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ON PRIMAL-DUAL THIRD DEGREE STOCHASTIC DOMINANCE

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1. STOCHASTIC DOMINANCE

In the stochastic dominance approach random variables are compared by pointwise comparison of some performance functions constructed from their distribution functions. Let X be a random variable representing some returns. The first performance function $F_1(X,r)$ is defined as the right-continuous cumulative distribution function itself: $F_1(X,r) = \mathbf{P}[X \le r]$ for $r \in R$. We say that X weakly dominates Y under the FSD rules ($X \succeq_{FSD} Y$), if $F_1(X,r) \le F_1(Y,r)$ for all $r \in R$, and X FSD dominates Y ($X \succ_{FSD} Y$), if at least one strict inequality holds. Actually, the stochastic dominance is a stochastic order thus defined on distributions rather than on random variables themselves. Nevertheless, it is a common convention, that in the case of random variables X and Y having distributions P_X and P_Y, the stochastic order relation $P_X \succeq P_Y$ might be viewed as a relation on random variables $X \succeq Y$ (Müller and Stoyan, 2002).

The second degree stochastic dominance relation is defined with the second performance function $F_2(X,r)$ given by areas below the cumulative distribution function itself, i.e.: $F_2(X,r) = \int_{-\infty}^{r} F_1(X,t)dt$ for $r \in R$. Similarly to FSD, we say that X weakly dominates Y under the SSD rules ($X \succeq_{SSD} Y$), if $F_2(X,r) \le F_2(Y,r)$ for all $r \in R$, while X SSD dominates Y ($X \succ_{SSD} Y$), when at least one inequality is strict. Certainly, $X \succ_{FSD} Y$ implies $X \succ_{SSD} Y$. Function $F_2(X,r)$, used to define the SSD relation can also be presented as follows (Ogryczak and Ruszczyński, 1999): $F_2(X,r) = \mathbf{E}[\max\{r - X, 0\}]$, thus representing the mean below-target deviations from real targets.

Alternatively, the stochastic dominance order can be expressed on the inverse cumulative functions (quantile functions) (Wang and Young, 1998). Namely, for random variable X, one may consider the performance function $F_{-1}(X,p)$ defined as is the left-continuous inverse of the cumulative distribution function $F_1(X,r)$, i.e., $F_{-1}(X,p) = \inf\ \{\eta : F_1(X,\eta) \ge p\}$. Obviously, X dominates Y under the FSD rules ($X \succ_{FSD} Y$), if $F_{-1}(X,p) \ge F_{-1}(Y,p)$ for all $p \in [0,1]$, where at least one strict inequality holds. Further, the second quantile function (or the so-called *Absolute Lorenz Curve* ALC) is defined by integrating F_{-1} as $F_{-2}(X,p) = \int_0^p F_{-1}(X,\alpha)d\alpha$ for $0 < p \le 1$. Actually, as shown in (Ogryczak and Ruszczyński, 2002), $F_{-2}(X,p) = \max_{r \in R}\ [pr - F_2(X,r)]$. Hence, by the theory of convex conjugate (dual) functions, the pointwise comparison of ALCs provides an alternative characterization of the SSD relation in the sense that $X \succeq_{SSD} Y$ if and only if $F_{-2}(X,\beta) \ge F_{-2}(Y,\beta)$ for all $0 < \beta \le 1$.

If $X \succ_{SSD} Y$, then X is preferred to Y within all risk-averse preference models that prefer larger outcomes. In terms of the expected utility theory the SSD relation represent all the preferences modeled with increasing and concave utility functions. It is therefore a matter of primary importance that a stochastic optimization model be consistent with the second degree stochastic dominance relation. However, in many applications one may deserve stronger risk averse. The classical higher degree stochastic dominance relations no longer maintain the equivalence of the primal and dual (inverse) models. This paper introduces a concept of the

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primal-dual higher degree stochastic dominance which preserve the equivalence of the primal and inverse dominance relations.

2. PRIMAL-DUAL TSD

Classical higher degree stochastic dominance relations depend on performance functions derived by integrating those of lower degrees. The third function $F_3(X,r)$ is given by integrating F_2, i.e.: $F_3(X,r) = \int_{-\infty}^{r} F_2(X,t)dt$ for $r \in R$ and it can also be presented as follows (Ogryczak and Ruszczyński, 2001): $F_3(X,r) = \mathbf{E}[\max\{r - X, 0\}^2]/2$, thus representing the mean square below-target deviations from real targets. The kth function $F_k(X,r)$ is defined as: $F_k(X,r) = \int_{-\infty}^{r} F_{k-1}(X,t)dt$ for $r \in R$. Similarly to FSD and SSD, we say that X weakly dominates Y under the kSD rules ($X \succeq_{kSD} Y$), if $F_k(X,r) \leq F_k(Y,r)$ for all $r \in R$. Certainly, $X \succ_{(k-1)SD} Y$ implies $X \succ_{kSD} Y$. One may also consider the higher degree quantile performance functions (Muliere and Scarsini, 1989). In particular, the third quantile function is defined by integrating as $F_{-3}(X,p) = \int_0^p F_{-2}(X,\alpha)d\alpha$ for $0 < p \leq 1$, while higher degree functions can respectively be built. Although, already the third degree inverse SD relation, $X \succeq_{TISD} Y$ iff $F_{-3}(X,p) \geq F_{-3}(Y,p)$ for all $0 < p \leq 1$, is not equivalent to the primal TSD. Moreover, function F_{-3} is neither monotonic nor convex as already F_{-2} is not always monotonic.

In order to build a primal-dual third degree stochastic dominance concept we need to normalize the corresponding second performance functions prior to their integration. We introduce a nondecreasing performance function $H_2(X,.) : R \rightarrow [0,1]$ and its generalized inverse $H_{-2}(X,.) = H_2^{-1}(X,.)$ such that $X \succeq_{SSD} Y$ iff $H_2(X,r) \leq H_2(Y,r)$ for all $r \in R$, and equivalently $H_{-2}(X,p) \geq H_{-2}(Y,p)$ for all $0 < p \leq 1$. In other words, we introduce alternative performance functions similar to a cdf and its inverse, respectively, but defining the second degree stochastic dominance instead of the FSD. The simplest way to define such performance functions is

$$H_2(X,\eta) = \sup\{p : F_2(X,\eta+\xi) \geq p\xi \ \forall_{\xi \geq 0}\}$$
$$H_{-2}(X,p) = \inf\{\eta : F_{-2}(X,p) \leq p\eta \ \}$$

When introducing the set of random variables $Q(\eta,p) = \{Z : \mathbf{P}[Z < \eta] = 0,\ \mathbf{P}[Z \leq \eta] \geq p\}$ the functions can be interpreted as follows. $H_2(X,\eta)$ represents then the largest p such that $\hat{X} \succeq_{SSD} X$ for some $\hat{X} \in Q(\eta,p)$ while $H_{-2}(X,p)$ represents the smallest η such that $\hat{X} \succeq_{SSD} X$ for some $\hat{X} \in Q(\eta,p)$.

By integration we get the third degree performance functions $H_3(X,r) = \int_{-\infty}^{r} H_2(X,t)dt$ for $r \in R$ and $H_{-3}(X,p) = \int_0^p H_{-2}(X,\alpha)d\alpha$ for $0 < p \leq 1$, respectively. Such functions are convex and they form a pair of conjugate functions. This allows us to define the third degree primal-dual stochastic dominance (TPDSD) as $X \succeq_{TPDSD} Y$ iff $H_3(X,r) \leq H_3(Y,r)$ for all $r \in R$, and equivalently $H_{-3}(X,p) \geq H_{-3}(Y,p)$ for all $0 < p \leq 1$. Obviously, $X \succeq_{SSD} Y$ implies $X \succeq_{TPDSD} Y$, but not vice versa. Similar approach one may apply to construct higher degree primal-dual stochastic dominance relations.

Various risk averse models can be build by using TPDSD performance functions as optimization criteria. Note that $H_{-2}(X,p) = F_{-2}(X,p)/p$ thus representing the TailVaR risk measures (known also as Average VaR or Conditional VaR). There is no simple formula for $H_2(X,r)$. Nevertheless, for both $H_{-3}(X,p)$ and $H_3(X,r)$ the corresponding integral approximations can be quite easily defined.

This paper presents initial analysis of the TPDSD relation and corresponding risk averse optimization models.

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