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Theory and Methodology

Comments on properties of the minmax solutions in goal programming

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Abstract

This note discusses the properties of solutions generated by the minmax models of goal programming (GP) and compromise programming (CP). GP approaches use a certain target point in the criterion (attribute) space to model decision maker's preferences. When the ideal (utopia) point is used as the target, the minmax GP model coincides with the minmax (Chebyshev) CP model. In a recent review of the current GP state-of-the-art, there have been included suggestions that the two equivalent models ensure Pareto efficiency of solutions and they guarantee a perfectly balanced allocation among the achievement of the individual targets. In this note, it is shown that the models, in general, do not ensure the efficiency of solutions and they do not guarantee the perfect equity among the individual achievements. Moreover, there are given sufficient and necessary conditions clarifying when the discussed properties of minmax solutions do occur. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

This note deals with multi-criteria optimization problems. Without loss of generality, it is assumed that all the criteria are maximized (that is, for each attribute 'more is better'). Similar to Tamiz et al. (1998), the following notations are used for the techniques under examination:

$\mathbf{x}$	decision vector (vector of decision variables)
Q	feasible set
q	number of attributes under consideration
$f_i(\mathbf{x})$	mathematical expression for the ith attribute (the ith criterion)
y_i	value of the ith attribute, $y_i = f_i(\mathbf{x})$
$\mathbf{y}$	outcome vector, $\mathbf{y} = (y_1, \dots, y_q)$
Y_a	set of attainable outcome vectors, $Y_a = \{(y_1, \dots, y_q) : \ y_i = f_i(\mathbf{x}), \ i = 1, 2, \dots, q; \ \mathbf{x} \in Q\}$
b_i	aspiration or reference level attached to the ith attribute

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b target (reference) point (in the outcome space), $\mathbf{b} = (b_1, \ldots, b_q)$

b_i^* ideal value corresponding to the *i*th attribute, $b_i^* = \max\{f_i(\mathbf{x}) : x \in Q\}$

$\mathbf{b}^*$ ideal (utopia) point (in the outcome space), $\mathbf{b}^* = (b_1^*, \ldots, b_q^*)$

An attainable outcome vector $\bar{\mathbf{y}} \in Y_a$ is called *nondominated* if there does not exist another attainable vector $\mathbf{y} \in Y_a$ such that $\mathbf{y} \geqslant \bar{\mathbf{y}}$ and $\mathbf{y} \neq \bar{y}$. The feasible decision vectors that generate nondominated outcome vectors are referred to as *efficient* (Pareto-optimal) solutions of the multi-criteria optimization problem. This means that each feasible decision vector for which one cannot improve any attribute value without worsening another is an efficient solution.

The minmax scalarization was widely studied in the multi-criteria optimization methodology (Steuer, 1986). The optimal set of the minmax scalarization always contains an efficient solution. Thus, if unique, the optimal solution of the minmax scalarization is efficient. In the case of multiple optimal solutions, one of them is efficient but also some of them may not be efficient. It is a serious flaw since practical large problems usually have multiple optimal solutions and typical optimization solvers generate one of them (essentially at random). Therefore, to overcome this flaw of the minmax scalarization, it is additionally regularized with the weighted aggregation to guarantee the efficiency of solutions, thus resulting in the so-called augmented minmax scalarization. In particular, the augmented minmax scalarization is used in the reference point method (RPM) which is an interactive technique in which the DM specifies preferences in terms of reference levels (Wierzbicki, 1977, 1982). Depending on the specified reference levels, scalarizing achievement function is built which, when minimized, generates an efficient solution to the problem. One of the simplest scalarizing achievement functions takes the following form (Steuer, 1986):

$$\max_{1 \leqslant i \leqslant q} \{v_i(b_i - f_i(\mathbf{x}))\} + \varepsilon \sum_{i=1}^{q} v_i(b_i - f_i(\mathbf{x})), \qquad (1)$$

where b_i denote reference levels, ε an arbitrarily small positive number and v_i are positive weights assigned to the corresponding attributes. In particular, similar to Tamiz et al. (1998), one may consider weights $v_i = w_i/k_i$ where k_i is the normalization constant attached to *i*th attribute and w_i is the preferential weight attached to the *i*th attribute.

As shown by Ogryczak (1994), RPM with the scalarizing function (1) can be expressed in terms of the goal programming (GP) implementation environment of deviational variables as the following RGP model:

$$\text{lex min} \left[\max_{1 \leqslant i \leqslant q} \{v_i(n_i - p_i)\}, \sum_{i=1}^{q} v_i(n_i - p_i) \right] \qquad (2)$$

subject to

$$f_i(\mathbf{x}) + n_i - p_i = b_i; \quad n_i, p_i \geqslant 0, \quad i = 1, 2, \ldots, q, \qquad (3)$$

$$\mathbf{x} \in Q. \qquad (4)$$

The RGP model always generates an efficient solution to the original multi-criteria problem simultaneously satisfying the RPM rules (Ogryczak, 1994). The RGP model (2)–(4) is similar to the standard minmax (fuzzy) GP model for maximized attributes (Ignizio, 1982)

$$\min \left[\max_{1 \leqslant i \leqslant q} \{v_i n_i\} \right] \qquad (5)$$

subject to (3) and (4).

However, the RGP model differs from (5) due to the use of negative weights ($-v_i$) attached to positive deviational variables and lexicographic minimization of the additional regularization term $\sum_{i=1}^{q} v_i(n_i - p_i)$.

Consider the case where reference levels are fixed at their ideal values b_i^* which results in the goal constraints

$$f_i(\mathbf{x}) + n_i - p_i = b_i^*, \quad n_i, p_i \geqslant 0, \quad i = 1, 2, \ldots, q, \qquad (6)$$

where by definition of the ideal values, all the positive deviational variables p_i are redundant (are equal to zero for all $\mathbf{x} \in Q$). The minmax GP model

$$\min \left[\max_{1 \leqslant i \leqslant q} \{v_i n_i\} \right] \tag{7}$$

subject to (6) and (4)

corresponds then to a compromise programming (CP) formulation with the Chebyshev (L_∞) metric (Zeleny, 1974). Since $p_i = 0$ for $i = 1, 2, \ldots, q$, in this particular situation the RGP model can be written as

$$\text{lex}\ \min \left[\max_{1 \leqslant i \leqslant q} \{v_i n_i\}, \sum_{i=1}^{q} v_i n_i \right] \tag{8}$$

subject to (6) and (4),

which differs from (7) due to the lexicographic minimization of the additional regularization term $\sum_{i=1}^{q} v_i n_i$.

The lexicographic minimization of the regularization term in (8) guarantees the efficiency of each optimal solution. In a recent GP state-of-the-art paper (Tamiz et al., 1998, p. 575) it is claimed that in the particular situation of $b_i = b_i^*$ the regularization term in (8) is redundant as the minmax GP model (7) itself guarantees the efficiency of solutions. Moreover, it is claimed the minmax GP solution is then perfectly equilibrated in the sense that the following chain of equalities holds (Tamiz et al., 1998, Eq. (10), p. 576):

$$\begin{aligned} v_1(b_1^* - f_1(\mathbf{x})) &= \cdots = v_i(b_i^* - f_i(\mathbf{x})) \\ &= \cdots = v_q(b_q^* - f_q(\mathbf{x})). \end{aligned} \tag{9}$$

The allegations are quite intuitive and valid in the case of bicriteria ($q = 2$) linear programming (Ballestero and Romero, 1991). However, in general case ($q > 2$), they are unjustified which will be shown with simple counterexamples in the following section.

2. The counterexamples

Since Tamiz et al. (1998) have considered multicriteria linear programming problems, we present counterexamples within this environment. Let us consider a problem with three attributes,

$$f_1(\mathbf{x}) = x_1, \quad f_2(\mathbf{x}) = x_2, \quad f_3(\mathbf{x}) = x_3, \tag{10}$$

maximized on the feasible set

$$\begin{aligned} Q = \{ (x_1, x_2, x_3) : x_1 + x_2 \leqslant 2, \\ 1.5 \leqslant x_3 \leqslant 2,\ x_i \geqslant 0,\ i = 1, 2, 3 \}. \end{aligned} \tag{11}$$

From this data, it is straightforward to obtain the efficient set as

$$\{ (x_1, x_2, x_3) : x_1 + x_2 = 2,\ x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 = 2 \}.$$

Further, the ideal point is $\mathbf{b}^* = (2, 2, 2)$. Let us consider the corresponding minmax GP problem (7) with equal weights: $v_1 = v_2 = v_3 = 1$. It can be written as the following linear programming problem (Tamiz et al., 1998, Eq. (8), p. 575):

$$\min\ z \tag{12}$$

subject to

$$n_i \leqslant z, \quad i = 1, 2, 3, \tag{13}$$

$$x_i + n_i - p_i = 2, \quad i = 1, 2, 3, \tag{14}$$

$$x_1 + x_2 \leqslant 2, \tag{15}$$

$$1.5 \leqslant x_3 \leqslant 2, \tag{16}$$

$$x_i,\ n_i,\ p_i \geqslant 0, \quad i = 1, 2, 3. \tag{17}$$

One can easily find that the decision vectors that are optimal for (12)–(17) form the set

$$\{ (x_1, x_2, x_3) : x_1 = 1,\ x_2 = 1,\ 1.5 \leqslant x_3 \leqslant 2 \}. \tag{18}$$

In this solution set only vector $(1, 1, 2)$, generated by the regularized problem (8), is an efficient solution while all the others are not efficient. In particular, the decision vector $(1, 1, 1.5)$ (together with $z = 1$, $p_1 = p_2 = p_3 = 0$, $n_1 = n_2 = 1$ and $n_3 = 0.5$) is a vertex optimal solution of the problem (12)–(17) and it is not efficient. Thus the minmax GP problem itself does not guarantee the efficiency of solutions. Therefore, the claim that when the reference point is fixed at the ideal point, the regularization term in RPM becomes redundant which makes the RPM model equivalent to the minmax GP (Tamiz et al., 1998, p. 575) is not correct.

Further, one can easily check that in the set (18) there is no solution satisfying the requirement of perfect equilibration (9). For any solution $\mathbf{x}$ from the set (18): $v_1(b_1^* - f_1(\mathbf{x})) = v_2(b_2^* - f_2(\mathbf{x})) = 1$

while $0 \leqslant v_3(b_3^* - f_3(\mathbf{x})) \leqslant 0.5$, and for the only efficient solution $(1,1,2)$ one gets $v_3(b_3^* - f_3(\mathbf{x})) = 0$. In fact, in the example there do not exist feasible solutions satisfying the requirement of perfect equilibration. Thus, even in the case of multi-criteria linear programming, setting the targets at the corresponding ideal values is not enough to guarantee the efficiency and perfect equilibration of the minmax GP solutions.

Note that even the unique minmax solution, and therefore the efficient one, may not be perfectly equilibrated. This can be demonstrated with three attributes (10) maximized on the feasible set

$$Q = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leqslant 7,\\ 3 \leqslant x_3 \leqslant 4,\ x_1 \geqslant 0,\ x_2 \geqslant 0\}.$$

The ideal point is $\mathbf{b}^* = (4,4,4)$ and the corresponding minmax GP problem (7) with equal weights ($v_1 = v_2 = v_3 = 1$) has the unique optimal solution $\bar{\mathbf{x}} = (2,2,3)$. It is easy to see that this solution does not satisfy the perfect equilibration condition (9).

To make the above examples as simple as possible we have considered them with unit weights ($v_1 = v_2 = v_3 = 1$). However, one can easily notice that for both examples there exist many other weight settings leading to the same results. Further, let us recall that v_i denote the final weights while Tamiz et al. (1998) consider the weighting scheme of the form w_i/k_i where k_i is the normalization constant attached to ith attribute and w_i is the (arbitrary) preferential weight attached to the ith attribute. Nevertheless, the above examples (without explicit use of the normalization constants k_i) comprise with the latter models when the preferential weights w_i are defined by the formula $w_i = v_i k_i$ thus resulting in $w_i/k_i = v_i$.

3. Sufficient and necessary conditions

The examples presented in the previous section show that, in general, the minmax scalarization does not guarantee the efficiency and perfect equilibration of the solutions. There arises a question as to when the minmax solutions have these two properties. Obviously, if the minmax scalarization generates solutions which are efficient and perfectly equilibrated, then such a solution must exist. Therefore, the necessary condition is the existence of a nondominated outcome vector $\bar{\mathbf{y}} \in Y_a$ such that

$$v_1(b_1 - \bar{y}_1) = \cdots = v_i(b_i - \bar{y}_i) = \cdots = v_q(b_q - \bar{y}_q). \tag{19}$$

We will show that this is also a sufficient condition.

Theorem 1. *For any target vector* $\mathbf{b}$ *and any positive weight coefficients* $v_i > 0,\ i = 1,2,\ldots,q,$ *if there exists a nondominated outcome vector* $\bar{\mathbf{y}} \in Y_a$ *satisfying the equilibration requirement* (19), *then* $\bar{\mathbf{y}}$ *is the unique optimal solution of the minmax problem*

$$\min \left\{ \max_{1 \leqslant i \leqslant q} \{v_i(b_i - y_i)\} : \mathbf{y} \in Y_a \right\}. \tag{20}$$

Proof. Let $\bar{\mathbf{y}} \in Y_a$ be a nondominated vector satisfying the equilibration requirement (19) with some target vector $\mathbf{b}$ and positive weight coefficients $v_i > 0$, $i = 1,2,\ldots,q$. This means, there exists a number α such that $v_i(b_i - \bar{y}_i) = \alpha$ for $i = 1,2,\ldots,q$.

Suppose that $\bar{\mathbf{y}}$ is not the unique optimal solution of the minmax problem (20). Then, there exists an attainable outcome vector $\mathbf{y} \in Y_a$ such that $\mathbf{y} \neq \bar{\mathbf{y}}$ and

$$\max_{1 \leqslant i \leqslant q} \{v_i(b_i - y_i)\} \leqslant \max_{1 \leqslant i \leqslant q} \{v_i(b_i - \bar{y}_i)\} = \alpha.$$

Hence

$$v_i(b_i - y_i) \leqslant \alpha = v_i(b_i - \bar{y}_i) \quad \text{for } i = 1,2,\ldots,q.$$

Thus, $\mathbf{y} \geqslant \bar{\mathbf{y}}$ and $\mathbf{y} \neq \bar{\mathbf{y}}$ which contradict the assumption that $\bar{\mathbf{y}}$ is nondominated. $\square$

Corollary 1. *For any target vector* $\mathbf{b}$ *and any positive weight coefficients* $v_i > 0,\ i = 1,2,\ldots,q,$ *if there exists a nondominated outcome vector* $\bar{\mathbf{y}} \in Y_a$ *satisfying the equilibration requirement* (19), *then each optimal solution of the minmax problem*

$$\min\left\{\max_{1\leqslant i\leqslant q}\{v_i(b_i - f_i(\mathbf{x}))\}:\quad \mathbf{x}\in Q\right\}$$

is efficient and perfectly equilibrated.

In the case when all the target values satisfy the inequalities $b_i \geqslant b_i^*$, the positive deviational variables in the goal constraint (3) become redundant and $n_i = (b_i - f_i(\mathbf{x}))$ for $i = 1, 2, \ldots, q$. Therefore, the following assertion is valid.

Corollary 2. *For any target values $b_i \geqslant b_i^*$, $i = 1, 2, \ldots, q$ and any positive weight coefficients $v_i > 0$, $i = 1, 2, \ldots, q$, if there exists a nondominated outcome vector $\bar{\mathbf{y}} \in Y_a$ satisfying the equilibration requirement* (19), *then each optimal solution of the minmax GP problem*

$$\min\left\{\max_{1\leqslant i\leqslant q}\{v_i n_i\} : f_i(\mathbf{x}) + n_i - p_i = b_i,\right.$$
$$\left. i = 1, \ldots, q, \quad \mathbf{x}\in Q\right\}$$

is efficient and perfectly equilibrated.

It follows from Corollary 2 that the minmax GP model with targets fixed at the corresponding ideal values indeed generates efficient and perfectly equilibrated solutions, provided that such efficient solutions exist. However, as shown in the previous section, linear programming structure of the decision problem is not enough to guarantee that a perfectly equilibrated efficient solution exists. Even when there exists a feasible perfectly equilibrated decision vector, one cannot be sure that a perfectly equilibrated efficient solution exists. For instance, when decreasing the lower bound on variable x_3 in (11) to get $1 \leqslant x_3 \leqslant 2$, the perfectly equilibrated solution $(1, 1, 1)$ becomes feasible but it is not efficient. Therefore, while applying the minmax GP approach (5) to a practical multi-criteria decision problem one cannot expect the guaranteed efficiency of generated solutions.

Note that Theorem 1 and Corollary 1 are valid for any multi-criteria optimization problems, including discrete and nonconvex ones. It follows from Corollary 1 that the RGP model (2)–(4) generate an efficient and perfectly equilibrated solution whenever such a solution exists. Moreover, the RGP model, due to the regularization, always generates an efficient solution to the original multi-criteria problem. Thus, by applying the RGP model (2)–(4) to a practical multi-criteria decision problem, one guarantees the efficiency of generated solutions and one may expect that the solution will be perfectly equilibrated if such a possibility exists.

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