Chapter 51
Sequential Algorithms for Max-Min Fair Bandwidth Allocation

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Abstract Telecommunications networks are facing an increasing demand for Internet services. Therefore, a problem of network dimensioning with elastic traffic arises, which requires the allocation of bandwidth to maximize service flows with fair treatment for all the services. In such applications, the so-called Max-Min Fairness (MMF) solution is widely used to formulate the resource allocation scheme. It assumes that the worst service performance is maximized, and the solution is additionally regularized with the lexicographic maximization of the second worst performance, the third worst, etc. Because of lexicographic maximization of ordered quantities, the MMF solution cannot be tackled by the standard optimization model. It can be formulated as a sequential lexico-graphic optimization procedure. Unfortunately, the basic sequential procedure is applicable only for convex models; thus it allows dealing with basic design problems, but fails if practical discrete restrictions commonly arising in tele-communications network design are to be taken into account. In this paper we analyze alternative sequential approaches allowing the solving of nonconvex MMF network dimensioning problems. Both of our approaches are based on sequential optimization of directly defined artificial criteria. The criteria can be introduced into the original model with some auxiliary variables and linear inequalities; thus the methods are easily implementable

51.1 Introduction

A fair method of bandwidth distribution among competing demands becomes a key issue in computer networks [3], and telecommunications network design in general [7, 8, 17, 19]. Due to the increasing demand for Internet services, the problem of network dimensioning with elastic traffic arises, which requires the

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allocation of bandwidth to maximize service flows with fair treatment for all the services [17]. The problem of network dimensioning with elastic traffic can be formulated as follows [16]. Given a network topology $G = <V,E>$, consider a set of pairs of nodes as the set $J = \{1, 2, ..., m\}$ of services representing the elastic flow from source v_j^s to destination v_j^d. For each service, we have given the set P_j of possible routing paths in the network from the source to the destination. This can be represented in the form of binary matrices $\Delta_e = (\delta_{ejp})_{j \in J; p \in P_j}$ assigned to each link $e \in E$, where $\delta_{ejp} = 1$ if link e belongs to the routing path $p \in P_j$ (connecting v_j^s with v_j^d) and $\delta_{ejp} = 0$ otherwise. For each service $j \in J$, the elastic flow from source v_j^s to destination v_j^d is a variable representing the model outcome and it will be denoted by x_j. This flow may be realized along various paths $p \in P_j$, and it is modeled as $x_j = \sum_{p \in P_j} x_{jp}$, where x_{jp} are nonnegative variables representing the elastic flow from source v_j^s to destination v_j^d along the routing path $p \in P_j$. The single-path model requires additional multiple choice constraints to enforce nonbifurcated flows.

The network dimensioning problem depends on allocating the bandwidth to several links in order to maximize the flows of all the services (demands). For each link $e \in E$, decision variables $\xi_e \geq 0$ represent the bandwidth allocated to link $e \in E$. Certainly, there are usually some bounds (upper limits) on possible expansion of the links capacities: $\xi_e \leq \bar{a}_e$ for all $e \in E$. Finally, the following constraints must be fulfilled:

$$0 \leq x_{jp} \leq Mu_{jp} \quad \forall j \in J; \; p \in P_j \tag{51.1a}$$

$$u_{jp} \in \{0, 1\} \quad \forall j \in J; \; p \in P_j \tag{51.1b}$$

$$\sum_{p \in P_j} u_{jp} = 1 \quad \forall j \in J \tag{51.1c}$$

$$\sum_{p \in P_j} x_{jp} = x_j \quad \forall j \in J \tag{51.1d}$$

$$\sum_{j \in J} \sum_{p \in P_j} \delta_{ejp} x_{jp} \leq \xi_e \quad \forall e \in E \tag{51.1e}$$

$$0 \leq \xi_e \leq \bar{a}_e \quad \forall e \in E \tag{51.1f}$$

$$\sum_{e \in E} c_e \xi_e \leq B \tag{51.1g}$$

where (51.1a), (51.1b), (51.1c), and (51.1d) represent single-path flow requirements using additional binary (flow assignment) variables u_{jp} and define the total service flows. Next, (51.1e) establishes the relation between service flows and links bandwidth. The quantity $y_e = \sum_{j \in J} \sum_{p \in P_j} \delta_{ejp} x_{jp}$ is the load of link e

and it cannot exceed the available link capacity. Further, while allocating the bandwidth to several links, the decisions must keep the cost within the available budget B (51.1 g), where for each link $e \in E$ the cost of allocated bandwidth is c_e.

The network dimensioning model can be considered with various objective functions, depending on the chosen goal. Typically, the fairness requirement is formalized with the lexicographic maxi-minimization (lexicographic Max-Min approach). Within telecommunications or network applications the lexicographic Max-Min approach has already appeared in [4] and now, under the name Max-Min Fair (MMF), is treated as one of the standard fairness concepts [2, 6, 10, 17, 20]. Indeed, the MMF approach generalizes equal sharing at a single link bandwidth to any network, also allowing the maximization of the second smallest flows provided that the smallest remains optimal, the third smallest, etc.

The lexicographic maxi-minimization can be seen as searching for a vector lexicographically maximal in the space of the feasible vectors with components rearranged in nondecreasing order. This can be mathematically formalized as follows. Let $\langle \mathbf{a} \rangle = \left(a_{\langle 1 \rangle}, a_{\langle 2 \rangle}, ..., a_{\langle m \rangle} \right)$ denote the vector obtained from $\mathbf{a}$ by rearranging its components in nondecreasing order. That means $a_{\langle 1 \rangle} \leq a_{\langle 2 \rangle} \leq ... \leq a_{\langle m \rangle}$, and there exists a permutation π of set J such that $a_{\langle j \rangle} = a_{\pi(j)}$ for $j = 1, 2, \ldots, m$. Lexicographically comparing such ordered vectors $\langle y \rangle$ one gets the so-called lex-min order. The MMF problem can be then represented in the following way:

$$\text{lex max} \left\{ \left(y_{\langle 1 \rangle}, y_{\langle 2 \rangle}, ..., y_{\langle m \rangle} \right) : y \in A \right\} \tag{51.2}$$

where A where depicts the set of attainable outcomes defined with constraints (51.1). Actually, we focus our analysis on the MMF bandwidth allocation problem, but the approaches developed can be applied to various lexicographic Max-Min optimization problems, i.e., to problem (51.2) with various attainable sets A.

The (point-wise) ordering of outcomes means that the lexicographic Max-Min problem (51.2) is, in general, hard to implement. Note that the quantity $y_{\langle 1 \rangle}$, representing the worst outcome, can easily be computed directly by the maximization:

$$y_{\langle 1 \rangle} = \max \left\{ r_1 : r_1 \leq y_j \quad \forall j \in J \right\}.$$

A similar simple formula does not exist for the further ordered outcomes $y_{\langle i \rangle}$. Nevertheless, for convex problems it is possible to build sequential algorithms for finding the consecutive values of the (unknown) MMF optimal outcome vector. While solving Max-Min problems for convex models, there exists at least one blocked outcome which is constant for the entire set of optimal solutions to the Max-Min problem. Hence, the MMF solution can be found by solving a sequence of properly defined Max-Min problems with fixed outcomes (flows) that have been blocked by some critical constraints (link capacities) [12]. Indeed, in the case of LP models this leads to efficient algorithms taking advantage of the duality theory for simple identification of blocked

Fig. 51.1 Sample network without any critical link and blocked flow for max-min solution
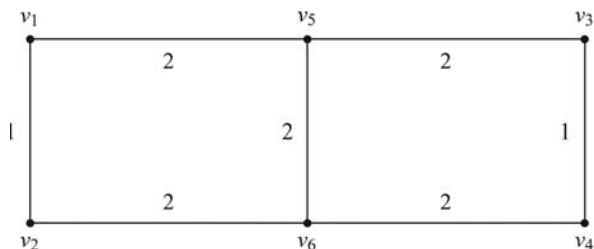
outcomes [1, 5, 18]. Unfortunately, in our network dimensioning model it applies only to the basic LP constraints (51.1d), (51.1e), (51.1f) and (51.1 g). In the case of a nonconvex feasible set, such a blocked quantity may not exist [11], which makes the approach not applicable to our case of nonbifurcated flows enforced by discrete constraints (51.1a), (51.1b), and (51.1c). This can be illustrated with the simplified network depicted in Fig. 51.1 with link capacity limits given in the figure, cost coefficients 4 for links (v_1,v_5), 3 for (v_3,v_5), and all others equal to 1, and the budget B = 11. We consider two demands: one connecting v_1 with v_2 along two possible paths (v_1,v_2) or (v_1,v_5,v_6,v_2); the second connecting v_3 with v_4 along two possible paths (v_3,v_4) or (v_3,v_5,v_6,v_4). The MMF solution is unique; it allocates flow 1 to path (v_1,v_2) (first demand) and flow 2 to path (v_3,v_5,v_6,v_4) (second demand). The Max-Min (single-path) problem leads us to the conclusion that one of two flows cannot exceed 1, but does not allow us to identify which one must be blocked. Note that the same difficulty arises also for the single path problem without any budget constraint, though the optimal solution is then not unique.

In this paper we analyze alternative sequential approaches to solving non-convex MMF network dimensioning problems. Both approaches are based on the lexicographic optimization of directly defined artificial criteria. The criteria can be introduced into the original model with some auxiliary variables and linear inequalities independently from the problem structure.

51.2 Cumulated Ordered Outcomes

The point-wise ordering of outcomes for lexicographic optimization within the MMF problem (51.2) is, in general, hard to implement. Following Yager [21], a direct formula, although requiring the use of integer variables, can be given for any $y_{\langle i \rangle}$. Namely, for any $k = 1, 2, ..., m$, the following formula is valid:

$$y_{\langle i \rangle} = \max r_i$$
$$\text{s.t.}$$
$$r_i - y_j \leq C z_{ij} \quad \forall j \in J \qquad (51.3)$$
$$z_{ij} \in \{0, 1\} \quad \forall j \in J$$
$$\sum_{j \in J} z_{ij} \leq i - 1$$

where C is a sufficiently large constant (larger than any possible difference between various individual outcomes y_j) which allows us to enforce inequality $r_i \leq y_j$ for $z_{ij} = 0$, while ignoring it for $z_{ij} = 1$. Note that for $i = 1$ all binary variables z_{1j} are forced to 0, thus reducing the optimization in this case to the standard LP model. However, for any other $i > 1$, all m binary variables z_{ij} are an important part of the model. Nevertheless, with the use of auxiliary integer variables, any MMF problem (either convex or nonconvex) can be formulated as the standard lexicographic maximization with directly defined objective functions:

$$\text{lex max}(r_1, r_2, ..., r_m)$$

$$\text{s.t.}$$

$$y \in A$$
$$r_i - y_j \leq C z_{ij} \quad \forall i, j \in J$$
$$z_{ij} \in \{0, 1\} \quad \forall i, j \in J \tag{51.4}$$
$$\sum_{j \in J} z_{ij} \leq i - 1 \quad \forall i \in J$$

We will refer to the above model as the Direct Ordered Outcomes (DOO) approach. Unfortunately, binary variables z_{ij} in the auxiliary constraints contribute to implementation difficulties of the DOO approach.

There is, however, a way to reformulate the MMF problem (51.2) so that only linear variables are used. Let us consider the cumulated criteria $\bar{\theta}_i(y) = \sum_{k=1}^{i} y_{\langle k \rangle}$, expressing, respectively, the worst (smallest) outcome, the total of the two worst outcomes, the total of the three worst outcomes, etc. Within the lexicographic optimization, a cumulation of criteria does not affect the optimal solution. Hence, the MMF problem (51.2) can be formulated as the standard lexicographic maximization with cumulated ordered outcomes:

$$\text{lex max}\left\{ \left(\bar{\theta}_1(y), \bar{\theta}_2(y), ..., \bar{\theta}_m(y) \right) : y \in A \right\}$$

Note that for any given vector $y \in R^m$ the cumulated ordered value $\bar{\theta}_i(y)$ can be found as the optimal value of the following LP problem:

$$\bar{\theta}_i(y) = \min \sum_{j \in J} y_j u_{ij}$$

$$\text{s.t.}$$

$$\sum_{j \in J} u_{ij} = k \tag{51.5}$$

$$0 \leq u_{ij} \leq 1 \quad \forall j \in J$$

The above problem is an LP for a given outcome vector y, while it becomes nonlinear for y being a variable. This difficulty can be overcome by taking advantage of the LP duality. Note that the LP duality of problem (51.5), with variable r_i corresponding to the equation $\sum_{j \in J} u_{ij} = k$ and variables d_{ij} corresponding to upper bounds on u_{ij}, leads us to the following formula:

$$\bar{\theta}_i(y) = \max ir_i - \sum_{j \in J} d_{ij}$$

s.t.

$$r_i - y_j \le d_{ij}, \ d_{ij} \ge 0 \quad \forall j \in J$$

(51.6)

It follows from (51.6) that $\bar{\theta}_k(y) = \max\left\{kr_k - \sum_{j \in J}\left(y_j - r_k\right)_+ : y \in A\right\}$, where $(.)_+$ denotes the nonnegative part of a number and r_k is an auxiliary (unbounded) variable. The latter, with the necessary adaptation to the minimized outcomes in the location problems, is equivalent to the computational formulation of the k–centrum model introduced in [15]. Hence, the LP dual transformation provides an alternative proof of that formulation.

Following (51.6), we may express the MMF problem (51.2) as a standard lexicographic optimization problem with predefined linear criteria:

$$\text{lex max}\left(r_1 - \sum_{j \in J} d_{1j}, 2r_2 - \sum_{j \in J} d_{2j}, ..., mr_m - \sum_{j \in J} d_{mj}\right)$$

s.t.

$$y \in A$$

$$d_{ij} \ge r_i - y_j \quad \forall i, j \in J$$

$$d_{ij} \ge 0 \quad \forall i, j \in J$$

(51.7)

We will refer to the above model as the Cumulated Ordered Outcomes (COO) approach.

Theorem 1. *An attainable outcome vector* $\in A$ *is an optimal solution of the MMF problem (51.2), if and only if it is an optimal solution of the COO model (51.7).*

Note that this direct lexicographic formulation of the COO model remains valid for nonconvex (e.g. discrete) models, where the standard sequential approaches [9] are not applicable. Model COO preserves the problem's convexity when the original problem is defined with convex feasible set A. In particular, for an LP original problem, it remains within the LP class while introducing $m^2 + m$ auxiliary variables and m^2 constraints. Thus, for many problems with not too large a number of services (demands) m, problem (51.7) can easily be solved directly.

51.3 Shortfalls to Ordered Targets

For some specific classes of discrete, or rather combinatorial, optimization problems, one may take advantage of the finiteness of the set of all possible outcome values. The ordered outcome vectors may be treated as describing a distribution of outcomes y. In the case where there exists a finite set of all possible outcomes, we can directly describe the distribution of outcomes with frequencies of outcomes. Let $V = \{v_1, v_2, ..., v_r\}$ (where $v_1 < v_2 < ... < v_r$) denote the set of all attainable outcomes. We introduce integer functions $h_k(y)$ $(k = 1, 2, ..., r)$ expressing the number of values v_k in the outcome vector y. Having defined functions h_k, we can introduce cumulative distribution functions:

$$\bar{h}_k(y) = \sum_{l=1}^{k} h_l(y), \quad k = 1, ..., r \tag{51.8}$$

Function $\bar{h}_k$ expresses the number of outcomes smaller than or equal to v_k. Since we want to maximize all the outcomes, we are interested in the minimization of all functions $\bar{h}_k$. Indeed, the following assertion is valid [11]: For outcome vectors $y', y'' \in V^m$, $\langle y' \rangle \geq \langle y'' \rangle$ if and only if $\bar{h}_k(y') \leq \bar{h}_k(y'')$ for all $k = 1, 2, ..., r$. This equivalence allows us to express the MMF problem (51.2) in terms of the standard lexicographic minimization problem with objectives $2\bar{h}(y)$:

$$\text{lex min}\left\{ \left(\bar{h}_1(y), ..., \bar{h}_r(y) \right) : y \in A \right\} \tag{51.9}$$

Theorem 2. *An attainable outcome vector $y \in A$ is an optimal solution of the MMF problem (51.2), if and only if it is an optimal solution of the lexicographic problem (51.9).*

The quantity $\bar{h}_k(y)$ can be computed directly by the minimization:

$$\bar{h}_k(y) = \min \sum_{j \in J} z_{kj}$$

$$\text{s.t. } v_{k+1} - y_j \leq C z_{kj}, \ z_{kj} \in \{0, 1\} \quad \forall j \in J$$

where C is a sufficiently large constant. Note that $\bar{h}_r(y) = m$ for any y, which means that the r-th criterion is always constant and therefore redundant in (51.9). Hence, the lexicographic problem (51.9) can be formulated as the following mixed integer problem:

$$\text{lex min} \left(\sum_{j \in J} z_{1j}, \sum_{j \in J} z_{2j}, ..., \sum_{j \in J} z_{r-1,j} \right)$$

$$\text{s.t.}$$

$$v_{k+1} - y_j \leq C z_{kj} \quad j \in J, \ k < r$$

$$z_{kj} \in \{0, 1\} \quad j \in J, \ k < r$$

$$y \in A$$

$$\tag{51.10}$$

Taking advantage of possible weighting and cumulating achievements in lexicographic optimization, one may eliminate auxiliary integer variables from the achievement functions. For this purpose we weight and cumulate vector $2\bar{h}(y)$ to get $\hat{h}_1(y) = 0$ and:

$$\hat{h}_k(y) = \sum_{l=1}^{k-1}(v_{l+1} - v_l)\bar{h}_l(y), \quad k = 2, ..., r \qquad (51.11)$$

Due to the positive differences $v_{l+1} - v_l > 0$, the lexicographic minimization problem (51.9) is equivalent to the lexicographic problem with objectives $2\hat{h}(y)$:

$$\text{lex min}\left\{\left(\hat{h}_1(y), ..., \hat{h}_r(y)\right) : y \in A\right\} \qquad (51.12)$$

which leads us to the following assertion.

Theorem 3. *An attainable outcome vector $y \in A$ is an optimal solution of the MMF problem (51.2), if and only if it is an optimal solution of the lexicographic problem (51.12).*

Actually, vector function $\hat{\mathbf{h}}(y)$ provides a unique description of the distribution of coefficients of vector y, i.e., for any $y', y'' \in V^m$ one gets: $\hat{\mathbf{h}}(y') = \hat{\mathbf{h}}(y) \Leftrightarrow \langle y'\rangle = \langle y\rangle$. Moreover,s $\hat{\mathbf{h}}(y') \leq \hat{\mathbf{h}}(y'')$ if and only if $\overline{\Theta}(y') \geq \overline{\Theta}(y)$ [11].

Note that $\hat{h}_1(y) = 0$ for any y, which means that the first criterion is constant and redundant in problem (51.12). Moreover, putting (51.8) into (51.11) allows us to express all achievement functions $\hat{h}_k(y)$ as piecewise linear functions of y:

$$\hat{h}_k(y) = \sum_{j\in J}\max\{v_k - y_j, 0\} \quad k = 1, ..., r \qquad (51.13)$$

Hence, the quantity $\hat{h}_k(y)$ can be computed directly by the following minimization:

$$\hat{h}_k(y) = \min \sum_{j\in J} t_{kj}$$

$$\text{s.t.} \qquad\qquad\qquad\qquad\qquad\qquad (51.14)$$

$$v_k - y_j \leq t_{kj}, \; t_{kj} \geq 0 \quad \forall j \in J$$

Therefore, the entire lexicographic model (51.12) can be formulated as follows:

$$\text{lex min}\left(\sum_{j\in J}t_{2j}, \sum_{j\in J}t_{3j}, ..., \sum_{j\in J}t_{rj}\right)$$

$$\text{s.t.}$$

$$v_k - y_j \leq t_{kj} \quad j \in J, \; k = 2, ..., r \qquad (51.15)$$

$$t_{kj} \geq 0 \quad j \in J, \; k = 2, ..., r$$

$$y \in A$$

We will refer to the above model as the Shortfalls to Ordered Targets (SOT) approach.

Note that the above formulation, unlike the problem (51.10), does not use integer variables and can be considered as an LP modification of the original constraints (51.1). Thus, this model preserves the problem's convexity when the original problem is defined with a convex set A. The size of problem (51.15) depends on the number of different outcome values. Thus, for many problems with not too large a number of outcome values, the problem can easily be solved directly. Note that in many problems of telecommunications network design, the objective functions express the quality of service, and one can easily consider a limited finite scale (grid) of the corresponding outcome values. One may also notice that model (51.15) opens a way for the fuzzy representation of quality measures within the MMF problems.

51.4 Computational Experiments

We have performed some initial tests of the sequential approaches to the MMF network dimensioning problem (51.1). We have not assumed any bandwidth granulation and thereby no grid of possible bandwidth values that can be allocated. Therefore, in the case of the Shortfalls to Ordered Targets approach, the resulting bandwidth allocation is only an approximation to the exact MMF solution.

For the experiments we used a set of 10 randomly generated problems for each tested size. The problems were generated as follows. First, we created a random but consistent network structure. Then we chose random node pairs to define services. For each service three different possible flow routes between the two end nodes were generated. Two of them were fully random and one was the shortest path between the nodes (with the smallest number of links). We decided to use the integer grid of the v_k values in the ordered values approach, that is, to check each integer value from the feasible set of objective values. In this case the number of targets depends on the range of the feasible objective values. We managed to restrict the number of targets to the range of 5–10, applying different link capacities for different problem sizes. We set the large budget limit to B, thus relaxing the budget constraints (51.1 g).

We analyzed the performance of the three sequential approaches: the Direct Ordered Outcomes (DOO) model (51.4), the Cumulated Ordered Outcomes (COO) model (51.7), and the Shortfalls to Ordered Targets (SOT) model (51.15), with the condition $y \in A$ representing the bandwidth allocation problem defined with constraints (51.1). Each model was computed using the standard sequential algorithm for lexicographic optimization with predefined objective functions. For the lexicographic maximization problem $\text{lex} \max\{(g_1(y), ..., g_m(y)) : y \in Y\}$, the algorithm reads as follows:

Step 0: Put $k := 1$.

Step 1: Solve problem P_k:
$$\max_{y \in Y}\{\tau_k : \tau_k \leq g_k(y),\ \tau_j^0 \leq g_j(y)\ \forall j < k\}$$
denote the optimal solution by (y^0, τ_k^0).

Step 2: If $k = m$, **STOP** (y^0 is MMF optimal).
Otherwise, put $k := k + 1$ and
go to **Step 1**.

For example, the algorithm for the COO model worked according to the above scheme with functions g_k defined as $kt_k - \sum_{j \in J} d_{kj}$. Let $k = 1$. Following (51.7), we built the initial problem P_1 with the objective $\tau_1 = t_1 - \sum_{j \in J} d_{1j}$ being maximized and m constraints of the form $t_1 - d_{1j} \leq y_j$, $j = 1...m$. The expression $y \in A$ of (51.7) was replaced by (51.1). Each new problem P_k in subsequent iterations ($k > 1$) was built by adding new constraints $\tau_{k-1}^0 \leq t_{k-1} - \sum_{j \in J} d_{k-1,j}$ and $t_k - d_{kj} \leq y_j$, $j = 1...m$ to problem P_{k-1}, where τ_{k-1}^0 was the optimal objective value of P_{k-1}. A similar algorithm was performed for the DOO and the SOT approaches. The difference was in the objectives and auxiliary constraints, as defined in (51.4) and (51.15), respectively. All the tests were performed on a Pentium IV 1.7 GHz computer employing the CPLEX 9.1 package.

Table 51.1 presents solution times for the three approaches being analyzed. The times are averages of 10 randomly generated problems. The upper index denotes the number of tests out of 10 for which the timeout of 120 s occurred. The minus sign '–' shows that the timeout occurred for all 10 test problems. One can notice that, while for smaller problems with the number of services equal to five, all three approaches perform very well, for bigger problems only the SOT approach gives acceptable results (in the sense of solving a majority of the problems within the 120 s time limit).

To examine how the number of targets in the SOT approach influences the test results, we also performed similar experiments increasing the capacities of the links and considering 15 to 25 targets. This did not significantly affect the DOO and COO approaches. For the SOT approach the computing times

Table 51. 1 Computation times (in s) for different solution approaches

	# of nodes	# of links	Number of services				
			5	10	20	30	45
	5	10	0.0	1.2			
DOO	10	20	0.0	6.8	–	–	–
(4)	15	30	0.0	3.9	–	–	–
	5	10	0.0	0.2			
COO	10	20	0.0	1.3	[3]62.9	–	–
(7)	15	30	0.1	1.0	[5]78.0	–	–
	5	10	0.1	0.1			
SOT	10	20	0.0	0.3	4.1	[2]35.0	[7]101
(15)	15	30	0.1	0.3	7.1	[4]72.4	[8]106

Table 51. 2 Computation times (in s) for problems with increased link capacities

			Number of services				
	# of nodes	# of links	5	10	20	30	45
	5	10	0.1	0.1			
SOT	10	20	0.1	1.3	23.8	[4]74.3	–
(15)	15	30	0.1	1.2	33.9	[8]108.0	–

increased (Table 51.2), but it still outperformed both the DOO and COO approaches.

51.5 Conclusion

As lexicographic maximization in the Max-Min Fair optimization is not applied to any specific order of the original outcomes, the MMF optimization can be very hard to implement in general nonconvex (possibly discrete) problems. We have shown that the introduction of some artificial criteria with auxiliary variables and linear inequalities allows one to model and to solve the MMF problems in a very efficient way. We have performed initial tests of computational performance of the presented models for the MMF network dimensioning problem. It turns out that both the models outperform the Direct Ordered Outcomes model. The Shortfall to Ordered Targets model enables to solve within 2 min a majority of the MMF single-path dimensioning problems for networks with 15 nodes and 30 links. Such performance is enough for the efficient analysis of a country's backbone network of ISPs (12 nodes and 18 links in the case of Poland [14]). Nevertheless, further research is necessary on the models and corresponding algorithms tailored to specific MMF network optimization problems. The models may also be applied to various MMF resource allocation problems, not necessarily related to networks.

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