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On solving the dual for portfolio selection by optimizing Conditional Value at Risk

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Abstract This note is focused on computational efficiency of the portfolio selection models based on the Conditional Value at Risk (CVaR) risk measure. The CVaR measure represents the mean shortfall at a specified confidence level and its optimization may be expressed with a Linear Programming (LP) model. The corresponding portfolio selection models can be solved with general purpose LP solvers. However, in the case of more advanced simulation models employed for scenario generation one may get several thousands of scenarios. This may lead to the LP model with huge number of variables and constraints thus decreasing the computational efficiency of the model. To overcome this difficulty some alternative solution approaches are explored employing cutting planes or nondifferential optimization techniques among others. Without questioning importance and quality of the introduced methods we demonstrate much better performances of the simplex method when applied to appropriately rebuilt CVaR models taking advantages of the LP duality.

Keywords Risk measures · Portfolio optimization · Computability · Linear programming

1 Introduction

Recently, the second order quantile risk measures have become popular in finance and banking. The simplest such measure, now commonly called the Conditional Value at Risk (CVaR) after Rockafellar and Uryasev [14], represents the mean shortfall at a

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specified confidence level. The CVaR measures optimization is consistent with the second degree stochastic dominance [11]. Several empirical analyses confirm its applicability to various financial optimization problems [1, 9]. The corresponding portfolio optimization models can be built as Linear Programming (LP) problems and solved with general purpose LP solvers. However, in the case of more advanced simulation models employed for scenario generation one may get several thousands of scenarios [3, 13] thus leading to the LP model with huge number of auxiliary variables and constraints and thereby hardly solvable by general LP tools. Actually, in the case of fifty thousand scenarios and two hundred instruments the model may require more than an hour computation time with the state-of-art LP solver (CPLEX code). To overcome this difficulty some alternative solution approaches are searched trying to approximate the returns with a factor representation [5], to reformulate the optimization problems as two-stage recourse problems [7], or to employ cutting planes [2]. Recently, Lim et al. [8] have developed the dedicated techniques of nondifferentiable optimization capable to solve effectively large scale CVaR models. We are afraid, however, that reported low performance of the general purpose LP solvers (simplex method) might be misleading for the CVaR models users who are less familiar with the LP modeling and solution procedures. Without questioning the importance and quality of the introduced methods we want to emphasize much better performances of the simplex method when applied to appropriately rebuilt CVaR models.

Actually, in the standard LP model for CVaR optimization, the number of constraints (matrix rows) is proportional to the number of scenarios, while the number of variables (matrix columns) is proportional to the total of the number of scenarios and the number of instruments. However, in the case of problems with numerous constraints the computational efficiency of the simplex method can easily be improved by taking advantages of the LP duality, i.e. by solving the dual to the original problem. This applies to various LP computable portfolio optimization models [6, 12] and it applies also to the CVaR model with large number of scenarios. In this note we show that the computational efficiency can be dramatically improved with an alternative model taking advantages of the LP duality. In the introduced dual model the number of structural constraints (matrix rows) is proportional to the number of instruments thus not affecting so seriously the simplex method efficiency by the number of scenarios.

2 The LP models

Let us consider portfolio optimization problem with security returns given by discrete random variables where coefficients r_{ij} denote return realizations for security i ($i = 1, 2, \dots, n$) under scenario j ($j = 1, 2, \dots, J$). Following formula (6) of [8], the CVaR portfolio optimization model can be formulated as the following LP problem:

$$\min_{\xi, z_j, x_i} \quad \xi + \sum_{j=1}^{J} v_j z_j \tag{1a}$$

$$\text{s.t.} \quad z_j + \sum_{i=1}^{n} r_{ij} x_i + \xi \geq 0 \quad \text{for } j = 1, \dots, J \tag{1b}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{1c}$$

$$x_i \geq 0 \quad \text{for } i = 1, \ldots, n, \qquad z_j \geq 0 \quad \text{for } j = 1, \ldots, J \tag{1d}$$

where ξ is unbounded variable. Variables x_i with (1c) represent shares of several securities in the portfolio. Note that with returns r_{ij} the corresponding losses are represented by $-r_{ij}$. Coefficients v_j are given as quantities $p_j/(1-\alpha)$ where p_j denotes the probability of scenario j and α is the confidence level (CVaR parameter). Actually, Lim et al. [8] assume all the scenarios equally probable ($p_j = 1/J$) and therefore they use a single coefficient $v = [(1-\alpha)J]^{-1}$ in model (6). Except from the core portfolio constraints (1c), model (1) contains J nonnegative variables z_j plus single ξ variable and J corresponding linear inequalities. Hence, its dimensionality is proportional to the number of scenarios J. Exactly, the LP model (1) contains $J+n+1$ variables and $J+1$ constraints. For limited number of scenarios such LP models are easily solvable by the general purpose LP solvers.

The use of advanced simulation models for scenario generation results in the corresponding LP model (1) containing a huge number of variables and constraints thus decreasing dramatically computational efficiency of the simplex method. Indeed, while the number of scenarios was fixed as 50,000 to provide an adequate approximation to the underlying unknown continuous return distribution the model (for 100 or 200 securities) became hardly solvable by the simplex method while still reasonably solvable the barrier method [8, Table 2].

The LP dual to model (1) takes the following form:

$$\max_{\eta, u_j} \quad \eta \tag{2a}$$

$$\text{s.t.} \quad \eta + \sum_{j=1}^{J} r_{ij} u_j \leq 0 \quad \text{for } i = 1, \ldots, n \tag{2b}$$

$$\sum_{j=1}^{J} u_j = 1 \tag{2c}$$

$$0 \leq u_j \leq v_j \quad \text{for } j = 1, \ldots, J \tag{2d}$$

Note that model may be considered a special case within the general theory of dual representations of coherent measures of risk, following from conjugate duality [10] which allows variables u_t to have the interpretation of probability distributions.

The dual LP model (2) contains J variables u_j corresponding to inequalities (1b), but the J constraints corresponding to variables z_j from (1) take the form of simple upper bounds (SUB) on u_j (2d) thus not affecting the problem complexity. Actually, the number of structural constraints in (2) is proportional to the total of portfolio size n and it is independent from the number of scenarios. Exactly, there are $J+1$ variables and $n+1$ constraints (2b) and (2c). This guarantees a high computational efficiency of the dual model even for very large number of scenarios.

Table 1 Computational times (in seconds) for the dual CVaR model (averages of 10 instances with 50,000 scenarios)

Number of securities	$\alpha = 0.95$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$	$\alpha = 0.6$
$n = 50$	5.3	7.1	8.8	9.8	10.0
$n = 100$	13.6	17.9	22.8	24.8	25.7
$n = 200$	38.9	52.6	68.00	74.9	75.8

We have run computational tests on large scale instances developed by Lim et al. [8]. They were originally generated from a multivariate normal distribution for 50, 100 or 200 securities with the number of scenarios 50,000. All computations were performed on a PC with the Intel Core i7 2.66 GHz processor and 6 GB RAM employing the simplex code of the CPLEX 12.1 package. An attempt to solve the primal model (1) with $\alpha = 0.95$ resulted in 580, 1443 and 5006 seconds of computation on average, for problems with 50, 100 and 200 securities, respectively. Solving the dual models (2) directly by the primal method (standard CPLEX settings) results in computation times 5.3, 13.6 and 38.9 CPU seconds, respectively. Moreover, the computation times remain very low for various confidence levels as shown in Table 1.

Note that the computation times reported in Table 2 of [8] are for the confidence level $\alpha = 0.95$ which corresponds to the first column of Table 1. Even if one takes into account, that the CPU used in [8] is approximately two times slower than the one used in our computations, the performances of the simplex method applied to the dual model are significantly better than those of the nondifferentiable optimization techniques as well as the barrier method reported in [8, Table 2]. Similar results were achieved in our earlier experiments using CPLEX 9.1 instead of the 12.1 package. One may notice, however, that the approach of Lim et al. [8] achieves near-optimality at the end of phase 2 itself with competitive effort as noted in Table 2 of [8].

While trying to take advantages of the CPLEX automatic dualization in the presolve phase (PREDUAL parameter [4]) instead of directly formulating the dual of the portfolio selection problem we got the solution times of 23.7, 45.0 and 106.4 seconds, for problems with 50, 100 and 200 securities, respectively, and $\alpha = 0.95$. Thus the times turned out to be very attractive when compared with the primal model solution times (about 20 times shorter) but significantly worse than those for directly rebuilt models (Table 1). Actually, the performances of the automatic dualization approach are quite comparable to those of Lim et al. [8]. Indeed, the nondifferentiable optimization approach adopted in [8] essentially solves the dual to the formulated linear program, albeit by using a nondifferentiable optimization technique.

3 Concluding remarks

The presented computational tests show that the simplex method can effectively solve large scale CVaR models provided that it is appropriately applied to the dual problem. Actually, the portfolio selection problems of fifty thousand scenarios and two hundred instruments can be solved with the general purpose LP solvers in less than a minute.

The simplex method performances on the dual CVaR model although beating those of the barrier algorithm and the nondifferentiable optimization techniques on problems up to 200 securities are not easily scalable with respect to the number of securities. Increasing number of securities results in increasing number of constraints of the dual model. The latter together with huge number of variables may significantly lower the simplex method efficiency. Therefore, our results do not question the importance and quality of the nondifferentiable optimization techniques introduced in [8]. They clarify the direct methods applicability for larger CVaR portfolio optimization problems while leaving only extremely huge problems defined by both the large number of scenarios and the large number of securities as requiring specialized optimization techniques. One may also notice that the dual transformation cannot be applied in the case of problems with more complicated nonconvex portfolio structure restrictions. In particular, it cannot be applied to mixed integer programming models related to nonconvex transaction costs.

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