2.1). Hence, its dimensionality is $(T+1)\times(n+1)$. The LP dual model takes then the following form:

$$\begin{aligned} &\min\ q \\ &\text{s.t.}\quad q - \sum_{t=1}^{T} r_{jt} u_t \geq 0 \quad \text{for } j = 1, \ldots, n \\ &\qquad \sum_{t=1}^{T} u_t = 1 \\ &\qquad u_t \ \geq 0 \quad \text{for } t = 1, \ldots, T \end{aligned} \tag{3.4}$$

with dimensionality $(n+1)\times(T+1)$. This guarantees a high computational efficiency of the dual model even for very large number of scenarios. Comparing the model to the dual CVaR model (3.2) one may notice that upper bounds are skipped. Indeed, the upper bounds p_t/β tend to the infinity with β approaching 0.

The SSD consistent and coherent MAD model with complementary risk measure $(\mu_\delta(\mathbf{x}) = \mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}) = \mathbb{E}\{\min\{\mu(\mathbf{x}), R_{\mathbf{x}}\}\})$, leads to the following LP problem:

$$\begin{aligned} &\max \sum_{j=1}^{n} \mu_j x_j - \sum_{t=1}^{T} p_t d_t \\ &\text{s.t.}\quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \ldots, n \\ &\qquad d_t - \sum_{j=1}^{n} (\mu_j - r_{jt}) x_j \geq 0,\ d_t \geq 0 \quad \text{for } t = 1, \ldots, T \end{aligned} \tag{3.5}$$

The above LP formulation uses $T+n$ variables and $T+1$ constraints while the LP dual model takes then the following form:

$$\begin{aligned} &\min\ q \\ &\text{s.t.}\quad q + \sum_{t=1}^{T} (\mu_j - r_{jt}) u_t \geq \mu_j \quad \text{for } j = 1, \ldots, n \\ &\qquad 0 \leq u_t \ \leq p_t \quad \text{for } t = 1, \ldots, T \end{aligned} \tag{3.6}$$

with dimensionality $n \times (T+1)$. Hence, high computational efficiency is still guaranteed even for very large number of scenarios.

We have run two groups of computational tests. The medium scale tests of 5 000, 7 000 and 10 000 scenarios and 76 securities were generated following the FTSE 100 related data (Fabian *et al.*, 2009). The large scale tests instances developed by Lim *et al.* (2009) were generated from a multivariate normal distribution for 50 or 100

Table 1. Computational times (in seconds) for the primal models.

Scenarios (T)	Securities (n)	CVaR (3.1) with various tolerance levels β						Minimax (3.3)	MAD (3.5)
		0.05	0.1	0.2	0.3	0.4	0.5		
5 000	76	2.1	2.5	3.6	5.0	6.4	7.3	0.5	18.0
7 000	76	3.9	4.2	6.6	9.0	11.5	13.7	0.8	7.3
10 000	76	5.8	7.8	13.0	17.3	22.2	26.8	1.2	14.4
50 000	50	3173.4	4687.9	—	—	—	—	7.7	—
50 000	100	—	—	—	—	—	—	24.1	—

Table 2. Computational times (in seconds) for the dual models.

Scenarios (T)	Securities (n)	CVaR (3.1) with various tolerance levels β						Minimax (3.3)	MAD (3.5)
		0.05	0.1	0.2	0.3	0.4	0.5		
5 000	76	0.6	0.8	0.8	0.8	0.8	0.8	0.5	0.5
7 000	76	1.0	1.0	1.2	1.2	1.2	1.2	0.6	0.8
10 000	76	1.4	1.7	1.9	2.0	2.0	2.1	0.9	1.1
50 000	50	14.8	19.2	24.2	27.1	27.7	28.7	3.7	25.4
50 000	100	40.4	53.9	70.5	77.6	80.7	78.0	8.4	80.6

securities with the number of scenarios 50 000 just providing an adequate approximation to the underlying unknown continuous price distribution. All computations were performed on a PC with the Pentium 4 2.6GHz processor and 3GB RAM employing the simplex code of the CPLEX 9.1 package.

In Tables 1 and 2 there are presented computation times for all the above primal and dual models. All results are presented as the averages of 10 different test instances of the same size. For the medium scale test problems the solution times of the dual CVaR models (3.2) ranging from 0.6 to 2.1 seconds are not much shorter than those for the primal models ranging from 2.1 to 26.8 seconds, respectively. However, an attempt to solve the primal CVaR model (3.1) of the large scale test problems was successful only for $\beta = 0.05$ and $\beta = 0.1$ and the times were dramatically longer than those for the dual model. For other values of β the timeout of 6000 seconds occurred (marked with '–'). For 100 securities the primal model was not solvable within the given time limit, while the dual models could be successfully solved in 40.4 to 80.7 seconds.

The Minimax models are computationally very easy. Running the computational tests we were able to solve the medium scale test instances of the dual model (3.4) in times below 1 second and the large scale test instances in up to 8.4 seconds on average. In fact, even the primal model could be solved in reasonable time up to 24.1 seconds for large scale test instances.

The MAD models are computationally similar to the CVaR models. Indeed, only medium scale test instances of the primal model (3.5) could be solved within the given time limit. Much shorter computing times could be achieved for the dual MAD model (3.6) — not more than 1.1 second for the medium scale and 80.6 seconds for the large scale test instances.

3.2. *Mean-risk models*

Let us consider now LP computable risk measures maximization with additional preferential constraints as specified in model (2.11). Note that introducing a lower bound on the required expected return in the primal portfolio optimization model (3.1) result only in a single additional variable in the dual model (3.2). Indeed, following (3.1), the corresponding CVaR portfolio optimization model according to (2.11) can be formulated as the following LP problem:

$$
\begin{aligned}
&\max \eta - \frac{1}{\beta}\sum_{t=1}^{T} p_t d_t \\
&\text{s.t. } \sum_{j=1}^{n} \mu_j x_j \geq \mu_0, \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1,\dots,n \\
&\qquad d_t - \eta + \sum_{j=1}^{n} r_{jt} x_j \geq 0,\ d_t \geq 0 \quad \text{for } t = 1,\dots,T
\end{aligned}
\tag{3.7}
$$

where η is unbounded variable. Hence, its dimensionality is proportional to the number of scenarios T. The LP model (3.7) contains $T + n + 1$ variables and $T + 2$ constraints. In the case of several thousands scenarios this may result huge number of variables and constraints thus decreasing the computational efficiency of the model. The computational efficiency can easily be achieved by taking advantages of the LP dual to model (3.7) that takes the following form:

$$
\begin{aligned}
&\min q - \mu_0 u_0 \\
&\text{s.t. } q - \mu_j u_0 - \sum_{t=1}^{T} r_{jt} u_t \geq 0 \quad \text{for } j = 1,\dots,n \\
&\qquad \sum_{t=1}^{T} u_t = 1 \\
&\qquad 0 \leq u_t \leq \frac{p_t}{\beta} \quad \text{for } t = 1,\dots,T
\end{aligned}
\tag{3.8}
$$

The dual LP model contains $T + 1$ variables u_t, but the T constraints corresponding to variables d_t from (3.7) take the form of simple upper bounds on u_t (for $t = 1,\dots,T$) thus not affecting the problem complexity. Actually, the number of constraints in (3.8) is proportional to the total of portfolio size n, thus it is independent from the number of scenarios. Exactly, there are $T + 1$ variables and $n + 1$ constraints. This again guarantees a high computational efficiency of the dual model even for very large number of scenarios. Similarly, other portfolio structure requirements are modeled with rather small number of constraints thus generating small number of additional variables in the dual model.

Similar to the CVaR model, introducing a lower bound on the required expected return in the Minimax portfolio optimization model (3.9) result only in a single additional variable in the dual model. Indeed, the Minimax portfolio optimization model with mean return requirement can be written as the following simple LP problem:

$$\begin{aligned} &\max \eta \\ &\text{s.t. } \sum_{j=1}^{n} \mu_j x_j \geq \mu_0, \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \dots, n \\ &\qquad -\eta + \sum_{j=1}^{n} r_{jt} x_j \geq 0, \quad \text{for } t = 1, \dots, T \end{aligned} \tag{3.9}$$

Except from the portfolio weights x_j, the model contains only one additional variable η. However, its dimensionality is $(T + 2) \times (n + 1)$. On the other hand, its dual takes the following form:

$$\begin{aligned} &\min q - \mu_0 u_0 \\ &\text{s.t. } q - \mu_j u_0 - \sum_{t=1}^{T} r_{jt} u_t \geq 0 \quad \text{for } j = 1, \dots, n \\ &\qquad \sum_{t=1}^{T} u_t = 1 \\ &\qquad u_t \geq 0 \quad \text{for } t = 1, \dots, T \end{aligned} \tag{3.10}$$

with dimensionality $(n+1)\times(T+2)$. This guarantees a high computational efficiency of the dual model even for very large number of scenarios. The model from the dual CVaR model (3.8) by omitting the upper bounds.

When introducing a lower bound on the required expected return into the coherent version of the MAD risk measure ($\mu_\delta(\mathbf{x}) = \mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}) = \mathbb{E}\{\min\{\mu(\mathbf{x}), R_{\mathbf{x}}\}\}$) optimization we get:

$$\begin{aligned} &\max \sum_{j=1}^{n} \mu_j x_j - \sum_{t=1}^{T} p_t d_t \\ &\text{s.t. } \sum_{j=1}^{n} \mu_j x_j \geq \mu_0, \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \dots, n \\ &\qquad d_t - \sum_{j=1}^{n} (\mu_j - r_{jt}) x_j \geq 0, \ d_t \geq 0 \quad \text{for } t = 1, \dots, T \end{aligned} \tag{3.11}$$

with $T+n$ variables and $T+2$ constraints. The LP dual model takes then the form:

$$\begin{aligned} &\min\ q-\mu_0 u_0 \\ &\text{s.t.}\ \ q-\mu_j u_0+\sum_{t=1}^{T}(\mu_j-r_{jt})u_t \ge \mu_j \quad \text{for } j=1,\dots,n \\ &\qquad 0 \le u_t\ \le p_t \quad \text{for } t=1,\dots,T \end{aligned} \tag{3.12}$$

with dimensionality $n \times (T+2)$. Hence, there is again guaranteed the high computational efficiency even for very large number of scenarios.

We have repeated the computational test of Table 2 for the dual mean-risk models with the required return μ_0 defined as the expected return of the portfolio with equal weights (market value). As one can see in Table 3 the solution times for the dual mean-risk CVaR models (3.8) are only slightly longer than those for the basic dual CVaR models (3.2). The additional constraint does not increase also solution times for the dual Minimax model (3.10). For the MAD model, while additional expected return constraint (3.12) increased slightly the solution times for the medium scale test instances, the test problems with 50 000 scenarios could be solved in 25.8 seconds on average for 50 securities and in 76.7 seconds for 100 instruments, respectively, which is about the same as for the original dual models (3.6).

To see how the value of the required expected return affects the solution times we have performed additional tests for the dual CVaR (with $\beta = 0.1$), Minimax and MAD models. This time μ_0 value (increased value) was set in the halfway between the expected return of the market value and the maximum possible return for single security portfolio. The results for one problem size of the medium scale problems

Table 3. Computational times (in seconds) for the dual mean-risk models.

Scenarios (T)	Securities (n)	CVaR (3.1) with various tolerance levels β						Minimax (3.3)	MAD (3.5)
		0.05	0.1	0.2	0.3	0.4	0.5		
5 000	76	0.9	0.9	1.1	1.3	1.4	1.6	0.5	2.1
7 000	76	1.2	1.4	1.8	2.0	2.2	2.3	0.7	3.1
10 000	76	2.0	2.3	2.9	3.4	3.9	4.0	1.0	10.8
50 000	50	14.9	19.4	24.0	26.8	28.2	27.9	3.9	25.8
50 000	100	40.0	54.6	68.7	77.7	78.2	78.8	8.2	76.7

Table 4. Computational times (in seconds) for the dual CVaR ($\beta = 0.1$), Minimax and MAD models with different required expected return value μ_0.

Scenarios	Securities	μ_0	CVaR	Minimax	MAD
10 000	76	no constraint	1.7	0.9	1.1
		market value	2.3	1.0	10.8
		increased value	2.3	1.1	13.9
50 000	100	no constraint	53.9	8.4	80.6
		market value	54.6	8.2	76.7
		increased value	49.9	10.50	55.4

(10 000 scenarios) and one of the large scale problems (100 securities) are shown in Table 4. The computational times are generally comparable with those for the market value constraints. One may notice a drop in computation times for large scale CVaR and MAD models with increased required expected return.

4. Gini's Mean Difference and Related Models

Yitzhaki (1982) introduced the portfolio optimization model using Gini's mean difference (GMD) as risk measure. The GMD is given as $\Gamma(\mathbf{x}) = \frac{1}{2}\int\int|\eta-\xi|dF_{\mathbf{x}}(\eta)dF_{\mathbf{x}}(\xi)$ although several alternative formulae exist. For a discrete random variable represented by its realizations y_t, the measure $\Gamma(\mathbf{x}) = \sum_{t'=1}^{T}\sum_{t''\neq t'-1}\max\{y_{t'}-y_{t''},0\}p_{t'}p_{t''}$ is LP computable (when minimized) leading to the following portfolio optimization model:

$$\begin{aligned} &\max -\sum_{t=1}^{T}\sum_{t'\neq t}p_t p_{t'} d_{tt'} \\ &\text{s.t.} \ \sum_{j=1}^{n}\mu_j x_j \geq \mu_0, \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j=1,\dots,n \\ &\qquad d_{tt'} \geq \sum_{j=1}^{n} r_{jt}x_j - \sum_{j=1}^{n} r_{jt'}x_j,\ d_{tt'} \geq 0 \quad \text{for } t,t'=1,\dots,T;\ t\neq t' \end{aligned} \tag{4.1}$$

which contains $T(T-1)$ nonnegative variables $d_{tt'}$ and $T(T-1)$ inequalities to define them. This generates a huge LP problem even for the historical data case where the number of scenarios is 100 or 200. Krzemienowski and Ogryczak (2005) have shown with the earlier experiments that the CPU time of 7 seconds on average for $T=52$ has increased to above 30 sec. with $T=104$ and even more than 180 sec. for $T=156$. However, similar to the CVaR models, variables $d_{tt'}$ are associated with the singleton coefficient columns. Hence, while solving the dual instead of the original primal, the corresponding dual constraints take the form of simple upper bounds (SUB) which are handled implicitly outside the LP matrix. For the simplest form of the feasible set (2.1) the dual GMD model takes the following form:

$$\begin{aligned} &\min q - \mu_0 u_0 \\ &\text{s.t.} \ \ q - \mu_j u_0 - \sum_{t=1}^{T}\sum_{t'\neq t}(r_{jt}-r_{jt'})u_{tt'} \geq 0 \quad \text{for } j=1,\dots,n \\ &\qquad 0 \leq u_{tt'} \ \leq p_t p_{t'} \quad \text{for } t,t'=1,\dots,T; t\neq t' \end{aligned} \tag{4.2}$$

where original portfolio variables x_j are dual prices to the inequalities. The dual model contains $T(T-1)$ variables $u_{tt'}$ but the number of constraints (excluding the SUB structure) $n+1$ is proportional to the number of securities. The above dual formulation can be further simplified by introducing variables:

$$\bar{u}_{tt'} = u_{tt'} - u_{t't} \quad \text{for } t, t' = 1, \ldots, T; t < t' \tag{4.3}$$

which allows us to reduce the number of variables to $T(T-1)/2$ by replacing (4.2) with the following:

$$\begin{aligned} &\min\ q - \mu_0 u_0 \\ &\text{s.t. } \ v - \mu_j u_0 - \sum_{t=1}^{T} \sum_{t'>t} (r_{jt} - r_{jt'}) \bar{u}_{tt'} \geq 0 \quad \text{for } j = 1, \ldots, n \\ &\qquad -p_t p_{t'} \leq \bar{u}_{tt'} \ \leq p_t p_{t'} \quad \text{for } t, t' = 1, \ldots, T; t < t' \end{aligned} \tag{4.4}$$

Such a dual approach may dramatically improve the LP model efficiency in the case of larger number of scenarios. Actually, as shown with the earlier experiments of Krzemienowski and Ogryczak (2005), the above dual formulations let us to reduce the optimization time below 10 seconds for $T = 104$ and $T = 156$. Nevertheless, the case of really large number of scenarios still may cause computational difficulties, due to huge number of variables ($T(T-1)/2$). This may require some column generation techniques (Desrosiers and Luebbecke, 2005) or nondifferentiable optimization algorithms (Lim *et al.*, 2009).

As shown by Yitzhaki (1982) for the SSD consistency of the GMD model one needs to maximize the complementary measure

$$\mu_\Gamma(\mathbf{x}) = \mu(\mathbf{x}) - \Gamma(\mathbf{x}) = \mathbb{E}\{R_\mathbf{x} \wedge R_\mathbf{x}\} \tag{4.5}$$

where the cumulative distribution function of $R_\mathbf{x} \wedge R_\mathbf{x}$ for any $\eta \in \mathbb{R}$ is given as $F_\mathbf{x}(\eta)(2 - F_\mathbf{x}(\eta))$. Hence, (4.5) is the expectation of the minimum of two independent identically distributed random variables (i.i.d.r.v.) $R_\mathbf{x}$ thus representing the *mean worse return*. This provides us with another LP model although it is not more compact than that of (4.1) and its dual (4.2). Alternatively, the GMD may be expressed with integral of the absolute Lorenz curve as

$$\Gamma(\mathbf{x}) = 2\int_0^1 (\alpha\mu(\mathbf{x}) - F_\mathbf{x}^{(-2)}(\alpha))d\alpha = 2\int_0^1 \alpha(\mu(\mathbf{x}) - M_\alpha(\mathbf{x}))d\alpha$$

and respectively

$$\mu_\Gamma(\mathbf{x}) = \mu(\mathbf{x}) - \Gamma(\mathbf{x}) = 2\int_0^1 F_\mathbf{x}^{(-2)}(\alpha)d\alpha = 2\int_0^1 \alpha M_\alpha(\mathbf{x})d\alpha \tag{4.7}$$

thus combining all the CVaR measures. In order to enrich the modeling capabilities, one may treat differently some more or less extreme events. In order to model downside risk aversion, instead of the Gini's mean difference, the *tail Gini's* measure introduced by Ogryczak and Ruszczyński (2002, 2002a) can be used:

$$\mu_{\Gamma_\beta}(\mathbf{x}) = \mu(\mathbf{x}) - \frac{2}{\beta^2}\int_0^\beta (\mu(\mathbf{x})\alpha - F_\mathbf{x}^{(-2)}(\alpha))d\alpha = \frac{2}{\beta^2}\int_0^\beta F_\mathbf{x}^{(-2)}(\alpha)d\alpha \tag{4.8}$$

In the simplest case of equally probable T scenarios with $p_t = 1/T$ (historical data for T periods), the tail Gini's measure for $\beta = K/T$ may be expressed as the weighted combination of CVaRs $M_{\beta_k}(\mathbf{x})$ with tolerance levels $\beta_k = k/T$ for $k = 1, 2, \ldots, K$ and properly defined weights (Ogryczak and Ruszczyński, 2002a). In a general case, we may resort to an approximation based on some reasonably chosen grid β_k, $k = 1, \ldots, m$ and weights w_k expressing the corresponding trapezoidal approximation of the integral in the formula (4.8). Exactly, for any $0 < \beta \leq 1$, while using the grid of m tolerance levels $0 < \beta_1 < \cdots < \beta_k < \cdots < \beta_m = \beta$ one may define weights:

$$w_k = \frac{(\beta_{k+1} - \beta_{k-1})\beta_k}{\beta^2}, \quad \text{for } k = 1, \ldots, m-1, \quad \text{and} \quad w_m = \frac{\beta - \beta_{m-1}}{\beta} \tag{4.9}$$

where $\beta_0 = 0$. This leads us to the Weighted CVaR (WCVaR) measure (Mansini *et al.*, 2007) defined as

$$M_{\mathbf{w}}^{(m)}(\mathbf{x}) = \sum_{k=1}^{m} w_k M_{\beta_k}(\mathbf{x}), \quad \sum_{k=1}^{m} w_k = 1, \quad w_k > 0 \quad \text{for } k = 1, \ldots, m \tag{4.10}$$

We emphasize that despite being only an approximation to (4.8), any WCVaR measure itself is a well defined LP computable measure with guaranteed SSD consistency and coherency, as a combination of the CVaR measures. Hence, it needs not to be built on a very dense grid to provide proper modeling of risk averse preferences. While analyzed on the real-life data from the Milan Stock Exchange the weighted CVaR models have usually performed better than the GMD itself, the Minimax or the extremal CVaR models (Mansini *et al.*, 2007).

Here we analyze only computational efficiency of the LP models representing the WCVaR portfolio optimization. For returns represented by their realizations we get the following LP optimization problem:

$$\begin{aligned} &\max \sum_{k=1}^{m} w_k \eta_k - \sum_{k=1}^{m} \frac{w_k}{\beta_k} \sum_{t=1}^{T} p_t d_{tk} \\ &\text{s.t.} \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \ldots, n \\ &\qquad d_{tk} - \eta_k + \sum_{j=1}^{n} r_{jt} x_j \geq 0, \; d_{tk} \geq 0 \quad \text{for } t = 1, \ldots, T; \; k = 1, \ldots, m \end{aligned} \tag{4.11}$$

where η_k (for $k = 1, \ldots, m$) are unbounded variables taking the values of the corresponding β_k-quantiles (in the optimal solution). Except from the core portfolio constraints (2.1), model (4.11) contains T nonnegative variables d_{tk} and T corresponding linear inequalities for each k. Hence, its dimensionality is proportional to the number of scenarios T and to the number of tolerance levels m. Exactly, the LP model contains $m \times T + n$ variables and $m \times T + 1$ constraints. It does not cause any computational difficulties for a few hundreds scenarios and a few tolerance levels, as in a simple computational analysis based on historical data (Mansini *et al.*, 2007). However, in the case of more advanced simulation models employed for scenario generation one may get several thousands scenarios. This may lead to the LP model (4.11) with huge number of variables and constraints thus decreasing the computational efficiency of the model. If the core portfolio constraints contain only linear relations, like (2.1), then the computational efficiency can easily be achieved by taking advantages of the LP dual to model (4.11). The LP dual model takes the following form:

$$\begin{aligned} &\min q \\ &\text{s.t.}\;\; q - \sum_{t=1}^{T} r_{jt} \sum_{k=1}^{m} u_{tk} \geq 0 \quad \text{for } j = 1, \ldots, n \\ &\qquad \sum_{t=1}^{T} u_{tk} = w_k \quad \text{for } k = 1, \ldots, m \\ &\qquad 0 \leq u_{tk} \leq \frac{p_t w_k}{\beta_k} \quad \text{for } t = 1, \ldots, T;\; k = 1, \ldots, m \end{aligned} \tag{4.12}$$

The dual LP model contains $m \times T$ variables u_{tk}, but the $m \times T$ constraints corresponding to variables d_{tk} from (4.11) take the form of simple upper bounds (SUB) on u_{tk} thus not affecting the problem complexity. Actually, the number of constraints in (4.12) is proportional to the total of portfolio size n and the number of tolerance levels m, thus it is independent from the number of scenarios. Exactly, there are $m \times T + 1$ variables and $m + n$ constraints. This guarantees a high computational efficiency of the dual model even for very large number of scenarios.

Similar to the CVaR model, introducing a lower bound on the required expected return in the primal portfolio optimization model (4.11) results only in a single additional variable in the dual model (4.12). Indeed, when introducing a lower bound on the required expected return into the WCVaR model we get:

$$\begin{aligned} &\max \sum_{k=1}^{m} w_k \eta_k - \sum_{k=1}^{m} \frac{w_k}{\beta_k} \sum_{t=1}^{T} p_t d_{tk} \\ &\text{s.t.}\;\; \sum_{j=1}^{n} \mu_j x_j \geq \mu_0, \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \ldots, n \\ &\qquad d_{tk} - \eta_k + \sum_{j=1}^{n} r_{jt} x_j \geq 0,\; d_{tk} \geq 0 \quad \text{for } t = 1, \ldots, T;\; k = 1, \ldots, m \end{aligned} \tag{4.13}$$

Its dimensionality is proportional to the number of scenarios T and to the number of tolerance levels m. Exactly, now the LP model contains $m \times T + n$ variables and $m \times T + 2$ constraints. In the case of large number of scenarios the computational efficiency can easily be improved by taking advantages of the LP dual to

Table 5. Computational times (in seconds) for the dual WCVaR models.

Scenarios (T)	Securities (n)	Model (4.12)		Model (4.14)	
		$m = 3$	$m = 5$	$m = 3$	$m = 5$
5 000	76	3.3	7.4	6.2	13.8
7 000	76	5.4	11.7	10.0	22.8
10 000	76	9.2	20.5	16.2	37.7
50 000	50	123.7	285.1	121.8	281.0
50 000	100	335.1	731.0	335.3	731.7

model (4.13):

$$\begin{aligned} &\min\ q - \mu_0 u_0 \\ &\text{s.t.}\quad q - \mu_j u_0 - \sum_{t=1}^{T} r_{jt} \sum_{k=1}^{m} u_{tk} \geq 0 \quad \text{for } j = 1, \ldots, n \\ &\qquad \sum_{t=1}^{T} u_{tk} = w_k \quad \text{for } k = 1, \ldots, m \\ &\qquad 0 \leq u_{tk} \leq \frac{p_t w_k}{\beta_k} \quad \text{for } t = 1, \ldots, T;\ k = 1, \ldots, m \end{aligned} \tag{4.14}$$

that contains $m \times T$ variables u_{tk}, but the $m \times T$ constraints corresponding to variables d_{tk} from (4.13) take the form of simple upper bounds on u_{tk} thus not affecting the problem complexity. Hence, again the number of constraints in (4.14) is proportional to the total of portfolio size n and the number of tolerance levels m, thus it is independent from the number of scenarios. Exactly, there are $m \times T + 2$ variables and $m + n$ constraints thus guaranteeing a high computational efficiency of for very large number of scenarios.

We have tested computational efficiency of the dual models (4.12) and (4.14) using the same randomly generated test instances as for testing of the CVaR and other basic models in Sec 3. Table 5 presents average computation times of the dual models for $m = 3$ with tolerance levels $\beta_1 = 0.1, \beta_2 = 0.25, \beta_3 = 0.5$ and weights $w_1 = 0.1, w_2 = 0.4$ and $w_3 = 0.5$, thus representing the parameters leading to good results on real life data (Mansini *et al.*, 2007), as well as for $m = 5$ with uniformly distributed tolerance levels $\beta_1 = 0.1, \beta_2 = 0.2, \beta_3 = 0.3, \beta_4 = 0.4, \beta_5 = 0.5$ and weights defined according to (4.9).

5. Concluding Remarks

The classical Markowitz model for portfolio selection using the variance as the risk measure is well suited for normal distributions of returns. However, many studies show that on various markets many stocks or other instruments do not follow normal distribution. This was revealed, in particular, for Japanese (Konno *et al.*, 2002) and Chinese (Chen and Wang, 2008) stock markets as well as for the Asian hedge funds (Hakamada *et al.*, 2007; Wong *et al.*, 2008). There were introduced several alternative risk measures which are appropriate for non-symmetric distributions and computationally attractive as (for discrete random variables) they result in solving linear programming (LP) problems. The LP solvability is very important for applications to real-life financial decisions where the constructed portfolios have to meet numerous side constraints and take into account transaction costs. A gamut of LP computable risk measures has been presented in the portfolio optimization literature although most of them are related to the absolute Lorenz curve and thereby the CVaR measures. We have shown that all the risk measures used in the LP solvable portfolio optimization models can be derived from the SSD shortfall criteria. This allows us to guarantee their SSD consistency for any distribution of outcomes.

The corresponding portfolio optimization models can be solved with general purpose LP solvers. However, in the case of more advanced simulation models employed for scenario generation one may get several thousands scenarios. This may lead to the LP model with huge number of variables and constraints thus decreasing the computational efficiency of the models. For the CVaR model, the number of constraints (matrix rows) is proportional to the number of scenarios. while the number of variables (matrix columns) is proportional to the total of the number of scenarios and the number of instruments. We have shown that the computational efficiency can be then dramatically improved with an alternative model taking advantages of the LP duality. In the introduced model the number of structural constraints (matrix rows) is proportional to the number of instruments thus not affecting seriously the simplex method efficiency by the number of scenarios. In particular, for the case of 50 000 scenarios, it has resulted in computation times below 30 seconds for 50 securities or below a minute for 100 instruments. Similar computational times have also been achieved for the dual reformulation of the MAD and the Minimax models.

Dual reformulation applied to the GMD portfolio optimization model results in a dramatic problem size reduction with the number of constraints equal to the number of instruments instead of the square of the number of scenarios. Although, the remaining high number of variables (square of the number of scenarios) still makes the problem computationally difficult for very large numbers of scenarios. This requires further research on column generation techniques or nondifferentiable optimization algorithms for the GMD model. On the other hand, the Weighted CVaR models approximating the tail Gini's measures after the dual reformulation have been quite efficiently solved even on 50 000 scenario instances with computation times below 5 minutes for 50 securities or about 10 minutes for 100 instruments.

This paper has been focused on computational efficiency of the LP computable portfolio optimization models and it has been shown that large numbers of scenarios do not cause any serious computational difficulties for the typical models. The optimal portfolio shares are not directly represented within the solution vector of the dual model problem but they are easily available as the dual variables (shadow prices) for respective inequalities. Moreover, the dual models may be considered a special case within the general theory of dual representations of coherent measures of risk (Ruszczyński and Shapiro, 2006; Miller and Ruszczyński, 2008) thus allowing variables to have the interpretation of probability distributions.

Certainly, possible wider usage of the LP computable portfolio optimization models still requires further research in many areas. The optimization results strongly depend on the quality of forecasted scenarios data and there is a need for powerful tools enabling effective forecasting while mining huge sets of financial data of various types (Li *et al.*, 2009). The LP computable portfolio optimization allows to model various risk averse preferences expressed with (primal or dual) utility functions (Ogryczak, 2002). The sensitivity analysis of models conducted with respect to utility functions rather than numeric values (Churilov *et al.*, 2004) seems to be a promising direction for further research.

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