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On Extending the LP Computable Risk Measures to Account Downside Risk*

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Abstract. A mathematical model of portfolio optimization is usually quantified with mean-risk models offering a lucid form of two criteria with possible trade-off analysis. In the classical Markowitz model the risk is measured by a variance, thus resulting in a quadratic programming model. Following Sharpe's work on linear approximation to the mean-variance model, many attempts have been made to linearize the portfolio optimization problem. There were introduced several alternative risk measures which are computationally attractive as (for discrete random variables) they result in solving linear programming (LP) problems. Typical LP computable risk measures, like the mean absolute deviation (MAD) or the Gini's mean absolute difference (GMD) are symmetric with respect to the below-mean and over-mean performances. The paper shows how the measures can be further combined to extend their modeling capabilities with respect to enhancement of the below-mean downside risk aversion. The relations of the below-mean downside stochastic dominance are formally introduced and the corresponding techniques to enhance risk measures are derived. The resulting mean-risk models generate efficient solutions with respect to second degree stochastic dominance, while at the same time preserving simplicity and LP computability of the original models. The models are tested on real-life historical data.

Keywords: portfolio optimization, mean-risk, stochastic dominance, downside risk, linear programming

1. Introduction

Following the seminal work by Markowitz [14], the portfolio selection problem is modeled as a mean-risk bicriteria optimization problem where the mean, representing the expected outcome, is maximized and the risk, a scalar measure of the variability of outcomes, is minimized. The classical Markowitz model uses the variance as the risk measure, but several other risk measures have been later considered thus creating the entire family of mean-risk (Markowitz type) models. While the original Markowitz model forms a quadratic programming problem, following the initial works on its linear programming (LP) approximation [23, 27], many attempts have been made to linearize the portfolio optimization procedure (c.f., [26] and references therein). The LP solvability is very important for applications to real-life financial decisions where the constructed portfolios have to meet numerous side constraints (including the minimum transaction lots [13]) and take into account transaction costs [11].

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The risk measures, although nonlinear, can be LP computable in the case of discrete random variables, i.e., in the case of returns defined by their realizations under the finite number of specified scenarios. This applies, in particular, to the mean absolute deviation from the mean. The mean absolute deviation was very early considered in the portfolio analysis ([24] and references therein) while quite recently Konno and Yamazaki [10] presented and analyzed the complete portfolio optimization model based on this risk measure—the so-called MAD model. Yitzhaki [29] introduced the mean-risk model using Gini's mean (absolute) difference as the risk measure (hereafter referred to as the GMD model).

The Markowitz mean-variance model provided the main theoretical background for the modern portfolio theory. Nevertheless, the optimization model itself is frequently criticized as not consistent with the stochastic dominance rules. The relation of stochastic dominance is one of the fundamental concepts of the decision theory (c.f. [12, 28]). Taking into account the axiomatic models of preferences for choice under risk, it introduces a partial order in the space of real random variables. While theoretically attractive, stochastic dominance order is computationally difficult, as a multiobjective model with a continuum of objectives. If the rates of return are multivariate normally distributed, then the MAD and the most of LP solvable mean-risk models are equivalent to the Markowitz mean-variance model which in this specific case is consistent with the stochastic dominance rules. However, the LP solvable mean-risk models do not require any specific type of return distributions. Opposite to the mean-variance approach, for general random variables some consistency with the second degree stochastic dominance (SSD) rules were shown for the Gini's mean difference [29] and for the MAD model [17].

The variance used as a risk measure causes that the variability of rate of return above the mean is penalized while the investors concern of an underperformance rather than the overperformance of a portfolio. This led Markowitz [15] to propose (below-mean) downside risk measures such as (downside) semivariance to replace variance as the risk measure. Many authors pointed out that the MAD model opens up opportunities for more specific modeling of the downside risk [3, 26]. Actually, the MAD model may easily be extended with some piece-wise linear penalty (risk) functions to provide opportunities for more specific modeling of the downside risk [9, 16]. In this paper we generalize the concept of recursive MAD measure [16] to introduce a construction which can be applied to various LP computable risk measures in order to extend their modeling capabilities with respect to enhancement of the below-mean downside risk aversion. This allows us to introduce a new risk measure representing the downside version of the Gini's mean difference.

The paper is organized as follows. In the next section we recall the basics of the stochastic dominance and mean-risk approaches. We also specify the meaning of consistency results for these two different approaches and discuss the specific consistency results for the MAD and GMD models. In Section 3 we analyze how the MAD and GMD risk measures can be further combined to enhance the below-mean downside risk aversion while preserving the SSD consistency of the original measures. For this purpose, we formally introduce the relations of the below-mean downside stochastic dominance and derive the corresponding techniques to enhance risk measures. The appropriate constructions are shown to be valid for various LP computable and SSD consistent risk measures but we focus our detailed analysis on the MAD and GMD measures. Section 4 provides a computational analysis of the our LP computable models comparing their performances on the asset allocation problem while using historical values of 81 sectorial S&P500 indices.

2. Stochastic dominance and mean-risk models

2.1. SD consistency concepts

The portfolio optimization problem considered in this paper follows the original Markowitz formulation and it is based on a single period model of investment. At the beginning of a period, an investor allocates the capital among various securities, thus assigning a nonnegative weight (share of the capital) to each security. During the investment period, a security generates a random rate of return. This results in a change of capital invested (observed at the end of the period) which is measured by the weighted average of the individual rates of return.

Let $J = \{1, 2, \ldots, n\}$ denote a set of securities considered for an investment. For each security $j \in J$, its rate of return is represented by a random variable R_j with a given mean $\mu_j = E\{R_j\}$. Further, let $\mathbf{x} = (x_j)_{j=1,2,\ldots,n}$ denote a vector of decision variables x_j expressing the weights defining a portfolio. To represent a portfolio, the weights must satisfy a set of constraints that form a feasible set $\mathcal{P}$. The simplest way of defining a feasible set is by a requirement that the weights must sum to one, i.e. $\sum_{j=1}^{n} x_j = 1$ and $x_j \geq 0$ for $j = 1, \ldots, n$. An investor usually needs to consider some other requirements expressed as a set of additional side constraints. Hereafter, it is assumed that $\mathcal{P}$ is a general LP feasible set given in a canonical form as a system of linear equations with nonnegative variables.

Each portfolio $\mathbf{x}$ defines a corresponding random variable $R_{\mathbf{x}} = \sum_{j=1}^{n} R_j x_j$ that represents a portfolio rate of return. The mean rate of return for portfolio $\mathbf{x}$ is given as:

$$\mu(\mathbf{x}) = \mathbb{E}\{R_{\mathbf{x}}\} = \sum_{j=1}^{n} \mu_j x_j$$

Following Markowitz [14], the portfolio optimization problem is modeled as a mean-risk bicriteria optimization problem

$$\max\{[\mu(\mathbf{x}), -\varrho(\mathbf{x})] : \quad \mathbf{x} \in \mathcal{P}\} \tag{1}$$

where the mean $\mu(\mathbf{x})$ is maximized and the risk measure $\varrho(\mathbf{x})$ is minimized. A feasible portfolio $\mathbf{x}^0 \in \mathcal{P}$ is called the efficient solution of problem (1) or the *μ/ϱ-efficient* portfolio if there is no $\mathbf{x} \in \mathcal{P}$ such that $\mu(\mathbf{x}) \geq \mu(\mathbf{x}^0)$ and $\varrho(\mathbf{x}) \leq \varrho(\mathbf{x}^0)$ with at least one inequality strict.

In the original Markowitz model [14] the risk is measured by the standard deviation or variance: $\sigma^2(\mathbf{x}) = \mathbb{E}\{(\mu(\mathbf{x}) - R_{\mathbf{x}})^2\}$. Several other risk measures have been later considered thus creating the entire family of mean-risk models. We restrict our analysis to the class of Markowitz-type mean-risk models where risk measures, similar to the standard deviation, are translation invariant and risk relevant dispersion parameters. Thus the risk measures, we consider, are not affected by any shift of the outcome scale and they are equal to 0 in the case of a risk-free portfolio while taking positive values for any risky portfolio. Moreover, in order to model possible taking advantages of a portfolio diversification, risk measure $\varrho(\mathbf{x})$ should be a convex function of $\mathbf{x}$.

The Markowitz model is frequently criticized as not consistent with axiomatic models of preferences for choice under risk [20]. Namely, except for the case of returns meeting the multivariate normal distribution, the mean-variance model may lead to inferior conclusions with respect to the stochastic dominance order. The concept of stochastic dominance order [28] is based on an axiomatic model of risk-averse preferences [20]. In stochastic dominance, uncertain returns (random variables) are compared by pointwise comparison of some performance functions constructed from their distribution functions. The first performance function $F_{\mathbf{x}}^{(1)}$ is defined as the right-continuous cumulative distribution function: $F_{\mathbf{x}}^{(1)}(\eta) = F_{\mathbf{x}}(\eta) = \mathbb{P}\{R_{\mathbf{x}} \leq \eta\}$ and it defines the first degree stochastic dominance (FSD). The second function is derived from the first as:

$$F_{\mathbf{x}}^{(2)}(\eta) = \int_{-\infty}^{\eta} F_{\mathbf{x}}(\xi)\, d\xi \quad \text{for real numbers } \eta,$$

and it defines the (weak) relation of *second degree stochastic dominance* (SSD):

$$R_{\mathbf{x}'} \succeq_{_{SSD}} R_{\mathbf{x}''} \Leftrightarrow F_{\mathbf{x}'}^{(2)}(\eta) \leq F_{\mathbf{x}''}^{(2)}(\eta) \quad \text{for all } \eta.$$

We say that portfolio $\mathbf{x}'$ *dominates* $\mathbf{x}''$ *under the SSD* ($R_{\mathbf{x}'} \succ_{_{SSD}} R_{\mathbf{x}''}$), if $F_{\mathbf{x}'}^{(2)}(\eta) \leq F_{\mathbf{x}''}^{(2)}(\eta)$ for all η, with at least one strict inequality. A feasible portfolio $\mathbf{x}^0 \in \mathcal{P}$ is called *SSD efficient* if there is no $\mathbf{x} \in \mathcal{P}$ such that $R_{\mathbf{x}} \succ_{_{SSD}} R_{\mathbf{x}^0}$. If $R_{\mathbf{x}'} \succ_{_{SSD}} R_{\mathbf{x}''}$, then $R_{\mathbf{x}'}$ is preferred to $R_{\mathbf{x}''}$ within all risk-averse preference models where larger outcomes are preferred. It is therefore a matter of primary importance that a model for portfolio optimization be consistent with the SSD relation, which implies that the optimal portfolio is SSD efficient.

The Markowitz model is not SSD consistent since its efficient set may contain SSD inferior portfolios characterized by a small risk but also very low return (c.f. [17] and references therein). Unfortunately, it is a common flaw of all Markowitz-type mean-risk models where risk is measured with some dispersion measures. Although, the necessary condition for the SSD relation is [5]

$$R_{\mathbf{x}'} \succeq_{_{SSD}} R_{\mathbf{x}''} \Rightarrow \mu(\mathbf{x}') \geq \mu(\mathbf{x}'') \tag{2}$$

this is not enough to guarantee the μ/ϱ dominance, due to the lack of similar consistency relation

$$R_{\mathbf{x}'} \succeq_{_{SSD}} R_{\mathbf{x}''} \Rightarrow \varrho(\mathbf{x}') \leq \varrho(\mathbf{x}'')$$

for the risk measures. For dispersion type risk measures $\varrho(\mathbf{x})$, it may occur that $R_{\mathbf{x}'} \succeq_{_{SSD}} R_{\mathbf{x}''}$ and simultaneously $\varrho(\mathbf{x}') > \varrho(\mathbf{x}'')$. In order to overcome this flaw of the Markowitz model, already Baumol [1] suggested to consider a safety measure, he called the expected gain-confidence limit criterion, $\mu(\mathbf{x}) - \lambda\sigma(\mathbf{x})$ to be maximized instead of the minimization of $\sigma(\mathbf{x})$ itself. Similar approach was introduced by Yitzhaki [29] with respect to the Gini's mean difference used as a risk measure. Hereafter, for any dispersion type risk measure $\varrho(\mathbf{x})$, the function $s(\mathbf{x}) = \mu(\mathbf{x}) - \varrho(\mathbf{x})$ will be referred to as the corresponding *safety* measure. Note that risk measures, we consider, are defined as translation invariant and risk relevant dispersion parameters. Hence, the corresponding safety measures are translation equivariant in the sense that any shift of the outcome scale results in an equivalent change of the safety measure value (with opposite sign safety measures are maximized), or in other words, the safety measures distinguish (and order) various risk-free portfolios (outcomes) according to their values. The safety measures, we consider, are risk relevant but in the sense that the value of a safety measure for any risky portfolio is less than the value for the risk-free portfolio with the same expected returns. Moreover, when risk measure $\varrho(\mathbf{x})$ is a convex function of $\mathbf{x}$, then the corresponding safety measure $s(\mathbf{x})$ is concave.

The SSD consistency of the safety measures may be formalized as follows. We say that the safety measure $\mu(\mathbf{x}) - \varrho(\mathbf{x})$ is *SSD consistent* or that the risk measure $\varrho(\mathbf{x})$ is *SSD safety consistent* if

$$R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''} \Rightarrow \mu(\mathbf{x}') - \varrho(\mathbf{x}') \geq \mu(\mathbf{x}'') - \varrho(\mathbf{x}'') \tag{3}$$

The relation of SSD (safety) consistency is called *strong* if, in addition to (3), the following holds

$$R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}''} \Rightarrow \mu(\mathbf{x}') - \varrho(\mathbf{x}') > \mu(\mathbf{x}'') - \varrho(\mathbf{x}'') \tag{4}$$

The following assertion [19] explains the safety consistency in terms of the bicriteria optimization.

Theorem 1. *If the risk measure $\varrho(\mathbf{x})$ is SSD safety consistent* (3), *then except for portfolios with identical values of $\mu(\mathbf{x})$ and $\varrho(\mathbf{x})$, every efficient solution of the bicriteria problem*

$$\max\{[\mu(\mathbf{x}), \mu(\mathbf{x}) - \varrho(\mathbf{x})] : \quad \mathbf{x} \in \mathcal{P}\} \tag{5}$$

is an SSD efficient portfolio. In the case of strong SSD safety consistency (4), *every portfolio* $\mathbf{x} \in \mathcal{P}$ *efficient to* (5) *is, unconditionally, SSD efficient.*

Following Theorem 1, one may consider the mean-safety bicriteria model (5) as a reasonable alternative to the corresponding mean-risk model (1). Note that a portfolio dominated in the mean-risk model (1) is also dominated in the corresponding mean-safety model (5). Hence, the efficient portfolios of problem (5) form a subset of the entire μ/ϱ–efficient set. The minimum risk portfolio (MRP), defined as the solution of $\min_{\mathbf{x}\in\mathcal{P}} \varrho(\mathbf{x})$, limits the curve to the mean-risk efficient frontier. Similar, the maximum safety portfolio (MSP), defined as the solution of $\max_{\mathbf{x}\in\mathcal{P}}[\mu(\mathbf{x}) - \varrho(\mathbf{x})]$, distinguishes a part of the mean-risk efficient frontier, which is also mean-safety efficient. By virtue of Theorem 1, in the case of a SSD safety consistent risk measure, this part of the efficient frontier represents portfolios which are SSD efficient.

2.2. *Basic LP computable risk measures*

While the original Markowitz model forms a quadratic programming problem, following Sharpe [23], many attempts have been made to linearize the portfolio optimization procedure (c.f., [26] and references therein). Certainly, to model advantages of a diversification, risk measures cannot be linear function of $\mathbf{x}$. Nevertheless, the risk measure can be LP computable in the case of discrete random variables, i.e., in the case of returns defined by their realizations under the specified scenarios. We will consider T scenarios with probabilities p_t (where $t = 1, \dots, T$). We will assume that for each random variable R_j there is known its realization r_{jt} under the scenario t. Typically, the realizations are derived from historical data treating T historical periods as equally probable scenarios ($p_t = 1/t$). The realizations of the portfolio returns $R_{\mathbf{x}}$ are given as

$$y_t = r_t(\mathbf{x}) = \sum_{j=1}^{n} r_{jt} x_j \tag{6}$$

and the expected value $\mu(\mathbf{x})$ can be computed as:

$$\mu(\mathbf{x}) = \sum_{t=1}^{T} y_t p_t = \sum_{t=1}^{T} \left[p_t \sum_{j=1}^{n} r_{jt} x_j \right]$$

Similarly, several risk measures can be expressed as convex piece-wise linear functions of y_t. Such measures are then LP computable with respect to the realizations y_t and, due to (6), with respect to $\mathbf{x}$.

Function $F_{\mathbf{x}}^{(2)}$, used to define the SSD relation can also be presented as follows [17]:

$$F_{\mathbf{x}}^{(2)}(\eta) = \mathbb{P}\{R_{\mathbf{x}} \leq \eta\} \mathbb{E}\{\eta - R_{\mathbf{x}} | R_{\mathbf{x}} \leq \eta\} = \mathbb{E}\{\max\{\eta - R_{\mathbf{x}}, 0\}\}$$

Hence, the SSD relation is the Pareto dominance for mean below-target deviations from infinite number (continuum) of targets. As a convex piece-wise linear function, it is LP computable for returns represented by their realizations y_t. Consequently, one can observe the growing popularity of the mean return—downside risk portfolio selection models [7] using various Lower Partial Moments for specific return targets. There is no universal risk measure equally good for all broad categories of risk and thus there is a need for caution while using one [25]. The below-target deviations are very useful in investment situations with clearly defined minimum acceptable returns (e.g. bankruptcy level) [4]. However, they do not represent any dispersion-type risk measure to be considered in the Markowitz-type mean-risk model. In particular, the below-target deviation may be equal to 0 for various risky portfolios thus violating the risk relevance requirement.

The simplest dispersion-type LP computable risk measures may be viewed as some approximations to the variance itself generated by the use of absolute values replacing the squares. In particular the *mean absolute deviation* from the mean is defined as:

$$\delta(\mathbf{x}) = \mathbb{E}\{|R_{\mathbf{x}} - \mu(\mathbf{x})|\} \tag{7}$$

For a discrete random variable represented by its realizations y_t, the mean absolute deviation (7) is a convex piece-wise linear function of **x** and thereby it is LP computable. Konno and Yamazaki [10] presented and analyzed the complete portfolio optimization model (MAD model) based on the risk measure defined as mean absolute deviation. Earlier, Yitzhaki [29] introduced the mean-risk model using Gini's mean (absolute) difference as the risk measure. For a discrete random variable represented by its realizations y_t, the *Gini's mean difference*

$$\Gamma(\mathbf{x}) = \frac{1}{2}\sum_{t=1}^{T}\sum_{t'=1}^{T} |y_{t'} - y_t| p_{t'} p_t \tag{8}$$

is obviously LP computable (when minimized). Note that formula (8) is similar to the alternative formula for variance $\sigma^2(\mathbf{x}) = \frac{1}{2}\sum_{t=1}^{T}\sum_{t'=1}^{T}(y_{t'} - y_t)^2 p_{t'} p_t$.

The mean absolute deviation is closely related to the function $F_{\mathbf{x}}^{(2)}$. Namely, the use of the mean expected return as an argument results in the risk measure known as the downside mean semideviation from the mean

$$\bar{\delta}(\mathbf{x}) = \mathbb{E}\{\max\{\mu(\mathbf{x}) - R_{\mathbf{x}}, 0\}\} = F_{\mathbf{x}}^{(2)}(\mu(\mathbf{x})) \tag{9}$$

The downside mean semideviation is always equal to the upside one [26] and, therefore, we will call it simply the *mean semideviation*. Certainly, the mean semideviation is a half of the mean absolute deviation from the mean, i.e. $\delta(\mathbf{x}) = 2\bar{\delta}(\mathbf{x})$. Hence the corresponding mean-risk model is equivalent to the MAD model. For a discrete random variable represented by its realizations y_t, minimization of the mean semideviation

$$\bar{\delta}(\mathbf{x}) = \frac{1}{2}\sum_{t=1}^{T}\left|\sum_{t'=1}^{T} y_{t'} p_{t'} - y_t\right| p_t = \sum_{t=1}^{T}\max\left\{\sum_{t'=1}^{T} y_{t'} p_{t'} - y_t,\ 0\right\} p_t \tag{10}$$

may be implemented with the LP models [3, 26].

As shown in [17], the mean semideviation is SSD safety consistent, i.e.

$$R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''} \Rightarrow \mu(\mathbf{x}') - \bar{\delta}(\mathbf{x}') \geq \mu(\mathbf{x}'') - \bar{\delta}(\mathbf{x}'') \tag{11}$$

Note that the corresponding safety measure can be expressed as

$$\mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}) = \mathbb{E}\{\mu(\mathbf{x}) - \max\{\mu(\mathbf{x}) - R_{\mathbf{x}}, 0\}\} = \mathbb{E}\{\min\{R_{\mathbf{x}}, \mu(\mathbf{x})\}\} \tag{12}$$

thus representing the *mean downside underachievement*.

The Gini's mean difference may be also expressed as the integral of $F_{\mathbf{x}}^{(2)}$ with respect to the probability measure induced by $R_{\mathbf{x}}$. In the case of discrete distributions we consider one gets

$$\Gamma(\mathbf{x}) = \sum_{t=1}^{T}\left[\sum_{t':y_{t'}<y_t} (y_t - y_{t'}) p_{t'}\right] p_t = \sum_{t=1}^{T} F_{\mathbf{x}}^{(2)}(y_t) p_t$$

Hence, $\Gamma(\mathbf{x})$ can be interpreted as the weighted sum of multiple mean below-target deviations but both the targets and the weights are distribution dependent.

Both Γ and $\bar{\delta}$ are well defined characteristics in the dual (quantile) model of the stochastic dominance [18, 19, 22]. However, the absolute semideviation is a rather rough measure compared to the Gini's mean difference. Note that $\bar{\delta}(\mathbf{x}) \leq \Gamma(\mathbf{x})$ and $\bar{\delta}(\mathbf{x})$ may be also interpreted as the Gini's mean difference of a two-point distribution approximating $R_{\mathbf{x}}$ in such a way that $\mu(\mathbf{x})$ and $\delta(\mathbf{x})$ remain unchanged. The Gini's mean difference is not only SSD safety consistent [29] but it satisfies also the requirements of the strong SSD safety consistency [18], i.e.

$$R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}''} \Rightarrow \mu(\mathbf{x}') - \Gamma(\mathbf{x}') > \mu(\mathbf{x}'') - \Gamma(\mathbf{x}'') \tag{13}$$

Note, that the corresponding safety measure

$$\mu(\mathbf{x}) - \Gamma(\mathbf{x}) = \mathbb{E}\{R_{\mathbf{x}} \wedge R_{\mathbf{x}}\} \tag{14}$$

expresses the expectation of the minimum of two i.i.d.r.v. $R_{\mathbf{x}}$ [29] thus representing the *mean worse return*. Actually, formula (14) provides an alternative way to compute the Gini's mean difference.

In multiple criteria optimization, the weighted sum is the simplest combination of several criteria. The weighted sum may be also used to combine dispersion type risk measures. Consider a set, say m, risk measures $\varrho_k(\mathbf{x})$ and their linear combination:

$$\varrho_{\mathbf{w}}^{(m)}(\mathbf{x}) = \sum_{k=1}^{m} w_k \varrho_k(\mathbf{x}), \qquad \sum_{k=1}^{m} w_k \leq 1, \quad w_k > 0 \quad \text{for } k = 1, \ldots, m \tag{15}$$

Note that

$$\mu(\mathbf{x}) - \varrho_{\mathbf{w}}^{(m)}(\mathbf{x}) = w_0 \mu(\mathbf{x}) + \sum_{k=1}^{m} w_k (\mu(\mathbf{x}) - \varrho_k(\mathbf{x}))$$

where $w_0 = 1 - \sum_{k=1}^{m} w_k \geq 0$. Hence, the following assertion is valid.

Theorem 2. *If all risk measures ϱ_k are SSD safety consistent, then every combined risk measure* (15) *is also SSD safety consistent in the sense that*

$$R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''} \Rightarrow \mu(\mathbf{x}') - \varrho_{\mathbf{w}}^{(m)}(\mathbf{x}') \geq \mu(\mathbf{x}'') - \varrho_{\mathbf{w}}^{(m)}(\mathbf{x}'')$$

Moreover, if at least one measure ϱ_{k_o} is strongly SSD safety consistent, then every combined risk measure (15) *is strongly SSD safety consistent.*

Theorem 2 allows us to combine various risk measure preserving their SSD consistency properties. Note that the combined risk measure (15), as a positive linear combination, is a convex piece-wise linear function of portfolio $\mathbf{x}$ whenever all the original risk measures

$\varrho_k(\mathbf{x})$ are such functions. Hence, in the case of returns represented by their realizations y_t, the weighted combination of LP computable risk measures remains LP computable.

The weighted combination allows us to regularize any SSD (safety) consistent risk measure to achieve the strong consistency. Recall that the Gini's mean difference satisfies the strong SSD (safety) consistency. According to Theorem 2, combining any risk measure with the Gini's mean difference results in a new measure satisfying the strong SSD consistency.

Theorem 3. *For any SSD safety consistent risk measure $\varrho(\mathbf{x})$ and any $0 < \varepsilon < 1$, the combined risk measure*

$$\varrho^{\varepsilon}(\mathbf{x}) = (1-\varepsilon)\varrho(\mathbf{x}) + \varepsilon\Gamma(\mathbf{x}) \tag{16}$$

is strongly SSD safety consistent in the sense that

$$R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}''} \Rightarrow \mu(\mathbf{x}') - \varrho^{\varepsilon}(\mathbf{x}') > \mu(\mathbf{x}'') - \varrho^{\varepsilon}(\mathbf{x}'')$$

Note that the weight ε corresponding to $\Gamma(\mathbf{x})$ within combination (16) may be arbitrarily small which means that the Gini's mean difference may be used like a regularization term to refine another risk measure. For instance, $\bar{\delta}^{\varepsilon}(\mathbf{x}) = (1-\varepsilon)\bar{\delta}(\mathbf{x}) + \varepsilon\Gamma(\mathbf{x})$ with arbitrarily small positive parameter ε represents the regularized mean semideviation which is strongly SSD safety consistent.

3. Enhanced below-mean downside risk measures

3.1. Below-mean downside stochastic dominance

The mean (downside) semideviation from the mean, used in the MAD model, is formally a downside risk measure. However, due to the symmetry of mean semideviations from the mean [26],

$$\bar{\delta}(\mathbf{x}) = \mathbb{E}\{\max\{\mu(\mathbf{x}) - R_{\mathbf{x}}, 0\}\} = \mathbb{E}\{\max\{R_{\mathbf{x}} - \mu(\mathbf{x}), 0\}\}$$

it is equally appropriate to interpret it as a measure of the upside risk. Similar, the Gini's mean difference is a symmetric risk measure (in the sense that for $R_{\mathbf{x}}$ and $-R_{\mathbf{x}}$ it has exactly the same value). To illustrate these properties, consider two finite random variables $R_{\mathbf{x}'}$ and $R_{\mathbf{x}''}$ defined as [9]:

$$\mathbb{P}\{R_{\mathbf{x}'} = \xi\} = \begin{cases} 0.2, & \xi = 0 \\ 0.1, & \xi = 1 \\ 0.4, & \xi = 2 \\ 0.3, & \xi = 7 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbb{P}\{R_{\mathbf{x}''} = \xi\} = \begin{cases} 0.3, & \xi = -1 \\ 0.4, & \xi = 4 \\ 0.1, & \xi = 5 \\ 0.2, & \xi = 6 \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

where $R_{\mathbf{x}''} = \mu(\mathbf{x}') - (R_{\mathbf{x}'} - \mu(\mathbf{x}'))$. Note that $\mu(\mathbf{x}') = \mu(\mathbf{x}'') = 3$ and $\mathbb{E}\{\max\{R_{\mathbf{x}'} - \eta, 0\} = \mathbb{E}\{\max\{\eta - R_{\mathbf{x}''}, 0\}$. Hence, $\bar{\delta}(\mathbf{x}') = \bar{\delta}(\mathbf{x}'') = 1.2$, $\sigma^2(\mathbf{x}') = \sigma^2(\mathbf{x}'') = 7.4$ and $\Gamma(\mathbf{x}') = \Gamma(\mathbf{x}'') = 1.42$. In other words, two random variables are identical from the viewpoint of the Markowitz as well as the MAD and the GMD models. One may notice that neither $R_{\mathbf{x}'}$ dominates $R_{\mathbf{x}''}$ nor $R_{\mathbf{x}''}$ dominates $R_{\mathbf{x}'}$ under the SSD rules but $F^{(2)}_{\mathbf{x}'}(\eta) \leq F^{(2)}_{\mathbf{x}''}(\eta)$ for all $\eta \leq 3$ and the inequality is strict for all $-1 < \eta < 3$. Thus $R_{\mathbf{x}'}$ is preferred to $R_{\mathbf{x}''}$ in a downside risk aversion context.

For better modeling of the risk averse preferences one may enhance the below-mean downside risk aversion in various measures. The below-mean risk downside aversion is a concept of risk aversion assuming that the variability of returns above the mean should not be penalized since the investors concern of an underperformance rather than the overperformance of a portfolio [15]. This can be implemented by focusing on the distribution of *below-mean downside underachievements* $\min\{R_{\mathbf{x}}, \mu(\mathbf{x})\}$ instead of the original distribution of returns $R_{\mathbf{x}}$. For general distributions, with not necessarily equal mean values, the below-mean downside stochastic dominance can be formulated as follows.

For any random variable $R_{\mathbf{x}}$ we introduce its below-mean downside underachievements

$$R^d_{\mathbf{x}} = \min\{R_{\mathbf{x}}, \mu(\mathbf{x})\} \tag{18}$$

It allows us to define the (weak) relation of *below-mean (downside) SSD* (BMSSD):

$$R_{\mathbf{x}'} \succeq_{BMSSD} R_{\mathbf{x}''} \Leftrightarrow R^d_{\mathbf{x}'} \succeq_{SSD} R^d_{\mathbf{x}''}$$

Similarly, one may define the (weak) relation of *below-mean (downside) FSD* (BMFSD):

$$R_{\mathbf{x}'} \succeq_{BMFSD} R_{\mathbf{x}''} \Leftrightarrow R^d_{\mathbf{x}'} \succeq_{FSD} R^d_{\mathbf{x}''}$$

The strict dominance relations are defined in the standard way. This means, we say that portfolio $\mathbf{x}'$ *dominates* $\mathbf{x}''$ *under the BMSSD* ($R_{\mathbf{x}'} \succ_{BMSSD} R_{\mathbf{x}''}$), if $R_{\mathbf{x}'} \succeq_{BMSSD} R_{\mathbf{x}''}$ while $R_{\mathbf{x}''} \not\succeq_{BMSSD} R_{\mathbf{x}'}$. A feasible portfolio $\mathbf{x}^0 \in \mathcal{P}$ is called *BMSSD efficient* if there is no $\mathbf{x} \in \mathcal{P}$ such that $R_{\mathbf{x}} \succ_{BMSSD} R_{\mathbf{x}^0}$.

Note that

$$\mathbb{P}\{R^d_{\mathbf{x}} \leq \eta\} = \begin{cases} \mathbb{P}\{R_{\mathbf{x}} \leq \eta\}, & \text{for } \eta < \mu(\mathbf{x}) \\ 1, & \text{for } \eta \geq \mu(\mathbf{x}) \end{cases} \tag{19}$$

while, according to (18) and (12),

$$\mu^d(\mathbf{x}) = \mathbb{E}\{R^d_{\mathbf{x}}\} = \mathbb{E}\{\min\{R_{\mathbf{x}}, \mu(\mathbf{x})\}\} = \mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}) \tag{20}$$

Hence, the second performance function $F^{(2)}_{\mathbf{x}^d}$ for the random variable $R^d_{\mathbf{x}}$ coincides with $F^{(2)}_{\mathbf{x}}(\eta)$ for $\eta \leq \mu(\mathbf{x})$ and takes the form of a straight line $\eta - (\mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}))$ for $\eta > \mu(\mathbf{x})$ (see figure 1).

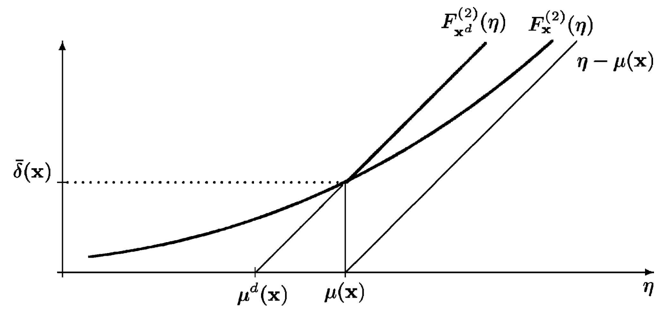

Figure 1. Function $F^{(2)}_{\mathbf{x}^d}$ for the below-mean downside distribution $R^d_{\mathbf{x}}$.

Directly from (19), it follows the equivalence:

$$R_{\mathbf{x}'} \succeq_{BMFSD} R_{\mathbf{x}''} \Leftrightarrow \mu(\mathbf{x}') \geq \mu(\mathbf{x}'') \quad \text{and} \quad F_{\mathbf{x}'}(\eta) \leq F_{\mathbf{x}''}(\eta) \quad \forall \eta \leq \mu(\mathbf{x}'') \tag{21}$$

However, the relation BMSSD turns out to be more subtle since only the following implication is valid:

$$\mu(\mathbf{x}') \geq \mu(\mathbf{x}'') \quad \text{and} \quad F^{(2)}_{\mathbf{x}'}(\eta) \leq F^{(2)}_{\mathbf{x}''}(\eta) \quad \forall \eta \leq \mu(\mathbf{x}'') \Rightarrow R_{\mathbf{x}'} \succeq_{BMSSD} R_{\mathbf{x}''} \tag{22}$$

As a counterexample to the equivalence in (22) one may consider

$$\mathbb{P}\{R_{\mathbf{x}'} = \xi\} = \begin{cases} 1.0, & \xi = 3 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbb{P}\{R_{\mathbf{x}''} = \xi\} = \begin{cases} 0.5, & \xi = 0 \\ 0.5, & \xi = 10 \\ 0, & \text{otherwise} \end{cases} \tag{23}$$

where $R_{\mathbf{x}'} \succ_{BMSSD} R_{\mathbf{x}''}$ despite $\mu(\mathbf{x}') = 3 < 5 = \mu(\mathbf{x}'')$. Hence, opposite to the standard SSD relation (compare (2)), the below-mean SSD allows a portfolio with a smaller expected return to dominate some more risky portfolios with larger expectations.

The relations (21) and (22) are sufficient to justify the implication:

$$R_{\mathbf{x}'} \succeq_{BMFSD} R_{\mathbf{x}''} \Rightarrow R_{\mathbf{x}'} \succeq_{BMSSD} R_{\mathbf{x}''}$$

Moreover, the below-mean stochastic dominance relations are consistent with the corresponding SD relations as stated in the following assertion.

Theorem 4. *The following implications are valid*:

$$R_{\mathbf{x}'} \succeq_{FSD} R_{\mathbf{x}''} \Rightarrow R_{\mathbf{x}'} \succeq_{BMFSD} R_{\mathbf{x}''} \tag{24}$$

$$R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''} \Rightarrow R_{\mathbf{x}'} \succeq_{BMSSD} R_{\mathbf{x}''} \tag{25}$$

Proof: Either $R_{\mathbf{x}'} \succeq_{FSD} R_{\mathbf{x}''}$ or $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$ imply $\mu(\mathbf{x}') \geq \mu(\mathbf{x}'')$. Hence, implications (24) and (25) follow from (21) and (22), respectively. □

Note that, according to (18) and (12), the SSD safety consistency for the mean semideviation (11) follows from the implications (2) applied to the distributions of downside underachievements. Hence, Theorem 4 generalizes Proposition 1 from [16].

Similar to the SSD consistency (3), the BMSSD consistency of the safety measures may be formalized. We will say that the safety measure $\mu(\mathbf{x}) - \varrho(\mathbf{x})$ is *BMSSD consistent* or that the risk measure $\varrho(\mathbf{x})$ is *BMSSD safety consistent* if

$$R_{\mathbf{x}'} \succeq_{_{BMSSD}} R_{\mathbf{x}''} \Rightarrow \mu(\mathbf{x}') - \varrho(\mathbf{x}') \geq \mu(\mathbf{x}'') - \varrho(\mathbf{x}'') \tag{26}$$

Certainly, if the risk measure $\varrho(\mathbf{x})$ is BMSSD safety consistent (26), then except for portfolios with identical values of $\mu(\mathbf{x})$ and $\varrho(\mathbf{x})$, every efficient solution of the bicriteria problem (5) is an BMSSD efficient portfolio.

By virtue of Theorem 4, every risk measure BMSSD safety consistent is also SSD safety consistent in the sense of (3). We will use this relation to extend LP computable risk measures allowing them to focus on below-mean downside risk while preserving their SSD safety consistency. Exactly, we will build BMSSD safety consistent measures. Note that $R_{\mathbf{x}'} \succ_{_{BMSSD}} R_{\mathbf{x}''}$ for (17), and therefore any BMSSD consistent risk measure must properly distinguish these distributions (opposite to $\bar{\delta}(\mathbf{x})$ and $\Gamma(\mathbf{x})$).

3.2. *Below-mean downside risk measures*

The simplest idea to build a below-mean SSD consistent risk measure is to apply the basic measure to the below-mean downside underachievements $R_{\mathbf{x}}^d$ instead of the original distribution $R_{\mathbf{x}}$. Let $\varrho^d(\mathbf{x})$ denote a risk measure defined as the result of application of the risk measure ϱ to the random variable $R_{\mathbf{x}}^d$. In particular, applying the mean semideviation (9) to $R_{\mathbf{x}}^d$ one gets

$$\bar{\delta}^d(\mathbf{x}) = \mathbb{E}\big\{\max\big\{\mu^d(\mathbf{x}) - R_{\mathbf{x}}^d, 0\big\}\big\} = \mathbb{E}\{\max\{\mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}) - R_{\mathbf{x}}, 0\}\}$$

Similarly, applying the Gini's mean difference to the distribution of downside underachievements $R_{\mathbf{x}}^d$, according to (14), we get

$$\Gamma^d(\mathbf{x}) = \mu^d(\mathbf{x}) - \mathbb{E}\big\{R_{\mathbf{x}}^d \wedge R_{\mathbf{x}}^d\big\}$$

For a discrete random variable represented by its realizations y_t, the above formulas take the forms:

$$\bar{\delta}^d(\mathbf{x}) = \sum_{t=1}^{T} \max\left\{\sum_{t'=1}^{T} y_{t'}^d p_{t'} - y_t^d,\ 0\right\} p_t \tag{27}$$

$$\Gamma^d(\mathbf{x}) = \frac{1}{2}\sum_{t=1}^{T}\sum_{t'=1}^{T} \left|y_{t'}^d - y_t^d\right| p_{t'} p_t \tag{28}$$

respectively, where y_t^d denote the realizations of the downside distribution $R_{\mathbf{x}}^d$:

$$y_t^d = r_t^d(\mathbf{x}) = \min\left\{y_t, \sum_{t'=1}^{T} y_{t'} p_{t'}\right\} = \min\left\{\sum_{j=1}^{n} r_{jt} x_j, \sum_{j=1}^{n} \mu_j x_j\right\} \tag{29}$$

Recall that in our initial example (17) of $R_{\mathbf{x}'} \succeq_{BMSSD} R_{\mathbf{x}''}$ we have $\mu(\mathbf{x}') = \mu(\mathbf{x}'') = 3$, $\bar{\delta}(\mathbf{x}') = \bar{\delta}(\mathbf{x}'') = 1.2$ and $\Gamma(\mathbf{x}') = \Gamma(\mathbf{x}'') = 1.42$. Similar, $\mu^d(\mathbf{x}') = \mu^d(\mathbf{x}'') = 1.8$ but $\bar{\delta}^d(\mathbf{x}') = 0.44 < \bar{\delta}^d(\mathbf{x}'') = 0.84$ and $\Gamma^d(\mathbf{x}') = 0.58 < \Gamma^d(\mathbf{x}'') = 0.84$ thus properly distinguishing these two distributions. Unfortunately, in general, the measures $\bar{\delta}^d$ and Γ^d are neither SSD safety consistent nor LP computable. According to (27) and (28), both the measures are convex and piece-wise linear functions of realizations y_t^d. However, realizations of the downside distribution (29) are (not linear) concave functions of original realizations y_t or portfolio shares x_j. Therefore, the measures $\bar{\delta}^d(\mathbf{x})$ and $\Gamma^d(\mathbf{x})$, in general, are not convex functions of portfolio $\mathbf{x}$ and therefore, (despite piece-wise linear) they are not LP computable. To overcome this weakness, one need to extend the measures ϱ^d as shown in the following theorem.

Theorem 5. *Let $\varrho^d(\mathbf{x})$ denote a risk measure defined as the result of application of the risk measure ϱ to the random variable $R_{\mathbf{x}}^d$. If ϱ is an LP computable SSD safety consistent risk measure, then the enhanced downside risk measure*

$$\varrho^{(2)}(\mathbf{x}) = \varrho^d(\mathbf{x}) + \bar{\delta}(\mathbf{x}) \tag{30}$$

is also LP computable and BMSSD safety consistent, i.e.

$$R_{\mathbf{x}'} \succeq_{BMSSD} R_{\mathbf{x}''} \Rightarrow \mu(\mathbf{x}') - \varrho^{(2)}(\mathbf{x}') \geq \mu(\mathbf{x}'') - \varrho^{(2)}(\mathbf{x}'') \tag{31}$$

Proof: The original risk measure ϱ is SSD safety consistent. Hence, the measure $\varrho^d(\mathbf{x})$ satisfies the following consistency relation

$$R_{\mathbf{x}'}^d \succeq_{SSD} R_{\mathbf{x}''}^d \Rightarrow \mu^d(\mathbf{x}') - \varrho^d(\mathbf{x}') \geq \mu^d(\mathbf{x}'') - \varrho^d(\mathbf{x}'')$$

By definition of the enhanced downside risk measure (30) and (20), one gets

$$\mu^d(\mathbf{x}) - \varrho^d(\mathbf{x}) = \mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}) - \varrho^d(\mathbf{x}) = \mu(\mathbf{x}) - \varrho^{(2)}(\mathbf{x})$$

which completes the proof of the BMSSD safety consistency (31).

In the case of discrete random variable represented by its realizations y_t, the risk measure is obviously piece-wise linear. In order to show its LP computability we need to prove its convexity. Note that the SSD safety consistency of measure ϱ guarantees that the corresponding safety measure s is concave and (weakly) increasing function of the realizations of the random variable. While applying it to the realizations of the downside distribution (29) we get a superposition of concave and (weakly) increasing functions. Hence,

$s^d(\mathbf{x}) = \mu^d(\mathbf{x}) - \varrho^d(\mathbf{x})$ is a piece-wise linear concave function and $\varrho^{(2)}(\mathbf{x}) = \mu(\mathbf{x}) - s^d(\mathbf{x})$ is a piece-wise linear convex function (thus LP computable when minimized). $\square$

Recall that, by virtue of Theorem 4, $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$ implies $R_{\mathbf{x}'} \succeq_{BMSSD} R_{\mathbf{x}''}$. Thus, Theorem 5 provides us with a construction which preserve the SSD safety consistency of original risk measures while enhancing them to meet the requirements of the BMSSD consistency.

Applying (30) to the mean semideviation allows us to define the enhanced risk measure for the original distribution of returns $R_{\mathbf{x}}$ as

$$\bar{\delta}^{(2)}(\mathbf{x}) = \bar{\delta}(\mathbf{x}) + \bar{\delta}^d(\mathbf{x}) = \bar{\delta}(\mathbf{x}) + \mathbb{E}\{\max\{\mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}) - R_{\mathbf{x}}, 0\}\} \tag{32}$$

Actually, the measure can be interpreted as a single mean semideviation (from the mean) applied with a penalty function: $\bar{\delta}^{(2)}(\mathbf{x}) = \mathbb{E}\{u(\max\{\mu(\mathbf{x}) - R_{\mathbf{x}}, 0\})\}$ where u is increasing and convex piece-wise linear penalty function with brakepoint $b = \bar{\delta}(\mathbf{x})$ and slopes: 1 and 2, respectively. Therefore, the risk measure $\bar{\delta}^{(2)}(\mathbf{x})$ we will refer to as the *mean penalized semideviation*. The corresponding enhanced below-mean safety measure takes the form

$$\mu(\mathbf{x}) - \bar{\delta}^{(2)}(\mathbf{x}) = \mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}) - \bar{\delta}^d(\mathbf{x})$$

It follows from Theorems 2 and 5 that the the following assertion is valid.

Corollary 1. *The mean penalized semideviation is an BMSSD safety consistent risk measure*, *i.e.*

$$R_{\mathbf{x}'} \succeq_{BMSSD} R_{\mathbf{x}''} \Rightarrow \mu(\mathbf{x}') - \bar{\delta}^{(2)}(\mathbf{x}') \geq \mu(\mathbf{x}'') - \bar{\delta}^{(2)}(\mathbf{x}'')$$

The penalized mean semideviation (32) was already introduced in connection with the so-called m-MAD model [16]. Theorem 5 provides the enhancement technique applicable for various risk measures. In particular it allows us to define the downside Gini's mean difference. Following (30), we get the *downside Gini's mean difference* defined as the enhanced risk measure:

$$\Gamma^{(2)}(\mathbf{x}) = \Gamma^d(\mathbf{x}) + \bar{\delta}(\mathbf{x}) \tag{33}$$

The corresponding enhanced safety measure takes the form:

$$\mu(\mathbf{x}) - \Gamma^{(2)}(\mathbf{x}) = \mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}) - \Gamma^d(\mathbf{x}) = \mathbb{E}\{\min\{R_{\mathbf{x}} \wedge R_{\mathbf{x}}, \mu(\mathbf{x})\}\} \tag{34}$$

Due to Theorem 5, the following assertion is valid.

Corollary 2. *The downside Gini's mean difference* (33) *is an BMSSD safety consistent risk measure*, *i.e.*

$$R_{\mathbf{x}'} \succeq_{BMSSD} R_{\mathbf{x}''} \Rightarrow \mu(\mathbf{x}') - \Gamma^{(2)}(\mathbf{x}') \geq \mu(\mathbf{x}'') - \Gamma^{(2)}(\mathbf{x}'')$$

Note that the enhanced measure $\Gamma^{(2)}(\mathbf{x})$ is no longer strongly SSD safety consistent as it ignores the distribution of the overachievements. Nevertheless, it remains strongly consistent with respect to the below-mean stochastic dominance in the sense that

$$R_{\mathbf{x}'} \succ_{_{BMSSD}} R_{\mathbf{x}''} \Rightarrow \mu(\mathbf{x}') - \Gamma^{(2)}(\mathbf{x}') > \mu(\mathbf{x}'') - \Gamma^{(2)}(\mathbf{x}'')$$

One may further combine the enhanced risk measures by application of the weighted sum (15). In particular, one may reduce the below-mean downside risk aversion by considering combinations $\varrho_w^{(2)}(\mathbf{x}) = w\varrho^{(2)}(\mathbf{x}) + (1-w)\bar{\delta}(\mathbf{x}) = \bar{\delta}(\mathbf{x}) + w\varrho^d(\mathbf{x})$ where $0 \leq w \leq 1$. By virtue of Theorem 2, such risk measures remain SSD safety consistent. Actually, due to (12), one may easily show the following consistency results:

for any $0 \leq w \leq 1$, the weighted penalized mean semideviation

$$\bar{\delta}_w^{(2)}(\mathbf{x}) = \bar{\delta}(\mathbf{x}) + w\bar{\delta}^d(\mathbf{x}) \tag{35}$$

is BMSSD safety consistent;

for any $0 < w \leq 1$, the weighted downside Gini's mean difference

$$\Gamma_w^{(2)}(\mathbf{x}) = \bar{\delta}(\mathbf{x}) + w\Gamma^d(\mathbf{x}) \tag{36}$$

is strongly BMSSD safety consistent.

3.3. The LP models

We provide here the detailed LP formulations for the models we have analyzed. In order to operationalize various portfolio optimization models, one needs to deal with specific investor preferences expressed in the models. The commonly accepted approach to implementation of the Markowitz-type mean-risk model is that based on the use of a specified lower bound μ_0 on expected returns which results in the following problem:

$$\min\{\varrho(\mathbf{x}) : \quad \mu(\mathbf{x}) \geq \mu_0, \quad \mathbf{x} \in \mathcal{P}\}. \tag{37}$$

However, we need to consider explicitly a separate problem

$$\max\{\mu(\mathbf{x}) - \varrho(\mathbf{x}) : \quad \mu(\mathbf{x}) \geq \mu_0, \quad \mathbf{x} \in \mathcal{P}\} \tag{38}$$

for the corresponding mean-safety model (5). Therefore, while describing a specific model, the pair of minimum-risk maximum-safety problems can be stated as:

$$\max\{\alpha\mu(\mathbf{x}) - \varrho(\mathbf{x}) \ : \ \mu(\mathbf{x}) \geq \mu_0, \quad \mathbf{x} \in \mathcal{P}\} \tag{39}$$

covering the minimization of risk measure $\varrho(\mathbf{x})$ for $\alpha = 0$ while for $\alpha = 1$ it represents the maximization of the corresponding safety measure $\mu(\mathbf{x}) - \varrho(\mathbf{x})$. Both optimizations are considered with a given lower bound on the expected return $\mu(\mathbf{x}) \geq \mu_0$. The models are summarized for the case of the simplest feasible set

$$\mathcal{P} = \left\{ \mathbf{x} \ : \ \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \qquad \text{for } j = 1, \ldots, n \right\}. \tag{40}$$

All the models (39) contain the following core LP constraints:

$$\mathbf{x} \in \mathcal{P} \quad \text{and} \quad z \geq \mu_0 \tag{41}$$

$$\sum_{j=1}^{n} \mu_j x_j = z \tag{42}$$

$$\sum_{j=1}^{n} r_{jt} x_j = y_t \quad \text{for } t = 1, \ldots, T \tag{43}$$

where z is an unbounded variable representing the mean return of the portfolio $\mathbf{x}$ and y_t ($t = 1, \ldots, T$) are unbounded variables to represent the realizations of the portfolio return under the scenario t. In addition to these common variables and constraints, each model contains its specific linear constraints to define the risk or safety measure.

MAD models. The standard MAD model, when implemented with the mean semideviation as the risk measure ($\varrho(\mathbf{x}) = \bar{\delta}(\mathbf{x})$), leads to the following LP problem:

$$\begin{aligned} \text{maximize} \quad & \alpha z - z_1 \\ \text{subject to} \quad & \text{(41)–(43) and} \\ & \sum_{t=1}^{T} p_t d_t^{(1)} = z_1 && (44) \\ & d_t^{(1)} + y_t \geq z, \quad d_t^{(1)} \geq 0 \quad \text{for } t = 1, \ldots, T && (45) \end{aligned}$$

where nonnegative variables $d_t^{(1)}$ represent downside deviations from the mean under several scenarios t and z_1 is a variable to represent the mean semideviation itself. The latter can be omitted by using the direct formula for mean semideviation in the objective function instead of Eq. (44). The above LP formulation uses $T + 1$ variables and $T + 1$ constraints to model the mean semideviation.

In order to incorporate downside risk aversion by the use of the weighted penalized mean semideviation (35), one needs to repeat constraints of type (44)–(45) for the second deviation level. This leads to the following formulation of the DMAD model:

$$\begin{aligned} \text{maximize} \quad & \alpha z - z_1 - w z_2 \\ \text{subject to} \quad & \text{(41)–(43), (44)–(45) and} \\ & \sum_{t=1}^{T} p_t d_t^{(2)} = z_2 \\ & d_t^{(2)} + z_1 + y_t \geq z, \quad d_t^{(2)} \geq 0 \quad \text{for } t = 1, \ldots, T \end{aligned}$$

This LP formulation uses $2(T + 1)$ variables and $2(T + 1)$ constraints.

GMD models. The model with risk measured by the Gini's mean difference ($\varrho(\mathbf{x}) = \Gamma(\mathbf{x})$), takes the form:

$$\begin{array}{ll} \text{maximize} & \alpha z - \sum_{t=1}^{T} \sum_{t' \neq t} p_t p_{t'} d_{t,t'} \\ \text{subject to} & \text{(41)–(43) and} \quad d_{t,t'} \geq y_t - y_{t'}, \; d_{t,t'} \geq 0 \quad \text{for } t, t' = 1, \ldots, T; \; t \neq t' \end{array}$$

which contains $T(T-1)$ nonnegative variables $d_{t,t'}$ and $T(T-1)$ inequalities to define them. However, variables $d_{t,t'}$ are associated with the singleton coefficient columns. Hence, while solving the dual instead of the original primal, the corresponding dual constraints take the form of simple upper bounds (SUB) which are handled implicitly outside the LP matrix. Such a dual approach may dramatically improve the LP model efficiency in the case of large number of scenarios.

In order to incorporate downside risk aversion by the use of the weighted downside Gini's mean difference (36), one may apply the Gini's measure to the downside deviations $d_t^{(1)}$ from the MAD model. This results in the following DGMD model:

$$\begin{array}{ll} \text{maximize} & \alpha z - z_1 - w \sum_{t=1}^{T} \sum_{t' \neq t} p_t p_{t'} d_{t,t'} \\ \text{subject to} & \text{(41)–(43), (44)–(45) and} \\ & d_{t,t'} \geq d_t^{(1)} - d_{t'}^{(1)}, \; d_{t,t'} \geq 0 \quad \forall t, t' = 1, \ldots, T; \; t \neq t' \end{array}$$

Again, while solving the dual instead of the original primal, the corresponding dual constraints take the form of simple upper bounds (SUB) which are handled implicitly outside the LP matrix.

Dual GMD models. For the simplest form of the feasible set (40) the dual GMD model takes the following form:

$$\text{minimize} \quad v - \mu_0 v_0 \tag{46}$$

subject to

$$v - \mu_j v_0 - \sum_{t=1}^{T} r_{jt} s_t \geq \alpha \mu_j \qquad \text{for } j = 1, \ldots, n \tag{47}$$

$$\sum_{t' \neq t} u_{t,t'} - \sum_{t' \neq t} u_{t',t} + s_t = 0 \qquad \text{for } t = 1, \ldots, T \tag{48}$$

$$v_0 \geq 0, \; 0 \leq u_{t,t'} \leq p_t p_{t'} \qquad \text{for } t, t' = 1, \ldots, T; t \neq t' \tag{49}$$

where original portfolio variables x_j are dual prices to (47) inequalities while y_t are dual prices to (48). The dual model contains $T(T-1)$ variables $u_{t,t'}$ but the number of constraints (excluding the SUB structure) is proportional to T. Moreover, the constraints (48) take the form of typical network flow thus allowing for a special algorithmic treatment within the simplex method [6].

The above dual formulation can be further simplified by introducing balance flow variables:

$$\bar{u}_{t,t'} = u_{t,t'} - u_{t',t} \quad \text{for } t, t' = 1, \ldots, T; t < t' \tag{50}$$

which allows us to reduce the number of variables by replacing (48)–(49) with the following constraints:

$$\sum_{t'>t} \bar{u}_{t,t'} - \sum_{t'<t} \bar{u}_{t',t} + s_t = 0 \quad \text{for } t = 1, \ldots, T$$
$$v_0 \geq 0, \; -p_t p_{t'} \leq \bar{u}_{t,t'} \leq p_t p_{t'} \quad \text{for } t, t' = 1, \ldots, T; t < t'$$

For the simplest form of the feasible set (40) the dual DGMD model takes the following form:

$$\text{minimize} \quad v - \mu_0 v_0 \tag{51}$$

subject to

$$v - \mu_j v_0 - \sum_{t=1}^{T} (r_{jt} - \mu_j) s_t \geq \alpha \mu_j \quad \text{for } j = 1, \ldots, n \tag{52}$$

$$v_0 \geq 0, \; s_t \geq 0 \quad \text{for } t = 1, \ldots, T \tag{53}$$

$$\sum_{t' \neq t} u_{t,t'} - \sum_{t' \neq t} u_{t',t} - s_t \geq -p_t \quad \text{for } t = 1, \ldots, T \tag{54}$$

$$0 \leq u_{t,t'} \leq w p_t p_{t'} \quad \text{for } t, t' = 1, \ldots, T; t \neq t' \tag{55}$$

where original portfolio variables x_j are dual prices to inequalities (52) while d_t^1 are dual prices to (54). The constraints (54) still take the form of network flow thus allowing for a special algorithmic treatment within the simplex method.

The number of variables can be further reduced by introducing balance flow variables (50) which replaces (54)–(55) with the following constraints:

$$\sum_{t'>t} \bar{u}_{t,t'} - \sum_{t'<t} \bar{u}_{t',t} - s_t \geq -p_t \quad \text{for } t = 1, \ldots, T$$
$$-w p_t p_{t'} \leq \bar{u}_{t,t'} \leq w p_t p_{t'} \quad \text{for } t, t' = 1, \ldots, T; t < t'$$

Moreover, a simple network structure of the constraints with respect to variables $\bar{u}_{t,t'}$ allows one to treat effectively a very large number of those variables with the column generation techniques. Although, our computational experiments shows that there is no such a need for typical portfolio optimization problems with T below 200.

Our preliminary analysis shows that the models can be effectively solved by standard (simplex based) LP techniques. A PC with a 1.2 GHz AMD Athlon processor and 256 MB RAM has been used to run an application written in C++ language by using the CPLEX 6.5 Callable Library [8]. Table 1 shows average solution times for the asset allocation problem (discussed in the next section) with $n = 81$ and $T = 52$, 104, 156, respectively.

Table 1. Average optimization time for different estimation periods (in seconds).

	Mean-risk ($\alpha = 0$)			Mean-safety ($\alpha = 1$)		
Model	$T = 52$	$T = 104$	$T = 156$	$T = 52$	$T = 104$	$T = 156$
MAD	6.3	6.4	6.7	6.3	6.4	6.7
DMAD	6.3	6.3	6.8	6.3	6.5	6.8
GMD	6.9	34.8	182.8	7.0	35.0	184.2
Dual GMD	6.3	6.5	7.1	6.4	6.4	7.2
DGMD	6.8	30.6	166.9	6.8	30.6	166.8
Dual DGMD	6.3	6.3	7.1	6.3	6.4	7.1

The computation time for the MAD and DMAD models has never exceeded 10 seconds. However, while dealing with the (primal) GMD and DGMD models, the CPU time has increased to above 30 sec. with $T = 104$ and even more than 180 sec. for $T = 156$. These have been dramatically improved when using the above dual formulations which let us reduce the optimization time below 10 seconds for the dual GMD and DGMD models. Thus, the optimization time of each particular downside risk model has not exceed 10 seconds which demonstrates high computational efficiency of the approaches.

4. Experimental analysis

4.1. Experimental framework

The present section is devoted to an analysis and comparison of the discussed LP models on real-life financial data. The models have been implemented and applied to investments in 81 S&P 500 sub-industries, defined according to the Global Industry Classification Standard (GICS).[1] In the following, we first describe the design of our experiments, then we present and discuss the results of in-sample analysis. Finally, an out-of-sample comparison of the models is provided.

We have tested and compared the MAD and GMD models with their downside enhancements. For the DMAD model versus the DGMD model we have decided to consider them with only one value of w equal to 1 (the maximal downside risk enhancement). Both mean-risk and mean-safety approaches have been compared. Recall that, according to Theorem 1, the latter have guaranteed the SSD efficiency.

For the experimental analysis we have prepared three sets of data, consisting of the weekly rates of return over the period 1990–2003, for the sub-industries under consideration. Each set of data was associated with the existing market trend on S&P 500 Index (figure 2). Datasets are as follows:

- Period A (01/01/90 – 09/12/94, horizontal trend): 258 weekly observations;
- Period B (12/12/94 – 01/09/00, upward trend): 299 weekly observations;
- Period C (04/09/00 – 30/05/03, downward trend): 143 weekly observations.

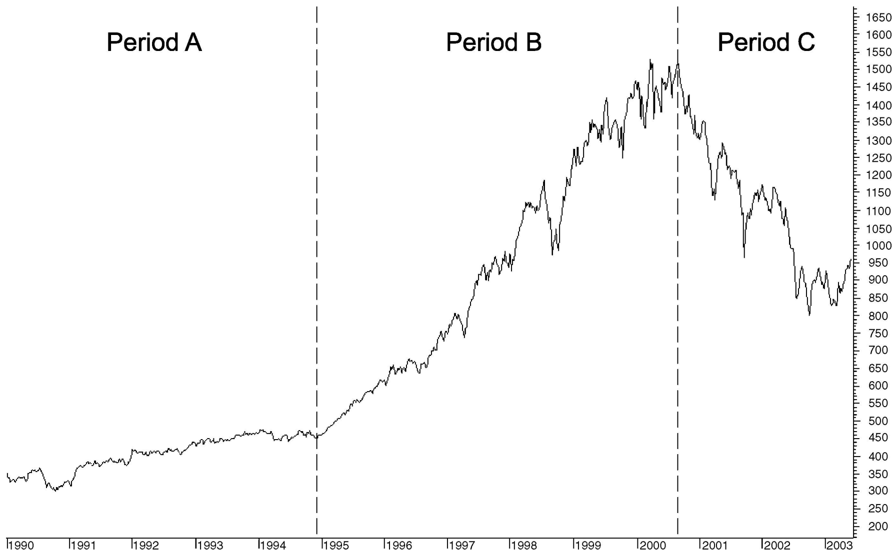

Figure 2. S&P 500 Index and the analyzed periods in 1990–2003.

We have chosen a weekly periodicity for the rates of return in order to reduce estimation errors. The rates of return have been computed as relative changes of the index values P_{jt}, i.e. $r_{jt} = (P_{j,t+1} - P_{jt})/P_{jt}$. However, while reporting the results, we convert the weekly rates of return onto a yearly basis.

Each dataset, corresponding to one of the periods from A to C, has been used to find the mean-risk/safety portfolios through solution of the described models. The target weekly required return has been set to two different values per each period for comparison purposes, i.e. corresponding to the yearly rates: 7.5%, 15% for Period A, 10%, 20% for Period B and 10%, 20% for Period C. The larger values were the highest ones for which it was possible to solve all the analyzed models in the corresponding periods without facing infeasibility problems (empty portfolios).

We have fixed the estimation (in-sample) period at 52 weeks (1 year). The ex-post behavior of all the selected portfolios has been examined out-of-sample at the end of the 4-week investment period following the portfolio selection date (the last date of the corresponding in-sample period). This approach allows us to conduct 203, 244 and 88 estimations for Period A, B and C, respectively.

4.2. *In-sample analysis*

For each dataset and all levels of the required rate of return, we have solved all the LP problems defined in the previous section. General characteristics of the optimal portfolios are shown in Table 2. All presented data are averages computed over the estimation periods. Table 2 is divided into two parts: the first one corresponds to the problem formulated as

Table 2. General characteristics of the optimal portfolios (average values).

			Mean-risk models ($\alpha = 0$)					Mean-safety models ($\alpha = 1$)				
						Shares					Shares	
Model	Per.	μ_0 (%)	obj. $\times 10^{-2}$	z (%)	div. (#)	(min)	(max)	obj. $\times 10^{-2}$	z (%)	div. (#)	(min)	(max)
MAD	A	7.5	−0.298	9.84	15.39	0.005	0.296	0.168	54.94	7.37	0.026	0.375
		15.0	−0.329	15.23	14.14	0.007	0.297	0.168	55.05	7.35	0.026	0.376
	B	10.0	−0.380	11.56	13.40	0.006	0.321	0.159	59.48	7.64	0.025	0.334
		20.0	−0.410	20.11	13.38	0.006	0.315	0.159	59.48	7.64	0.025	0.334
	C	10.0	−0.467	10.45	14.48	0.008	0.198	−0.053	40.14	8.93	0.024	0.272
		20.0	−0.524	20.02	13.82	0.006	0.212	−0.055	40.58	9.09	0.023	0.271
DMAD	A	7.5	−0.484	9.44	15.00	0.005	0.305	−0.139	34.17	10.74	0.010	0.286
		15.0	−0.530	15.08	13.94	0.007	0.299	−0.143	34.77	10.57	0.011	0.280
	B	10.0	−0.611	11.25	13.43	0.005	0.323	−0.211	40.28	10.36	0.011	0.314
		20.0	−0.661	20.07	13.14	0.005	0.311	−0.211	40.41	10.37	0.011	0.311
	C	10.0	−0.767	10.44	13.84	0.006	0.206	−0.440	28.91	11.36	0.010	0.234
		20.0	−0.854	20.00	13.34	0.006	0.215	−0.449	30.10	11.23	0.010	0.235
GMD	A	7.5	−0.459	9.68	15.56	0.004	0.315	−0.084	40.08	10.15	0.012	0.289
		15.0	−0.499	15.07	14.46	0.007	0.307	−0.087	40.44	10.04	0.014	0.287
	B	10.0	−0.563	11.12	14.20	0.005	0.334	−0.136	43.96	10.25	0.011	0.301
		20.0	−0.607	20.01	13.89	0.006	0.318	−0.136	44.01	10.25	0.011	0.300
	C	10.0	−0.724	10.32	14.22	0.007	0.193	−0.377	33.38	11.35	0.010	0.227
		20.0	−0.808	20.00	14.01	0.006	0.197	−0.385	34.37	11.16	0.010	0.232
DGMD	A	7.5	−0.528	9.65	15.60	0.004	0.304	−0.199	30.75	11.48	0.008	0.290
		15.0	−0.578	15.10	14.34	0.006	0.298	−0.204	31.51	11.35	0.009	0.281
	B	10.0	−0.666	11.24	13.48	0.005	0.330	−0.282	37.29	10.73	0.010	0.314
		20.0	−0.719	20.04	13.38	0.006	0.318	−0.282	37.44	10.77	0.010	0.312
	C	10.0	−0.844	10.27	14.31	0.006	0.197	−0.528	27.67	11.60	0.011	0.218
		20.0	−0.941	20.00	13.68	0.007	0.210	−0.540	29.09	11.52	0.011	0.225

the minimization of the risk measure ($\alpha = 0$), while the second refers to the maximization of the corresponding safety measure ($\alpha = 1$). Each part consists of five columns showing: the objective function value (obj.), the portfolio percent average return (z), the portfolio diversification (div.) represented by the number of selected sub-industries, the minimum and maximum share within the portfolio, respectively. The average return is reported as converted onto a yearly basis. Rows of the table correspond to all the tested models over the three periods for various levels of the required return (μ_0).

The next table presents some more detailed characteristics. Table 3 shows ranges for the mean returns, the diversification, the minimum and maximum share held by sub-industries obtained for each model over all the periods for various required rates of return when $\alpha = 0$ and $\alpha = 1$, respectively.

Table 3. Mean returns, diversification, minimal, and maximal shares of the optimal portfolios (ranges).

			Mean-risk models ($\alpha = 0$)				Mean-safety models ($\alpha = 1$)			
Model	Per.	μ_0 (%)	z %	div. (#)	min. share $\times 10^{-4}$	max. share $\times 10^{-2}$	z (%)	div. (#)	min. share $\times 10^{-4}$	max. share $\times 10^{-2}$
MAD	A	7.5	7.5–20.4	4–25	0.3–943.0	10.2–59.4	7.5–225.9	2–14	0.6–1262.2	15.2–96.6
		15.0	15.0–20.4	3–23	0.6–1506.5	11.2–58.6	15.0–225.9	2–14	0.6–1506.5	15.3–96.6
	B	10.0	10.0–33.3	7–23	0.1–664.7	13.4–68.3	20.3–118.6	4–15	0.6–1560.3	12.9–63.9
		20.0	20.0–33.3	7–25	0.2–379.4	12.9–65.7	20.3–118.6	4–15	0.6–1560.3	12.9–63.9
	C	10.0	10.0–21.7	7–23	0.2–852.6	12.5–38.2	12.7–81.6	4–14	2.1–1812.3	17.1–42.3
		20.0	20.0–21.7	9–22	0.7–263.5	12.1–34.8	20.0–81.6	4–14	0.7–1812.3	17.1–42.3
DMAD	A	7.5	7.5–18.0	5–23	0.4–391.4	12.0–57.5	7.5–131.2	5–18	0.1–569.7	13.1–60.5
		15.0	15.0–18.0	3–23	0.1–1509.4	12.0–57.3	15.0–131.2	3–18	0.1–1509.4	13.1–60.5
	B	10.0	10.0–25.3	8–21	0.3–309.3	13.9–67.6	15.2–99.8	6–18	0.2–864.2	14.4–60.8
		20.0	20.0–25.3	9–22	0.2–239.9	11.1–61.9	20.0–99.8	6–18	1.2–864.2	14.4–60.0
	C	10.0	10.0–16.2	9–21	0.6–259.5	12.0–36.2	10.0–54.3	8–16	1.2–599.8	14.7–35.2
		20.0	20.0–20.0	9–19	0.4–406.6	12.9–33.8	20.0–54.3	8–16	0.6–599.8	15.0–35.2
GMD	A	7.5	7.5–17.0	5–25	0.1–253.3	9.7–56.5	7.5–165.0	5–17	0.2–501.6	14.7–56.3
		15.0	15.0–17.0	3–24	0.7–1654.8	11.4–56.0	15.0–165.0	3–17	0.2–1654.8	14.5–56.3
	B	10.0	10.0–20.9	9–23	0.0–297.4	10.8–63.7	16.7–99.8	5–18	0.8–644.3	13.7–53.5
		20.0	20.0–20.9	8–24	0.1–263.6	10.7–61.0	20.0–99.8	5–18	0.8–644.3	13.7–53.5
	C	10.0	10.0–14.6	7–22	0.6–887.9	9.4–30.0	10.0–72.3	7–18	1.9–466.2	14.0–34.2
		20.0	20.0–20.0	8–19	0.8–367.7	11.5–37.2	20.0–72.3	7–18	1.9–466.2	14.0–37.2
DGMD	A	7.5	7.5–18.6	4–24	0.1–763.8	11.0–56.2	7.5–87.7	4–20	0.3–763.8	11.8–61.9
		15.0	15.0–18.6	3–24	0.1–1506.5	11.4–57.9	15.0–87.7	3–20	0.3–1506.5	11.8–61.9
	B	10.0	10.0–26.0	8–25	0.1–325.9	13.2–66.5	13.2–87.7	6–20	0.0–557.9	14.0–60.5
		20.0	20.0–26.0	8–24	0.3–302.2	11.7–64.5	20.0–87.7	6–20	0.0–557.9	14.0–59.7
	C	10.0	10.0–15.3	8–23	2.1–283.8	11.8–36.8	10.0–52.2	8–17	1.0–499.0	13.4–33.9
		20.0	20.0–20.0	7–20	0.5–398.6	11.1–34.2	20.0–52.2	7–17	1.0–499.0	13.4–34.2

Having analyzed the results, we have observed that when the required return increases the risk as well as the safety, both indicated by the objective function values, increases and decreases, respectively (notice that the risk measures are represented with the negative sign in the objective function of our general model (39)). The portfolio average returns tend to go up with an increase in the required return. The diversification is quite stable for all the tested models, however, the mean-safety models provide the lower diversification in terms of average values and ranges as well. The efficient portfolios with respect to the mean-safety measures consist of shares with the larger minimum share than the portfolios generated by the mean-risk models.

In Table 4 we put together the mean returns of the minimum risk portfolio (MRP) and the maximum safety portfolio (MSP) for all the tested models in each period. All data

Table 4. MRP and MSP mean returns (average values).

	MMAD		DMAD		GMD		DGMD	
Period	MRP	MSP	MRP	MSP	MRP	MSP	MRP	MSP
A	6.26	54.92	5.41	34.09	6.24	40.02	5.83	30.63
B	6.45	59.48	6.18	40.28	6.26	43.96	6.21	37.29
C	2.58	40.14	4.61	28.87	3.08	33.30	2.83	27.56

are average values computed over the estimation periods. Through the analysis of Table 4 some conclusions on the market trend can be drawn: the MRP return tends to increase from Period A to Period B and then fall in Period C. It is worth noticing that by comparing Table 4 with Tables 2 and 3 we can conclude that the mean-safety models have generated the corresponding MSPs with the higher mean returns but with the lower diversification in comparison with the mean-risk models.

As an additional insight into the models comparison, the efficient frontiers found by the models over different periods can be presented (figure 3). We have generated the efficient frontiers obtained by different models for Period A, B and C, respectively. Recall that $\bar{\delta}(\mathbf{x}) \leq \Gamma(\mathbf{x})$ as well as the enhanced downside risk measures are always larger (in terms of absolute values) than the original ones. Therefore, the efficient frontiers corresponding

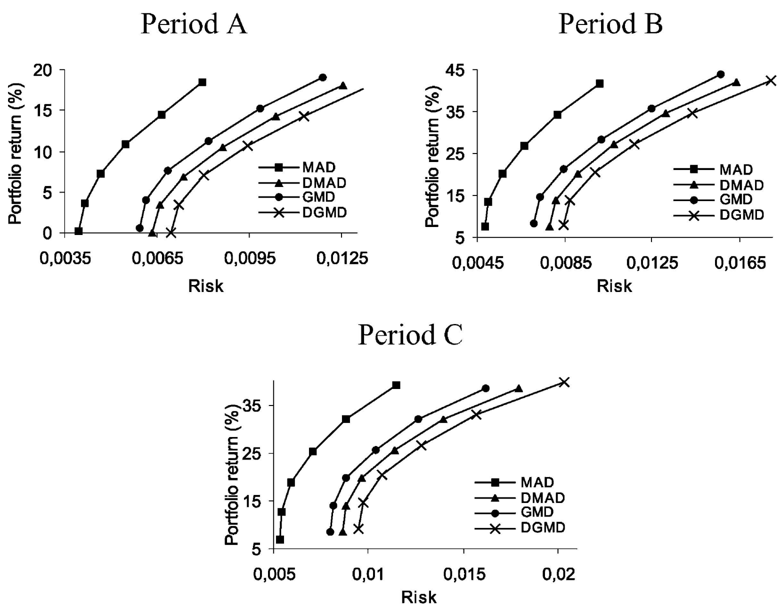

Figure 3. Efficient frontiers.

to the downside risk measures (represented by the DMAD and DGMD models) are shifted most to the right for all the periods. Except from this obvious shifts, the frontiers turn out to be very similar with respect to the shape. It is interesting to notice that in all cases the DMAD frontier tends to be located between those of GMD and DGMD thus justifying the DMAD as quite a good approximation to GMD models (which is not true for the original MAD).

4.3. Out-of-sample analysis

In a real life environment, model comparisons are usually done by means of the ex-post analysis. We used the approach based on the representation of the ex-post returns of the

Table 5. Ex-post performance criteria average values of the mean-risk models ($\alpha = 0$).

Model	Per.	μ_0 (%)	$r_{\min}$ (%)	r_{av} (%)	$r_{\max}$ (%)	σ	$\bar{\sigma}$	r_{med} (%)	$r \geq \mu_0$ (%)	$r \geq z$ (%)
MAD	A	7.5	−61.03	16.64	244.81	0.3827	0.2226	11.43	57.14	50.25
		15.0	−61.78	19.36	349.54	0.4740	0.2489	10.67	44.83	44.83
	B	10.0	−61.92	18.70	217.28	0.5046	0.3085	7.87	48.36	46.31
		20.0	−63.89	18.69	164.01	0.4929	0.3111	10.48	43.03	43.03
	C	10.0	−91.41	13.85	247.56	0.5674	0.3556	7.75	46.59	45.45
		20.0	−91.44	15.25	244.12	0.5799	0.3634	15.07	42.05	42.05
DMAD	A	7.5	−59.06	15.36	201.51	0.3726	0.2220	9.22	50.74	47.78
		15.0	−60.64	19.06	344.36	0.4756	0.2494	10.76	43.35	42.36
	B	10.0	−64.32	18.78	232.02	0.5148	0.3169	9.66	49.59	47.54
		20.0	−65.76	17.62	163.29	0.4979	0.3111	9.59	38.93	38.93
	C	10.0	−91.59	16.87	221.99	0.5822	0.3704	10.90	50.00	50.00
		20.0	−92.18	16.39	209.72	0.5938	0.3764	14.82	44.32	44.32
GMD	A	7.5	−58.83	15.15	203.03	0.3663	0.2131	8.58	53.69	50.25
		15.0	−63.81	17.27	341.86	0.4544	0.2365	10.14	44.83	44.83
	B	10.0	−63.59	19.11	207.05	0.5032	0.3111	7.82	47.95	47.13
		20.0	−64.89	18.45	167.14	0.4928	0.3080	10.74	42.62	42.62
	C	10.0	−90.77	16.92	249.76	0.5825	0.3693	11.79	51.14	51.14
		20.0	−91.57	14.04	181.24	0.5604	0.3661	12.84	42.05	42.05
DGMD	A	7.5	−61.07	15.94	204.42	0.3740	0.2213	10.57	54.68	48.77
		15.0	−62.35	18.18	346.58	0.4620	0.2413	10.18	44.34	44.34
	B	10.0	−65.41	19.11	218.49	0.5130	0.3139	9.86	50.00	47.95
		20.0	−63.97	17.99	163.29	0.4959	0.3104	9.92	41.39	40.98
	C	10.0	−91.38	14.21	230.67	0.5708	0.3619	9.45	48.86	48.86
		20.0	−91.69	16.42	254.23	0.5990	0.3772	6.37	42.05	42.05

selected portfolios over a given period and on their comparison against the required levels of return. One should bear in mind that the portfolio performances are usually affected by market trends which makes very difficult to draw some uniform conclusions.

We have decided to use some performance criteria to compare different models in the out-of-sample periods. For this purpose, we have computed the following ex-post parameters:

– the minimum, average and maximum portfolio return ($r_{\min}$, r_{av} and $r_{\max}$, respectively),
– the standard deviation (σ),
– the downside semi-standard deviation ($\bar{\sigma}$),
– the median (r_{med}),
– the percentage ratio of the number of times the mean portfolio return is greater or equal to the required return ($r \geq \mu_0$),

Table 6. Ex-post performance criteria average values of the mean-safety models ($\alpha = 1$).

Model	Per.	μ_0 (%)	$r_{\min}$ (%)	r_{av} (%)	$r_{\max}$ (%)	σ	$\bar{\sigma}$	r_{med} (%)	$r \geq \mu_0$ (%)	$r \geq z$ (%)
MAD	A	7.5	−66.75	35.58	485.74	0.8329	0.4061	18.74	56.65	31.53
		15.0	−66.75	36.70	485.74	0.8581	0.4134	18.22	52.71	31.53
	B	10.0	−83.81	35.24	334.11	0.8258	0.4766	16.62	54.51	33.20
		20.0	−83.81	35.24	334.11	0.8258	0.4766	16.62	47.95	33.20
	C	10.0	−93.70	14.55	304.49	0.6095	0.3799	12.08	51.14	30.68
		20.0	−93.70	15.49	304.49	0.6244	0.3841	12.08	40.91	29.55
DMAD	A	7.5	−62.05	24.08	324.11	0.5298	0.2974	13.35	56.65	36.45
		15.0	−62.05	26.73	344.36	0.5972	0.3163	14.22	49.26	36.95
	B	10.0	−73.25	22.49	225.58	0.5736	0.3474	10.66	51.23	32.79
		20.0	−73.25	22.24	210.60	0.5663	0.3458	10.66	44.67	32.79
	C	10.0	−92.88	15.13	216.50	0.5693	0.3790	13.72	52.27	36.36
		20.0	−92.88	15.98	216.50	0.5895	0.3823	14.10	43.18	36.36
GMD	A	7.5	−63.95	27.22	458.48	0.6622	0.3222	13.26	55.17	36.45
		15.0	−63.95	29.08	458.48	0.7057	0.3355	14.49	49.26	35.96
	B	10.0	−74.83	25.07	237.66	0.5930	0.3645	15.70	53.28	32.79
		20.0	−74.83	24.98	237.66	0.5913	0.3639	15.70	47.13	32.79
	C	10.0	−93.76	11.06	196.86	0.5399	0.3652	10.48	52.27	34.09
		20.0	−93.76	12.24	196.86	0.5621	0.3718	10.48	42.05	32.95
DGMD	A	7.5	−63.32	23.04	219.62	0.4838	0.2843	12.43	56.16	37.93
		15.0	−63.32	25.18	346.58	0.5480	0.3005	12.83	48.77	38.42
	B	10.0	−72.12	21.38	229.33	0.5477	0.3317	12.19	52.87	31.15
		20.0	−72.12	21.00	229.33	0.5396	0.3293	12.25	42.21	31.15
	C	10.0	−93.89	15.24	287.02	0.5963	0.3749	10.76	51.14	38.64
		20.0	−93.89	16.09	287.02	0.6118	0.3805	11.20	39.77	36.36

Table 7. Model comparison with their downside extensions in terms of the ex-post portfolio returns.

		Mean-risk models ($\alpha = 0$)		Mean-safety models ($\alpha = 1$)	
Period	μ_0 (%)	$r_{\text{DMAD}} \geq r_{\text{MAD}}$ (%)	$r_{\text{DGMD}} \geq r_{\text{GMD}}$ (%)	$r_{\text{DMAD}} \geq r_{\text{MAD}}$ (%)	$r_{\text{DGMD}} \geq r_{\text{GMD}}$ (%)
A	7.5	45.81	55.67	48.77	56.16
	15.0	48.77	53.69	49.75	56.16
B	10.0	48.77	46.72	44.26	40.16
	20.0	47.95	43.44	45.08	39.34
C	10.0	56.82	34.09	55.68	68.18
	20.0	51.14	54.55	54.55	67.05

– the percentage ratio of the number of times the mean portfolio return is greater or equal to the in-sample return ($r \geq z$).

The minimum, maximum, average and median ex-post portfolio returns have been converted from monthly onto yearly basis. The standard deviation σ as well as the downside semi-standard deviation $\bar{\sigma}$ have been computed with respect to a given ex-post portfolio return.

In Tables 5–6 we present the average values of each criterion for the analyzed models over the three periods for all levels of the required rate of return. The first corresponds to the optimal portfolios of the mean-risk models ($\alpha = 0$), while the second refers to the optimal portfolios found by the mean-safety models ($\alpha = 1$).

Both tables show that the average portfolio returns for all the models exceed the lower required rate of return and in case of the mean-safety models also higher required rate of return, if we exclude Period C. Moreover, the average portfolio returns for the mean-safety models are higher with the exception for Period C. Furthermore, if we take into account the ratio from the last but one column, it will turn out that the mean-safety models are better than the corresponding mean-risk models. It is worth noticing that the DMAD and DGMD models provide better average portfolio returns in Period C (downward trend) than the MAD and GMD models, respectively. This can be clearly seen through the analysis of Table 7, where we compare the MAD and GMD models with their downside extensions. In Table 7 we put together percentage ratios of the number of times the mean portfolio return generated by the MAD and GMD models is greater or equal to the mean portfolio return of the corresponding downside enhancements. Those ratios are always grater than 50% in Period C (with the only exception for the DGMD mean-risk model with μ_0 of 10%) and also in Period A for all the DGMD models. Hence, apart from the case of a definitely upward trend, the DGMD mean-safety models seem to be reasonable portfolio optimization tools.

5. Concluding remarks

The Markowitz model of portfolio optimization quantifies the problem in a lucid form of only two criteria: the mean, representing the expected outcome, and the risk—a scalar measure of the variability of outcomes. The classical Markowitz model uses the variance as the risk measure, thus resulting in a quadratic optimization problem. There were introduced several alternative risk measures which are computationally attractive as (for discrete random variables) they result in solving linear programming (LP) problems. The LP solvability is very important for applications to real-life financial decisions where the constructed portfolios have to meet numerous side constraints and take into account transaction costs.

Typical LP computable risk measures, like the mean semideviation (from the mean), used in the MAD model, as well as the Gini's mean difference are symmetric (in the sense that for $R_{\mathbf{x}}$ and $-R_{\mathbf{x}}$ they have exactly the same values). For better modeling of the risk averse preferences one may enhance the below-mean downside risk aversion in various measures. The below-mean risk downside aversion is a concept of risk aversion assuming that the variability of returns above the mean should not be penalized since the investors concern of an underperformance rather than the overperformance of a portfolio. This can be formalized with the below-mean downside stochastic dominance formulated for general distributions by focusing on the corresponding distributions of downside underachievements. In particular, the below-mean (downside) second degree stochastic dominance (BMSSD), opposite to the standard SSD, allows a portfolio with a smaller expected return to dominate some more risky portfolios with larger expectations. Thus, although consistent with the SSD itself, the BMSSD significantly enriches the risk aversion modeling capabilities. While applying these constructions to the LP solvable risk measures (or rather the safety measures corresponding the original risk measures) we have managed to explain the risk measure of the m–MAD model [16] and, more important, we have managed to introduce a new measure of the downside Gini's mean difference.

The theoretical results are valid for various LP computable risk measures. However, we have focused on the analysis of the enhanced MAD and GMD models. Computer simulation of the assets allocation problem built on historical values of 81 S&P500 sectorial indices has shown that the mean-safety models performs on average better than the corresponding mean-risk models. Moreover, in the case of stable or decreasing market, the below-mean downside risk aversion enhancement has further improved average performances of the mean-safety models. These promising results show a need for comprehensive experimental studies analyzing practical performances of the enhanced below-mean downside risk measures within specific areas of financial applications.

Note

1. GICS is the industry classification structure used for Standard & Poor's U.S. industry index calculations.

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