International Journal of Approximate Reasoning 50 (2009) 915–928

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

On efficient WOWA optimization for decision support under risk

Włodzimierz Ogryczak *, Tomasz Śliwiński

Warsaw University of Technology, Faculty of Electronics and IT, Institute of Control & Computation Engineering, Nowowiejska 15/19, 00-665 Warsaw, Poland

ARTICLE INFO

Article history:
Received 15 April 2008
Received in revised form 2 October 2008
Accepted 12 February 2009
Available online 6 March 2009

Keywords:
Decision under risk
Preference modeling
Aggregation operators
OWA
WOWA
Linear programming

ABSTRACT

The problem of averaging outcomes under several scenarios to form overall objective functions is of considerable importance in decision support under uncertainty. The so-called Weighted OWA (WOWA) aggregation offers a well-suited approach to this problem. The WOWA aggregation, similar to the classical ordered weighted averaging (OWA), uses the preferential weights assigned to the ordered values (i.e. to the worst value, the second worst and so on) rather than to the specific criteria. This allows one to model various preferences with respect to the risk. Simultaneously, importance weighting of scenarios can be introduced. In this paper, we analyze solution procedures for optimization problems with the WOWA objective functions related to decisions under risk. Linear programming formulations are introduced for optimization of the WOWA objective with monotonic preferential weights thus representing risk averse preferences. Their computational efficiency is demonstrated.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

Consider a decision problem under uncertainty where the decision is based on the maximization of a scalar (real valued) outcome. The final outcome is uncertain and only its realizations under various scenarios are known. Exactly, for each scenario $S_i (i = 1, \ldots, m)$ the corresponding outcome realization is given as a function of the decision variables $y_i = f_i(\mathbf{x})$. We are interested in larger outcomes under each scenario. Hence, the decision under uncertainty can be considered a multiple criteria optimization problem:

$$\max_{\mathbf{x} \in \mathcal{F}} \quad (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})), \tag{1}$$

where $\mathbf{x}$ denotes a vector of decision variables to be selected within the feasible set $\mathcal{F} \subset R^q$ of constraints under consideration and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ is a vector function that maps the feasible set $\mathcal{F}$ into the criterion space R^m.

From the perspective of decisions under uncertainty, model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. In order to make it operational, one needs to assume some solution concept specifying what it means to maximize multiple objective functions. The solution concepts are defined by aggregation functions $a : R^m \to R$. Thus the multiple criteria problem (1) is replaced with the (scalar) maximization problem

$$\max_{\mathbf{x} \in Q} \quad a(\mathbf{f}(\mathbf{x})).$$

The most commonly used aggregation is based on the weighted mean where positive importance weights $p_i (i = 1, \ldots, m)$ are allocated to several scenarios

* Corresponding author.
E-mail addresses: wogrycza@ia.pw.edu.pl, W.Ogryczak@ia.pw.edu.pl (W. Ogryczak), tsliwins@ia.pw.edu.pl (T. Śliwiński).

0888-613X/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.ijar.2009.02.010

$$A_{\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} y_i p_i. \tag{2}$$

The weights are typically normalized to the total $1\left(\sum_{i=1}^{m} p_i = 1\right)$ with possible interpretation as scenarios (subjective) probabilities. The weighted mean allowing to define the importance of scenarios does not allow one to model the decision maker's preferences regarding the distribution of outcomes. In particular, it does not allow one to model risk averse preferences [17].

The preference weights can be effectively introduced within the fuzzy optimization methodology with the so-called Ordered Weighted Averaging (OWA) aggregation developed by Yager [37]. In the OWA aggregation the weights are assigned to the ordered values (i.e. to the largest value, the second largest and so on) rather than to the specific criteria. This guarantees a possibility to model various preferences with respect to the risk. Since its introduction, the OWA aggregation has been successfully applied to many fields of decision making [41,42,16].

The OWA operator allows one to model various aggregation functions from the maximum through the arithmetic mean to the minimum. Thus, it enables modeling of various preferences from the optimistic to the pessimistic one [7]. On the other hand, the OWA does not allow one to allocate any importance weights to specific scenarios. Actually, the weighted mean (2) cannot be expressed in terms of the OWA aggregations. Several attempts have been made to incorporate importance weighting into the OWA operator [40,8]. Finally, Torra [29] has incorporated importance weighting into the OWA operator within the Weighted OWA (WOWA) aggregation. The WOWA averaging is defined by two weighting vectors: the preferential weights $\mathbf{w}$ and the importance weights $\mathbf{p}$. It covers both the weighted means (defined with $\mathbf{p}$) and the OWA averages (defined with $\mathbf{w}$) as special cases. Actually, the WOWA average is a particular case of Choquet integral using a distorted probability as the measure [30]. Since its introduction, the WOWA operator has been successfully applied to many fields of decision making [33] including multicriteria optimization [18] and metadata aggregation problems [15].

Example 1. As an illustration we will use simple portfolio optimization problem. An investor has to allocate his capital among various securities, thus assigning a nonnegative share of the capital to each security. During the investment period, each security generates a random rate of return. This results in a change of the capital invested (observed at the end of the period) depending on the earlier allocation decisions.

Following the (discrete) scenario analysis approach the portfolio optimization problem can be formulated as follows [16]. There is given a set $J = \{1, 2, \ldots, q\}$ of securities for an investment. We assume, as usual, that for each security $j \in J$ there is given a vector of data $(c_{ij})_{i=1,\ldots,m}$, where c_{ij} is the observed (or forecasted) rate of return of security j under scenario i (hereafter referred to as outcome). We consider discrete distributions of returns defined by the finite set $I = \{1, 2, \ldots, m\}$ of scenarios. The outcome data forms an $m \times q$ matrix $\mathbf{C} = (c_{ij})_{i=1,\ldots,m;j=1,\ldots,q}$ whose columns correspond to securities while rows $\mathbf{c}_i = (c_{ij})_{j=1,2,\ldots,q}$ correspond to outcomes for different scenarios. Further, let $\mathbf{x} = (x_j)_{j=1,2,\ldots,q}$ denote the vector of decision variables defining a portfolio. Each variable x_j expresses the portion of the capital invested in the corresponding security. Under scenario i portfolio $\mathbf{x}$ generates return $\sum_{j=1}^{q} c_{ij} x_j$

$$\mathbf{y} = \mathbf{Cx} = (\mathbf{c}_1\mathbf{x}, \mathbf{c}_2\mathbf{x}, \ldots, \mathbf{c}_m\mathbf{x}).$$

The portfolio selection problem can be considered as an LP problem with m objective functions $f_i(\mathbf{x}) = \mathbf{c}_i\mathbf{x} = \sum_{j=1}^{q} c_{ij} x_j$ to be maximized [16]:

$$\max_{\mathbf{x}} \quad \left\{ \mathbf{Cx} : \sum_{j=1}^{q} x_j = 1, \; x_j \geqslant 0 \text{ for } j = 1, \ldots, q \right\}. \tag{3}$$

Hence, our portfolio optimization problem can be considered a special case of the multiple criteria problem (1) and one may seek an optimal portfolio with some criteria aggregation.

Consider a simplified problem with 2 securities and 3 scenarios. The rates of return (in percents) are given in Table 1. Portfolio $\mathbf{x} = (x_1, x_2)$ generates then rate of return $-9x_1 + 7x_2$ under Scenario 1, $6x_1 + 7x_2$ under Scenario 2, and $9x_1 - 5x_2$ under Scenario 3. For instance, portfolio $(0.5, 0.5)$ generates rate of return -1% under Scenario 1, 13.5% under Scenario 2, and 2% under Scenario 2. The multiple criteria LP model (3) for problem from Table 1 takes the following form:

$$\max_{x_1, x_2} \quad \{(-9x_1 + 7x_2, 6x_1 + 7x_2, 9x_1 - 5x_2) : x_1 + x_2 = 1, \; x_1, x_2 \geqslant 0\}.$$

Introducing importance weights p_1, p_2 and p_2 for the corresponding scenario one may optimize the weighted average $p_1(-9x_1 + 7x_2) + p_2(6x_1 + 7x_2) + p_3(9x_1 - 5x_2)$ getting the scalarized LP problem

Table 1
Rates of return (in percents) for simplified instance of the portfolio optimization problem.

	Security 1	Security 2	Portfolio $\mathbf{x}$
Scenario 1	−9	−7	$-9x_1 + 7x_2$
Scenario 2	−6	−7	$6x_1 + 7x_2$
Scenario 2	−9	−5	$9x_1 - 5x_2$

$$\max_{x_1,x_2} \quad \{(-9p_1+6p_2+9p_3)x_1+(7p_1+7p_2-5p_3)x_2 : x_1+x_2=1,\ x_1,x_2 \geqslant 0\}.$$

Such a model results, however, in very risky optimal solutions defined as single security portfolios. Indeed, portfolio $(1,0)$ is the unique optimal solution when $-9p_1+6p_2+9p_3>7p_1+7p_2-5p_3$ and portfolio (0, 1) when $-9p_1+6p_2+9p_3<7p_1+7p_2-5p_3$.

The risk aversion preferences may be modeled with the OWA preferential weights. For instance, with preferential weights $\mathbf{w}=(0,0,1)$ one gets max–min aggregation

$$\max_{x_1,x_2} \quad \{\min\{-9x_1+7x_2, 6x_1+7x_2, 9x_1-5x_2\} : x_1+x_2=1,\ x_1,x_2 \geqslant 0\}$$

leading to the optimal portfolio (0.4,0.6) guaranteeing rather low but positive return under each scenario (0.6% under Scenarios 1 or 3, and 7.6% under Scenario 2).

The WOWA aggregation enables one to model both the importance of scenarios as well as the risk averse preferences. We show further that the corresponding WOWA optimization problem can be modeled with auxiliary linear inequalities and effectively solved.

While many researchers have paid attention to the problem of OWA weights determination [1–3,35] the OWA and WOWA optimization problems have not gain much attention. The weighting of the ordered outcome values causes that the OWA optimization problem is nonlinear even for linear programming (LP) formulation of the original constraints and criteria. Yager [38] has shown that the OWA optimization can be converted into a mixed integer programming problem. We have shown [21,24] that the OWA optimization with monotonic weights can be formed as a standard linear program of higher dimension. Recently, similar concepts we have outlined for the WOWA optimization [22,23]. In this paper, we analyze in details solution models for optimization problems with the WOWA objective functions modeling decisions under risk. A linear programming formulation is introduced and analyzed for optimization of the WOWA objective with increasing preferential weights representing risk averse preferences. The paper is organized as follows. In the next section, we introduce formally the WOWA operator and derive some alternative computational formulas based on direct application of the preferential weights to the conditional means with respect to the importance weights. There is also introduced a generalized WOWA aggregation where the preferential weights are allocated to an arbitrarily defined grid of ordered outcomes. Further, in Section 3, we analyze the orness/andness properties of the WOWA operator with monotonic preferential weights and the corresponding risk profiles. In Section 4, we introduce the LP formulations for maximization of the WOWA and the generalized WOWA aggregations with increasing preferential weights. Finally, in Section 5 we demonstrate computational efficiency of the introduced models.

2. The Weighted OWA aggregation

2.1. The WOWA operator

Let $\mathbf{w}=(w_1,\ldots,w_m)$ be a weighting vector of dimension m such that $w_i \geqslant 0$ for $i=1,\ldots,m$ and $\sum_{i=1}^m w_i=1$. The corresponding OWA aggregation of outcomes $\mathbf{y}=(y_1,\ldots,y_m)$ can be mathematically formalized as follows [37]. First, we introduce the ordering map $\Theta : R^m \to R^m$ such that $\Theta(\mathbf{y})=(\theta_1(\mathbf{y}),\theta_2(\mathbf{y}),\ldots,\theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \geqslant \theta_2(\mathbf{y}) \geqslant \cdots \geqslant \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y})=y_{\tau(i)}$ for $i=1,\ldots,m$. Further, we apply the weighted sum aggregation to ordered achievement vectors $\Theta(\mathbf{y})$, i.e. the OWA aggregation has the following form:

$$A_{\mathbf{w}}(\mathbf{y})=\sum_{i=1}^m w_i\theta_i(\mathbf{y}). \tag{4}$$

The OWA aggregation (4) allows to model various aggregation functions from the maximum ($w_1=1, w_i=0$ for $i=2,\ldots,m$) through the arithmetic mean ($w_i=1/m$ for $i=1,\ldots,m$) to the minimum ($w_m=1, w_i=0$ for $i=1,\ldots,m-1$).

Again, let $\mathbf{w}=(w_1,\ldots,w_m)$ be an m-dimensional vector of preferential weights $w_i \geqslant 0$ for $i=1,\ldots,m$ and $\sum_{i=1}^m w_i=1$. Further, let $\mathbf{p}=(p_1,\ldots,p_m)$ be an m-dimensional vector of importance weights such that $p_i \geqslant 0$ for $i=1,\ldots,m$ and $\sum_{i=1}^m p_i=1$. The corresponding Weighted OWA aggregation of vector $\mathbf{y}=(y_1,\ldots,y_m)$ is defined [29,32] as follows:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y})=\sum_{i=1}^m \omega_i\theta_i(\mathbf{y}) \quad \text{with } \omega_i=w^*\left(\sum_{k\leqslant i} p_{\tau(k)}\right)-w^*\left(\sum_{k<i} p_{\tau(k)}\right), \tag{5}$$

where w^* is an increasing function interpolating points $(\frac{i}{m},\sum_{k\leqslant i} w_k)$ together with the point (0.0) and τ representing the ordering permutation for $\mathbf{y}$ (i.e. $y_{\tau(i)}=\theta_i(\mathbf{y})$). Moreover, function w^* is required to be a straight line when the points can be interpolated in this way. We will focus our analysis on the piecewise linear interpolation function w^* which is the simplest form of the required interpolation.

Example 2. Consider outcome vectors $\mathbf{y}'=(3,1,2,4,5)$ and $\mathbf{y}''=(1,1,2,6,4)$ where individual outcomes correspond to five scenarios. While introducing preferential weights $\mathbf{w}=(0.05,0.1,0.15,0.2,0.5)$ one may calculate the OWA averages:

$A_{\mathbf{w}}(\mathbf{y}') = 0.05 \cdot 5 + 0.1 \cdot 4 + 0.15 \cdot 3 + 0.2 \cdot 2 + 0.5 \cdot 1 = 2$ and $A_{\mathbf{w}}(\mathbf{y}'') = 0.05 \cdot 6 + 0.1 \cdot 4 + 0.15 \cdot 2 + 0.2 \cdot 1 + 0.5 \cdot 1 = 1.7$. Further, let us introduce importance weights $\mathbf{p} = (0.1, 0.1, 0.2, 0.5, 0.1)$ which means that results under the third scenario are 2 times more important then those under scenario 1, 2 or 5, while the results under scenario 4 are even 5 times more important. To take into account the importance weights in the WOWA aggregation (5) we introduce piecewise linear function

$$w^*(\xi) = \begin{cases} 0.05\xi/0.2 & \text{for } 0 \leqslant \xi \leqslant 0.2, \\ 0.05 + 0.10(\xi - 0.2)/0.2 & \text{for } 0.2 < \xi \leqslant 0.4, \\ 0.15 + 0.15(\xi - 0.4)/0.2 & \text{for } 0.4 < \xi \leqslant 0.6, \\ 0.3 + 0.2(\xi - 0.6)/0.2 & \text{for } 0.6 < \xi \leqslant 0.8, \\ 0.5 + 0.5(\xi - 0.8)/0.2 & \text{for } 0.8 < \xi \leqslant 1.0 \end{cases}$$

and calculate weights ω_i according to formula (5) as w^* increments corresponding to importance weights of the ordered outcomes, as illustrated in Fig. 1. In particular, one get $\omega_1 = w^*(p_5) = 0.025$ and $\omega_2 = w^*(p_5 + p_4) - w^*(p_5) = 0.275$ for vector $\mathbf{y}'$ while $\omega_1 = w^*(p_4) = 0.225$ and $\omega_2 = w^*(p_4 + p_5) - w^*(p_4) = 0.075$ for vector $\mathbf{y}''$. Finally, $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}') = 0.025 \cdot 5 + 0.275 \cdot 4 + 0.1 \cdot 3 + 0.35 \cdot 2 + 0.25 \cdot 1 = 2.475$ and $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}'') = 0.225 \cdot 6 + 0.075 \cdot 4 + 0.2 \cdot 2 + 0.25 \cdot 1 + 0.25 \cdot 1 = 2.55$.

Note that one may alternatively compute the WOWA values by using the importance weights to replicate corresponding scenarios and calculate then OWA aggregations. In the case of our importance weights $\mathbf{p}$ we need to consider five copies of scenario 4 and two copies of scenario 3 thus generating corresponding vectors $\tilde{\mathbf{y}}' = (3, 1, 2, 2, 4, 4, 4, 4, 4, 5)$ and $\tilde{\mathbf{y}}'' = (1, 1, 2, 2, 6, 6, 6, 6, 6, 4)$ of ten equally important outcomes. Original five preferential weights must be then applied respectively to the average of the two largest outcomes, the average of the next two largest outcomes etc. Indeed, we get $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}') = 0.05 \cdot 4.5 + 0.1 \cdot 4 + 0.15 \cdot 4 + 0.2 \cdot 2.5 + 0.5 \cdot 1.5 = 2.475$ and $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}'') = 0.05 \cdot 6 + 0.1 \cdot 6 + 0.15 \cdot 5 + 0.2 \cdot 2 + 0.5 \cdot 1 = 2.55$. We will further formalize this approach and take its advantages to build the LP computational models.

2.2. Alternative WOWA formulas

Function w^* can be defined by its generation function g with the formula $w^*(\alpha) = \int_0^\alpha g(\xi)\,d\xi$. Introducing breakpoints $\alpha_i = \sum_{k \leqslant i} p_{\tau(k)}$ and $\alpha_0 = 0$ allows us to express

$$\omega_i = \int_0^{\alpha_i} g(\xi)\,d\xi - \int_0^{\alpha_{i-1}} g(\xi)\,d\xi = \int_{\alpha_{i-1}}^{\alpha_i} g(\xi)\,d\xi$$

and the entire WOWA aggregation as

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} \theta_i(\mathbf{y}) \int_{\alpha_{i-1}}^{\alpha_i} g(\xi)\,d\xi = \int_0^1 g(\xi) \overline{F}_{\mathbf{y}}^{(-1)}(\xi)\,d\xi, \tag{6}$$

where $\overline{F}_{\mathbf{y}}^{(-1)}$ is the stepwise function $\overline{F}_{\mathbf{y}}^{(-1)}(\xi) = \theta_i(\mathbf{y})$ for $\alpha_{i-1} < \xi \leqslant \alpha_i$. It can also be mathematically formalized as follows. First, we introduce the right-continuous cumulative distribution function (cdf):

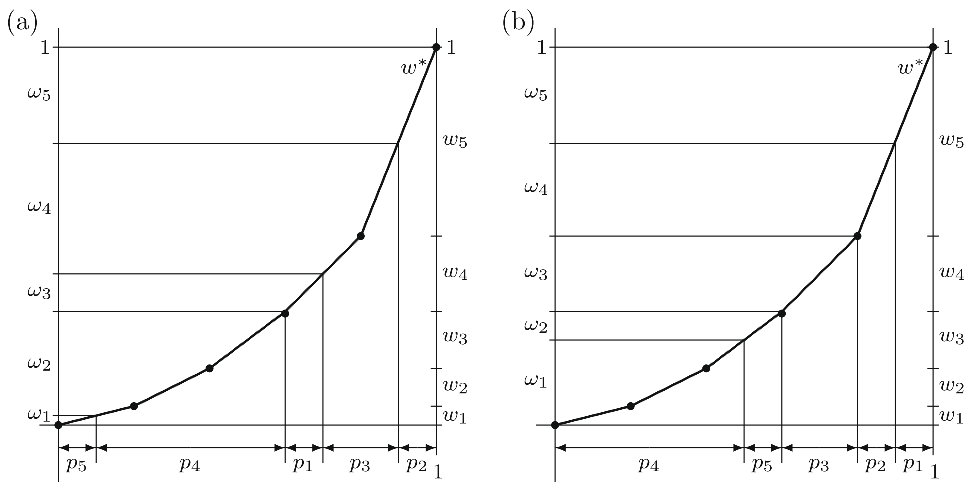

Fig. 1. Definition of weights ω_i for Example 2: (a) vector y', (b) vector y''.

$$F_{\mathbf{y}}(d) = \sum_{i=1}^{m} p_i \delta_i(d) \quad \text{where } \delta_i(d) = \begin{cases} 1 & \text{if } y_i \leqslant d, \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

which for any real (outcome) value d provides the measure of outcomes smaller or equal to d. Next, we introduce the quantile function $F_{\mathbf{y}}^{(-1)} = \inf\{\eta : F_{\mathbf{y}}(\eta) \geqslant \xi\}$ for $0 < \xi \leqslant 1$ as the left-continuous inverse of the cumulative distribution function $F_{\mathbf{y}}$, and finally $\overline{F}_{\mathbf{y}}^{(-1)}(\xi) = F_{\mathbf{y}}^{(-1)}(1-\xi)$.

Formula (6) provides the most general expression of the WOWA aggregation allowing for expansion to continuous case. The original definition of WOWA allows one to build various interpolation functions w^* [31] thus to use different generation functions g in formula (6). Let us focus our analysis on the simplest piecewise linear interpolation function w^*. Note, however, that the piecewise linear functions may be built with various number of breakpoints, not necessarily m. Thus, any nonlinear function can be well approximated by a piecewise linear function with appropriate number of breakpoints. Therefore, we will consider weights vectors $\mathbf{w}$ of dimension n not necessarily equal to m. Any such piecewise linear interpolation function w^* can be expressed with the stepwise generation function

$$g(\xi) = nw_k \quad \text{for } (k-1)/n < \xi \leqslant k/n, \quad k = 1, \ldots, n. \tag{8}$$

This leads us to the following specification of formula (6):

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} w_k n \int_{(k-1)/n}^{k/n} \overline{F}_{\mathbf{y}}^{(-1)}(\xi)\, d\xi. \tag{9}$$

Note that $n\int_{(k-1)/n}^{k/n} \overline{F}_{\mathbf{y}}^{(-1)}(\xi)\, d\xi$ represents the average within the kth portion of $1/n$ largest outcomes, the corresponding conditional mean [20,25]. Hence, formula (10) defines WOWA aggregations with preferential weights $\mathbf{w}$ as the corresponding OWA aggregation but applied to the conditional means calculated according to the importance weights $\mathbf{p}$ instead of the original outcomes. Fig. 2 illustrates application of formula (10) for computation of the WOWA aggregations in Example 2.

We will treat formula (9) as a formal definition of the WOWA aggregation of m-dimensional outcomes $\mathbf{y}$ defined by the m-dimensional importance weights $\mathbf{p}$ and the n-dimensional preferential weights $\mathbf{w}$.

Formula (9) may be reformulated to use the left-tail averages

$$\begin{aligned} A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) &= \sum_{k=1}^{n} w_k n \int_{(k-1)/n}^{k/n} F_{\mathbf{y}}^{(-1)}(1-\xi)\, d\xi = \sum_{k=1}^{n} w_k n \left(\int_0^{(k-1)/n} F_{\mathbf{y}}^{(-1)}(1-\xi)\, d\xi - \int_0^{k/n} F_{\mathbf{y}}^{(-1)}(1-\xi)\, d\xi \right) \\ &= \sum_{k=1}^{n} n(w_k - w_{k-1}) \int_0^{k/n} F_{\mathbf{y}}^{(-1)}(1-\xi)\, d\xi, \end{aligned} \tag{10}$$

where $w_0 = 0$. Further, taking into account $1 - k/n = (n-k)/n$ we get

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} w'_k L\left(\mathbf{y}, \mathbf{p}, \frac{k}{n}\right) \tag{11}$$

with weights

$$w'_k = n(w_{n-k+1} - w_{n-k}) \quad \text{for } k = 1, \ldots, n-1 \text{ and } w'_n = nw_1 \tag{12}$$

and $L(\mathbf{y}, \mathbf{p}, \xi)$ defined as function of ξ by left-tail integrating of $F_{\mathbf{y}}^{(-1)}$, i.e.

$$L(\mathbf{y}, \mathbf{p}, 0) = 0 \quad \text{and} \quad L(\mathbf{y}, \mathbf{p}, \xi) = \int_0^{\xi} F_{\mathbf{y}}^{(-1)}(\alpha)\, d\alpha \quad \text{for } 0 < \xi \leqslant 1. \tag{13}$$

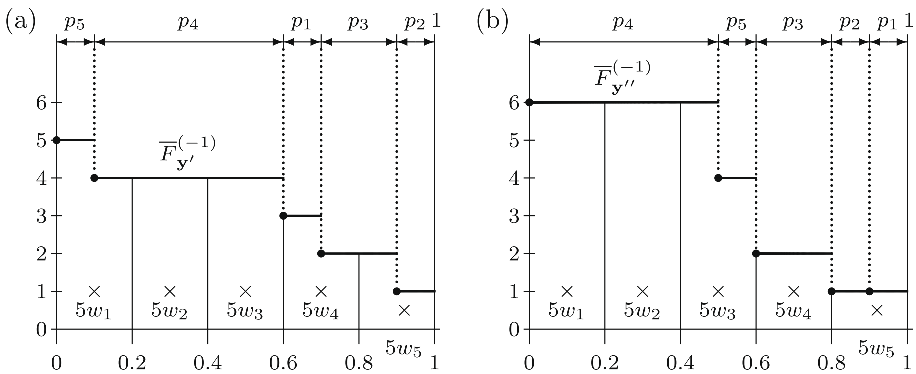

Fig. 2. Formula (10) applied to calculations in Example 2 (a) vector $\mathbf{y}'$, (b) vector $\mathbf{y}''$.

Graph of function $L(\mathbf{y},\mathbf{p},\xi)$ takes the form of convex piecewise linear curve, connecting (0,0) and the point $(1, A_{\mathbf{p}}(\mathbf{y}))$ as $L(\mathbf{y},\mathbf{p},1) = \int_0^1 F_{\mathbf{y}}^{(-1)}(\alpha)\,d\alpha = A_{\mathbf{p}}(\mathbf{y})$. It is called Absolute Lorenz Curve (ALC) [19], due to its relation to the classical Lorenz curve [12] used in income economics as a cumulative population versus income curve to compare equity of income distributions. Indeed, the Lorenz curve may be viewed [5] as function $LC(\xi) = \frac{1}{A_{\mathbf{p}}(\mathbf{y})}\int_0^{\xi} F_{\mathbf{y}}^{(-1)}(\alpha)\,d\alpha$ thus equivalent to function $L(\mathbf{y},\mathbf{p},\xi)$ normalized by the distribution average. Therefore, the classical Lorenz model is focused on equity while ignoring the average result and any perfectly equal distribution of income has the diagonal line as the Lorenz curve (the same independently from the income value). Within the ALC model both equity and values of outcomes are represented. Fig. 3 shows the absolute Lorenz curves (13) for data from Example 2. We will use formula (11) to prove some properties of the WOWA aggregation as well as to develop linear programming optimization models.

2.3. Generalized WOWA aggregation

WOWA aggregation follows the OWA preference model thus requiring the preferential weights to be defined for all k/n-quantiles ($k = 1,2,\ldots,n$). Although in various practical problems of decisions under risk the preferences might be modeled in relation with a number of preselected quantiles. For instance, the risk measure Value-at-Risk (VaR) representing specific β-quantile values $VaR_{\beta}(\mathbf{y}) = F_{\mathbf{y}}^{(-1)}(\beta)$ is commonly used in banking for β equal to 0.01 or 0.05. Formula (6) allows us to define a generalized WOWA aggregation where the preferential weights w_k are allocated to an arbitrarily defined grid of ordered outcomes defined by quantile breakpoints $\beta_0 = 0 < \beta_1 < \cdots < \beta_{n-1} < \beta_n = 1$, i.e. the aggregation defined with a piecewise linear function w_{β}^* introduced by the stepwise generation function

$$g_{\beta}(\xi) = \frac{w_k}{\beta_k - \beta_{k-1}} \quad \text{for } \beta_{k-1} < \xi \leqslant \beta_k, \quad k = 1,\ldots,n. \tag{14}$$

Such defined $w_{\beta}^*(\alpha) = \int_0^{\alpha} g_{\beta}(\xi)d\xi$ is an increasing piecewise linear function interpolating points $\left(\beta_k, \sum_{i\leqslant k} w_i\right)$ together with the point (0.0). Formula (14) applied within (6) leads us to the following generalization of the formula (10):

$$A_{\mathbf{w},\beta,\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} \frac{w_k}{\beta_k - \beta_{k-1}} \int_{\beta_{k-1}}^{\beta_k} \overline{F}_{\mathbf{y}}^{(-1)}(\xi)\,d\xi. \tag{15}$$

For instance, the generalized WOWA aggregations $A_{\mathbf{w},\beta,\mathbf{p}}(\mathbf{y})$ with preferential weights $\mathbf{w} = (0.05, 0.15, 0.8)$ allocated to the grid $\beta = (0.2, 0.5, 1.0)$ calculated for vectors $\mathbf{y}', \mathbf{y}''$ and importance weights from Example 2 results in: $A_{\mathbf{w},\beta,\mathbf{p}}(\mathbf{y}') = 0.05 \cdot 4.5 + 0.15 \cdot 4 + 0.8 \cdot 2.4 = 2.745$ and $A_{\mathbf{w},\beta,\mathbf{p}}(\mathbf{y}'') = 0.05 \cdot 6 + 0.15 \cdot 6 + 0.8 \cdot 2 = 2.8$. This is illustrated in Fig. 4.

Similar to (11), the generalized WOWA aggregation may be expressed with the tail averages as

$$A_{\mathbf{w},\beta,\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} \frac{w_k}{\beta_k - \beta_{k-1}} (L(\mathbf{y},\mathbf{p},1-\beta_{k-1}) - L(\mathbf{y},\mathbf{p},1-\beta_k)) = \sum_{k=1}^{n} w_k'' L(\mathbf{y},\mathbf{p},1-\beta_{k-1}), \tag{16}$$

where $L(\mathbf{y},\mathbf{p},\xi)$ is defined by left-tail integrating of $F_{\mathbf{y}}^{(-1)}$ according to formula (13) and weights w_k'' are defined as

$$w_k'' = \frac{w_k}{\beta_k - \beta_{k-1}} - \frac{w_{k-1}}{\beta_{k-1} - \beta_{k-2}} \quad \text{for } k = 2,\ldots,n \text{ and } w_1'' = \frac{w_1}{\beta_1}. \tag{17}$$

Note that contrary to (11), for the case of general breakpoints β_k we cannot take advantages of the grid symmetry replacing values $1-\beta_k$ with other breakpoints. Therefore, we stay with weights w_k'' assigned to tail averages $L(\mathbf{y},\mathbf{p},1-\beta_{k-1})$.

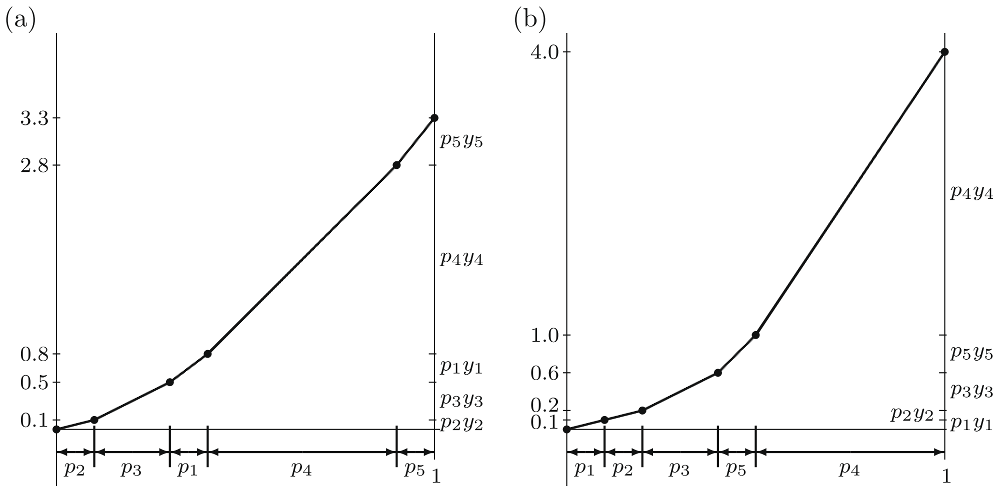

Fig. 3. Absolute Lorenz curves (13) for Example 2: (a) vector $\mathbf{y}'$, (b) vector $\mathbf{y}''$.

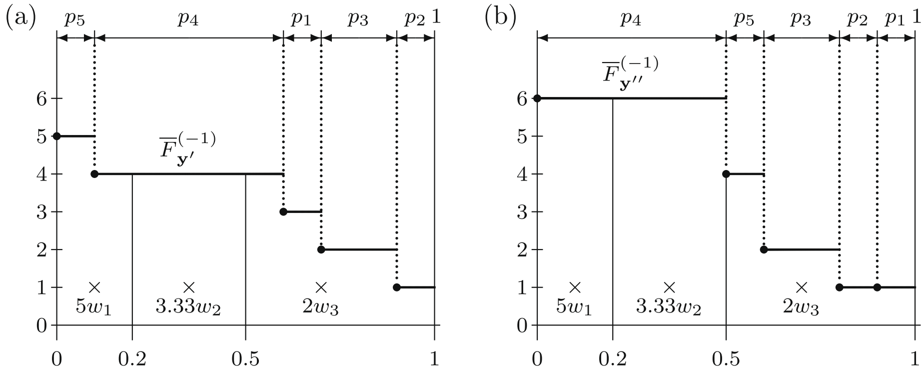

Fig. 4. Formula (15) applied to calculations of the generalized WOWA aggregations in Example 2: (a) vector $\mathbf{y}'$, (b) vector $\mathbf{y}''$.

As a very particular case of the generalized WOWA, for a single breakpoint between 0 and 1, i.e. $n = 2$, for $\beta_1 = 1 - \beta$ with some $0 < \beta < 1$ and weights $w_1 = 0, w_2 = 1$ one gets

$$A_{\mathbf{w},\beta,\mathbf{p}}(\mathbf{y}) = \frac{1}{\beta}\int_{1-\beta}^{1} \overline{F}_{\mathbf{y}}^{(-1)}(\xi)\,d\xi = \frac{1}{\beta}\int_{0}^{\beta} F_{\mathbf{y}}^{(-1)}(\xi)\,d\xi \tag{18}$$

thus representing the risk measure known as Tail Value-at-Risk or Conditional Value-at-Risk (CVaR) and becoming recently very popular in financial applications [4].

3. The orness measures and risk preferences

The OWA aggregation may model various preferences from the optimistic (max) to the pessimistic (min). Yager [37] introduced a well appealing concept of the orness measure to characterize the OWA operators. The degree of orness associated with the OWA operator $A_{\mathbf{w}}(\mathbf{y})$ is defined as

$$\text{orness}(\mathbf{w}) = \sum_{i=1}^{m} \frac{m-i}{m-1} w_i. \tag{19}$$

For the max aggregation representing the fuzzy 'or' operator with weights $\mathbf{w} = (1, 0, \ldots, 0)$ one gets $\text{orness}(\mathbf{w}) = 1$ while for the min aggregation representing the fuzzy 'and' operator with weights $\mathbf{w} = (0, \ldots, 0, 1)$ one has $\text{orness}(\mathbf{w}) = 0$. For the average (arithmetic mean) one gets $\text{orness}((1/m, 1/m, \ldots, 1/m)) = 1/2$. Actually, one may consider a complementary measure of andness defined as $\text{andness}(\mathbf{w}) = 1 - \text{orness}(\mathbf{w})$. OWA aggregations with orness greater or equal 1/2 are considered or-like whereas the aggregations with orness smaller or equal 1/2 are treated as and-like. The former correspond to rather optimistic preferences while the latter represents rather pessimistic preferences. The OWA aggregations with monotonic weights are either or-like or and-like. Exactly, decreasing weights $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_m$ define an or-like OWA operator, while increasing weights $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_m$ define an and-like OWA operator [11].

Yager [39] proposed to define the OWA weighting vectors via the regular increasing monotone (RIM) quantifiers, which provide a dimension independent description of the aggregation. A fuzzy subset Q of the real line is called a RIM quantifier if Q is (weakly) increasing with $Q(0) = 0$ and $Q(1) = 1$. The OWA weights can be defined with a RIM quantifier Q as $w_i = Q(i/m) - Q((i-1)/m)$, and the orness measure can be extended to a RIM quantifier (according to $m \to \infty$) as follows [39]:

$$\text{orness}(Q) = \int_0^1 Q(\alpha)\,d\alpha. \tag{20}$$

Thus, the orness of a RIM quantifier is equal to the area under it. The measure takes the values between 0 (achieved for $Q(1) = 1$ and $Q(\alpha) = 0$ for all other α) and 1 (achieved for $Q(0) = 0$ and $Q(\alpha) = 1$ for all other α). In particular, $\text{orness}(Q) = 1/2$ for $Q(\alpha) = \alpha$ which is generated by equal weights $w_k = 1/n$. Formula (20) allows one to define the orness of the WOWA aggregation (5) which can be viewed with the RIM quantifier $Q(\alpha) = w^*(\alpha)$ [10]. Let us consider piecewise linear function $Q = w^*$ defined by weights vectors $\mathbf{w}$ of dimension n according to the stepwise generation function (8). One may easily notice that decreasing weights $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n$ generate a strictly increasing concave curve $Q(\alpha) \geqslant \alpha$ thus guaranteeing the or-likeness of the WOWA operator. Similarly, increasing weights $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$ generate a strictly increasing convex curve $Q(\alpha) \leqslant \alpha$ thus guaranteeing the and-likeness of the WOWA operator. With respect to decisions under risk the and-likeness of the scenarios aggregation represents the risk averse preferences and the WOWA objective functions with increasing preferential weights (interpreted as probabilities) represent the risk averse aggregations of outcomes under several scenarios.

The classical model of choice under uncertainty, following the von Neumann and Morgenstern's Expected Utility (EU) theory [34], is based on maximization of quantities $\int_{-\infty}^{\infty} u(\xi)\,dF_{\mathbf{y}}(\xi)$ where the risk preferences are represented by the utility function u. The risk averse preferences are characterized by increasing concave utility functions. The most general mathematical model of the risk averse preferences is then given by the Second Stochastic Dominance (SSD) relation [14]: $F_{\mathbf{y}'} \succeq_{SSD} F_{\mathbf{y}''}$ iff $\int_{-\infty}^{\alpha} F_{\mathbf{y}'}(\xi)\,d\xi \leqslant \int_{-\infty}^{\alpha} F_{\mathbf{y}''}(\xi)\,d\xi$ for all α. The SSD dominance $F_{\mathbf{y}'} \succ_{SSD} F_{\mathbf{y}''}$ guarantees that $F_{\mathbf{y}'}$ is preferred to $F_{\mathbf{y}''}$ within all risk averse preference models that prefer larger outcomes ($F_{\mathbf{y}'}$ generates greater or equal expected utility for all increasing concave utility functions).

Following formula (6), the WOWA averages may be interpreted within the Rank-Dependent Expected Utility (RDEU) model of choice under uncertainty [27] (known also as Anticipated Utility [26] or the dual theory of choice under uncertainty [36]) which is based on the axiomatic foundation [28] alternative to that for the classical EU theory. Indeed, according to (6),

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \int_0^1 g(\xi)\overline{F}_{\mathbf{y}}^{(-1)}(\xi)\,d\xi = \int_0^1 g(1-\xi)F_{\mathbf{y}}^{(-1)}(\xi)\,d\xi = \int_0^1 F_{\mathbf{y}}^{(-1)}(\xi)\,d\phi(\xi)$$

with the rank-dependent utility function $\phi(\xi) = 1 - w^*(1-\xi)$. Increasing weights $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$ generate a strictly increasing concave rank-dependent utility function ϕ thus guaranteeing the risk averse preferences in terms of the RDEU model.

Actually, the absolute Lorenz curves (13) represent a dual characterization of the SSD relation [19] as $F_{\mathbf{y}'} \succeq_{SSD} F_{\mathbf{y}''}$ iff $\int_0^{\beta} F_{\mathbf{y}'}^{(-1)}(\xi)\,d\xi \geqslant \int_0^{\beta} F_{\mathbf{y}''}^{(-1)}(\xi)\,d\xi$ for all $0 \leqslant \beta \leqslant 1$. Formula (11) represents the WOWA aggregation with increasing preferential weights as the weighted (positive) combination of the n ALC values. Therefore, the WOWA objective functions with increasing preferential weights are SSD consistent and they represent also the risk averse preferences in term of the classical decision theory (despite they do not represent expected utility).

Proposition 1. *WOWA aggregation defined by increasing preferential weights $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$ represents risk averse preferences in terms of the SSD order, i.e.*

$$F_{\mathbf{y}'} \succeq_{SSD} F_{\mathbf{y}''} \Rightarrow A_{\mathbf{w},\mathbf{p}}(\mathbf{y}') \geqslant A_{\mathbf{w},\mathbf{p}}(\mathbf{y}'').$$

Similarly, the generalized WOWA aggregation (15) can be viewed with the RIM quantifier $Q(\alpha) = w^*_{\beta}(\alpha)$ defined by weights vectors $\mathbf{w}$ of dimension n according to the stepwise generation function (14), i.e., $w^*_{\beta}(\alpha) = \int_0^{\alpha} g_{\beta}(\xi)\,d\xi$.

Relatively increasing weights $w_1/\beta_1 \leqslant w_2/(\beta_2 - \beta_1) \leqslant \cdots \leqslant w_n/(\beta_n - \beta_{n-1})$ generate a strictly increasing convex curve $Q(\alpha) \leqslant \alpha$ thus guaranteeing the and-likeness of the generalized WOWA operator.

The generalized WOWA can be directly expressed within the RDEU model with the rank-dependent utility function $\phi(\xi) = 1 - w^*_{\beta}(1-\xi)$. Relatively increasing weights $w_1/\beta_1 \leqslant w_2/(\beta_2 - \beta_1) \leqslant \cdots \leqslant w_n/(\beta_n - \beta_{n-1})$ generate a strictly increasing concave rank-dependent utility function ϕ thus guaranteeing the risk averse preferences in terms of the RDEU model. Moreover, following formula (16), the generalized WOWA aggregations with relatively increasing preferential weights are SSD consistent and they represent also the risk averse preferences in term of the classical decision theory.

Proposition 2. *Generalized WOWA aggregation defined by relatively increasing preferential weights $w_1/\beta_1 \leqslant w_2/(\beta_2 - \beta_1) \leqslant \cdots \leqslant w_n/(\beta_n - \beta_{n-1})$ represents risk averse preferences in terms of the SSD order, i.e.*

$$F_{\mathbf{y}'} \succeq_{SSD} F_{\mathbf{y}''} \Rightarrow A_{\mathbf{w},\beta,\mathbf{p}}(\mathbf{y}') \geqslant A_{\mathbf{w},\beta,\mathbf{p}}(\mathbf{y}'').$$

We will focus our analysis on the WOWA aggregation defined by increasing weights $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$ or the generalized WOWA characterized by relatively increasing weights $w_1/\beta_1 \leqslant w_2/(\beta_2 - \beta_1) \leqslant \cdots \leqslant w_n/(\beta_n - \beta_{n-1})$. Following Propositions 1 and 2, respectively, maximization of such WOWA aggregations models risk averse preferences.

4. Linear programming models for WOWA optimization

4.1. WOWA models

Consider maximization of a risk averse WOWA aggregation defined by increasing weights $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$

$$\max_{\mathbf{x}\in Q} \quad A_{\mathbf{w},\mathbf{p}}(\mathbf{f}(\mathbf{x})). \tag{21}$$

Due to formula (11), the problem may be expressed as

$$\max_{\mathbf{x}\in Q} \quad \sum_{k=1}^{n} w'_k L\left(\mathbf{f}(\mathbf{x}), \mathbf{p}, \frac{k}{n}\right) \tag{22}$$

with positive weights w'_k defined by (12).

According to (13), values of function $L(\mathbf{y}, \mathbf{p}, \xi)$ for any $0 \leqslant \xi \leqslant 1$ can be given by optimization:

$$L(\mathbf{y}, \mathbf{p}, \xi) = \min_{s_i} \left\{ \sum_{i=1}^{m} y_i s_i : \sum_{i=1}^{m} s_i = \xi,\ 0 \leqslant s_i \leqslant p_i\ \forall\, i \right\}. \tag{23}$$

The above problem is an LP for a given outcome vector $\mathbf{y}$. Although, when used within the problem (22) it leads to nonlinear optimization as both y_i and s_i represent variables. This difficulty can be overcome by taking advantage of the LP dual to (23). Introducing dual variable t corresponding to the equation $\sum_{i=1}^m s_i = \xi$ and variables d_i corresponding to upper bounds on s_i one gets the following LP dual expression of $L(\mathbf{y}, \mathbf{p}, \xi)$

$$L(\mathbf{y}, \mathbf{p}, \xi) = \max_{t,d_i} \left\{ \xi t - \sum_{i=1}^{m} p_i d_i : t - d_i \leqslant y_i,\ d_i \geqslant 0\ \forall\, i \right\}. \tag{24}$$

This LP model enables the following statements.

Proposition 3. *WOWA aggregation $A_{\mathbf{w},\mathbf{p}}(\mathbf{y})$ defined by increasing preferential weights $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$ is a piecewise linear concave function of $\mathbf{y}$.*

Proof. Note that for any given $\mathbf{p}$ and ξ, due to formula (24), $L(\mathbf{y}, \mathbf{p}, \xi)$ is a piecewise linear concave function of $\mathbf{y}$. Hence, due to increasing preferential weights, following formula (11) the entire WOWA aggregation is a piecewise linear concave function of $\mathbf{y}$ as a linear combination of functions $L(\mathbf{y}, \mathbf{p}, \xi)$ for $\xi = k/n$, $k = 1, 2, \ldots, n$ with nonnegative weights w'_k. □

Proposition 4. *Maximization of a risk averse WOWA aggregation (21) with increasing preferential weights $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$ may be implemented as the following LP expansion of the original constraints:*

$$\begin{aligned} \max_{t_k, d_{ik}, x_j} \quad & \sum_{k=1}^{n} w'_k \left[\tfrac{k}{n} t_k - \sum_{i=1}^{m} p_i d_{ik} \right] \\ \text{s.t.} \quad & \mathbf{x} \in F,\ t_k - d_{ik} \leqslant f_i(\mathbf{x}),\ d_{ik} \geqslant 0 \quad \forall\, i, k. \end{aligned} \tag{25}$$

Consider multiple criteria problems (1) with linear objective functions $f_i(\mathbf{x}) = \mathbf{c}_i\mathbf{x}$ and polyhedral feasible sets:

$$\max \quad \{(y_1, y_2, \ldots, y_m)) : \mathbf{y} = \mathbf{C}\mathbf{x},\ \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geqq \mathbf{0}\}, \tag{26}$$

where $\mathbf{C}$ is an $m \times q$ matrix (consisting of rows $\mathbf{c}_i$), $\mathbf{A}$ is a given $v \times q$ matrix and $\mathbf{b} = (b_1, \ldots, b_v)^T$ is a given right hand side vector. For such problems, we get the following LP formulation of the WOWA maximization (21):

$$\max_{t_k, d_{ik}, y_i, x_j} \quad \sum_{k=1}^{n} \frac{k}{n} w'_k t_k - \sum_{k=1}^{n} \sum_{i=1}^{m} w'_k p_i d_{ik} \tag{27}$$

$$\text{s.t.} \quad \sum_{j=1}^{q} a_{rj} x_j = b_r \quad \text{for } r = 1, \ldots, v, \tag{28}$$

$$y_i - \sum_{j=1}^{q} c_{ij} x_j = 0 \quad \text{for } i = 1, \ldots, m, \tag{29}$$

$$d_{ik} \geqslant t_k - y_i,\ d_{ik} \geqslant 0 \quad \text{for } i = 1, \ldots, m;\ k = 1, \ldots, n, \tag{30}$$

$$x_j \geqslant 0 \quad \text{for } j = 1, \ldots, q. \tag{31}$$

Model (27)–(31) is an LP problem with $mn + m + n + q$ variables and $mn + m + v$ constraints. Thus, for problems with not too large number of scenarios (m) and preferential weights (n) it can be solved directly. Note that WOWA model (27)–(31) differs from the analogous deviational model for the OWA optimizations [21] only due to coefficients within the objective function (27) and the possibility of different values of m and n.

The number of constraints in problem (27)–(31) is similar to the number of variables. Nevertheless, for the simplex approach it may be better to deal with the dual of (27)–(31) than with the original problem. Note that variables d_{ik} in the primal are represented with singleton columns. Hence, the corresponding rows in the dual represent only simple upper bounds.

Introducing the dual variables: $u_r (r = 1, \ldots, v)$, $v_i (i = 1, \ldots, m)$ and $z_{ik} (i = 1, \ldots, m;\ k = 1, \ldots, n)$ corresponding to the constraints (28)–(30), respectively, we get the following dual:

$$\begin{aligned} \min_{z_{ik}, v_i, u_r} \quad & \sum_{r=1}^{v} b_r u_r \\ \text{s.t.} \quad & \sum_{r=1}^{v} a_{rj} u_r - \sum_{i=1}^{m} c_{ij} v_i \geqslant 0 \quad \text{for } j = 1, \ldots, q, \\ & v_i - \sum_{k=1}^{n} z_{ik} = 0 \quad \text{for } i = 1, \ldots, m, \\ & \sum_{i=1}^{m} z_{ik} = \tfrac{k}{n} w'_k \quad \text{for } k = 1, \ldots, n, \\ & 0 \leqslant z_{ik} \leqslant p_i w'_k \quad \text{for } i = 1, \ldots, m;\ k = 1, \ldots, n. \end{aligned} \tag{32}$$

The dual problem (32) contains: $m+n+q$ structural constraints, $m+v$ unbounded variables and mn bounded variables. Since the average complexity of the simplex method depends on the number of constraints, the dual model (32) can be directly solved for quite large values of m and n. Moreover, the columns corresponding to mn variables z_{ik} form the transportation/assignment matrix thus allowing one to employ special techniques of the simplex SON algorithm [6] for implicit handling of these variables. Such techniques increase dramatically efficiency of the simplex method but they require a special tailored implementation. We have not tested this approach within our initial computational experiments based on the use of a general purpose LP code.

4.2. Generalized WOWA models

Linear programming models can also be introduced for the generalized WOWA. Consider the maximization of a risk averse generalized WOWA aggregation with relatively increasing weights $w_1/\beta_1 \leqslant w_2/(\beta_2-\beta_1) \leqslant \cdots \leqslant w_n/(\beta_n-\beta_{n-1})$

$$\max_{\mathbf{x}\in Q} \quad A_{\mathbf{w},\beta,\mathbf{p}}(\mathbf{f}(\mathbf{x})). \tag{33}$$

Following (16), it may be expressed as the problem $\max_{\mathbf{x}\in Q}\sum_{k=1}^n w''_k L(\mathbf{f}(\mathbf{x}),\mathbf{p},1-\beta_{k-1})$ with positive weights w''_k defined by formula (17). Taking advantages of the dual LP expression for the tail averages (24), one gets the following statements.

Proposition 5. *Generalized WOWA aggregation $A_{\mathbf{w},\beta,\mathbf{p}}(\mathbf{y})$ defined by relatively increasing preferential weights $w_1/\beta_1 \leqslant w_2/(\beta_2-\beta_1) \leqslant \cdots \leqslant w_n/(\beta_n-\beta_{n-1})$ is a piecewise linear concave function of $\mathbf{y}$.*

Proposition 6. *Maximization of a risk averse generalized WOWA aggregation (33) with relatively increasing preferential weights $w_1/\beta_1 \leqslant w_2/(\beta_2-\beta_1) \leqslant \cdots \leqslant w_n/(\beta_n-\beta_{n-1})$ may be implemented as the following LP expansion of the original constraints:*

$$\begin{aligned} &\max_{t_k,d_{ik},x_j} && \sum_{k=1}^n w''_k\Big[(1-\beta_{k-1})t_k - \sum_{i=1}^m p_i d_{ik}\Big] \\ &\text{s.t.} && \mathbf{x}\in F, \quad t_k - d_{ik} \leqslant y_i, \quad d_{ik} \geqslant 0 \quad \forall i,k. \end{aligned} \tag{34}$$

For multiple criteria problems (26) with linear objective functions $f_i(\mathbf{x}) = \mathbf{c}_i\mathbf{x}$ defining the return realizations under several scenarios, problem (34) takes the following LP form:

$$\begin{aligned} &\max_{t_k,d_{ik},x_j} && \sum_{k=1}^n (1-\beta_{k-1})w''_k t_k - \sum_{k=1}^n\sum_{i=1}^m w''_k p_i d_{ik} \\ &\text{s.t.} && \sum_{j=1}^q a_{rj}x_j = b_r \quad \text{for } r=1,\ldots,v \\ & && d_{ik} \geqslant t_k - \sum_{j=1}^q c_{ij}x_j, \quad d_{ik} \geqslant 0 \quad \text{for } i=1,\ldots,m;\ k=1,\ldots,n, \\ & && x_j \geqslant 0 \quad \text{for } j=1,\ldots,q. \end{aligned} \tag{35}$$

Since, in the generalized WOWA model the number of breakpoints n is usually much smaller that the number of scenarios m, we have eliminated variables y_i. This allows us to eliminate also m equities defining those variables. Obviously, such an elimination is also possible for the standard WOWA model (27)–(31) although not important for comparable orders of n and m. Actually, our computational experiments has demonstrated that for the case of $n = m$ explicit use of variables y_i results in shorter computation times.

The LP model (35) with $mn+n+q$ variables and $mn+v$ constraints can be directly solved for problems with not too large number of scenarios (m) and preferential weights (n). Alternatively, it can be replaced with the corresponding LP dual:

$$\begin{aligned} &\min_{z_{ik},u_r} && \sum_{r=1}^v b_r u_r \\ &\text{s.t.} && \sum_{r=1}^v a_{rj}u_r - \sum_{k=1}^n \bar{c}_j z_{ik} \geqslant 0 \quad \text{for } j=1,\ldots,q, \\ & && \sum_{i=1}^m z_{ik} = (1-\beta_{k-1})w''_k \quad \text{for } k=1,\ldots,n, \\ & && 0 \leqslant z_{ik} \leqslant p_i w''_k \quad \text{for } i=1,\ldots,m;\ k=1,\ldots,n, \end{aligned} \tag{36}$$

where $\bar{c}_j = \sum_{i=1}^m c_{ij}$. The dual problem (36) contains $mn+v$ variables but only $n+q$ structural constraints.

In particular, for a limiting case of $n=2$, $\beta_1 = 1-\beta$ with some $0<\beta<1$ and weights $w_1 = 0, w_2 = 1$ allowing the generalized WOWA (18) to represent the $\mathrm{CVaR}_\beta(\mathbf{y})$ risk measure, the corresponding LP computational models take the following forms of primal

$$\begin{aligned}
\max_{t,d_i,x_j} \quad & t - \frac{1}{\beta}\sum_{i=1}^{m} p_i d_i \\
\text{s.t.} \quad & \sum_{j=1}^{q} a_{rj}x_j = b_r \quad \text{for } r = 1,\ldots,v, \\
& d_i \geqslant t - \sum_{j=1}^{q} c_{ij}x_j, \quad d_i \geqslant 0 \quad \text{for } i = 1,\ldots,m, \\
& x_j \geqslant 0 \quad \text{for } j = 1,\ldots,q,
\end{aligned} \tag{37}$$

and dual, respectively

$$\begin{aligned}
\min_{z_i,u_r} \quad & \sum_{r=1}^{v} b_r u_r \\
\text{s.t.} \quad & \sum_{r=1}^{v} a_{rj}u_r - \bar{c}_j z_i \geqslant 0 \quad \text{for } j = 1,\ldots,q, \\
& \sum_{i=1}^{m} z_i = 1, \; 0 \leqslant z_i \leqslant p_i/\beta \quad \text{for } i = 1,\ldots,m.
\end{aligned} \tag{38}$$

5. Computational tests

In order to analyze the computational performances of the LP models for the WOWA optimization, similarly to [21], we have solved randomly generated problems of portfolio optimization as presented in Example 1 with the objectives aggregated by the WOWA

$$\max_{\mathbf{x}} \quad \left\{ A_{\mathbf{w},\mathbf{p}}(\mathbf{Cx}) : \sum_{j=1}^{q} x_j = 1, \; x_j \geqslant 0 \text{ for } j = 1,\ldots,q \right\}, \tag{39}$$

where matrix $\mathbf{C} = (c_{ij})_{i=1,\ldots,m;j=1,\ldots,q}$ represents returns of security j under scenario i, the importance weights p_i are assigned to several scenarios while the preferential weights w_k are increasing to represent the risk averse preferences. Both the primal (27)–(31) and the dual (32) forms of the computational model have been tested.

Example 3. For illustration of model building let us consider the simplified problem with 2 securities and 3 scenarios (see Table 1) with the WOWA aggregation defined by importance weights $\mathbf{p} = (0.2, 0.2, 0.6)$ and preferential weights $\mathbf{w} = (0.1, 0.2, 0.7)$. The primal model (27)–(31) takes then the following form:

$$\begin{aligned}
\max_{t_k,d_{ik},y_i,x_j} \quad & 0.5t_1 + 0.2t_2 + 0.3t_3 - 0.3d_{11} - 0.3d_{21} - 0.9d_{31} \\
& -0.06d_{12} - 0.06d_{22} - 0.18d_{32} - 0.06d_{13} - 0.06d_{23} - 0.18d_{33} \\
\text{s.t.} \quad & x_1 + x_2 = 1; \quad x_1 \geqslant 0; \quad x_2 \geqslant 0, \\
& y_1 = -9x_1 + 7x_2; \quad y_2 = 6x_1 + 7x_2; \quad y_3 = 9x_1 - 5x_2, \\
& d_{ik} \geqslant t_k - y_i; \quad d_{ik} \geqslant 0 \quad \text{for } i = 1,2; \; k = 1,2,3,
\end{aligned}$$

while the respective dual (32) can be written as follows:

$$\begin{aligned}
\min_{z_{ik},v_i,u} \quad & u \\
\text{s.t.} \quad & u \geqslant -9v_1 + 6v_2 + 9v_3; \quad u \geqslant 7v_1 + 7v_2 - 5v_3, \\
& v_1 = z_{11} + z_{12} + z_{13}; \quad v_2 = z_{21} + z_{22} + z_{23}; \quad v_3 = z_{31} + z_{32} + z_{33}, \\
& z_{11} + z_{21} + z_{31} = 0.5; \quad z_{12} + z_{22} + z_{32} = 0.2; \quad z_{13} + z_{23} + z_{33} = 0.3, \\
& 0 \leqslant z_{11} \leqslant 0.3; \quad 0 \leqslant z_{21} \leqslant 0.3; \quad 0 \leqslant z_{31} \leqslant 0.9, \\
& 0 \leqslant z_{12} \leqslant 0.06; \quad 0 \leqslant z_{22} \leqslant 0.06; \quad 0 \leqslant z_{32} \leqslant 0.18, \\
& 0 \leqslant z_{13} \leqslant 0.06; \quad 0 \leqslant z_{23} \leqslant 0.06; \quad 0 \leqslant z_{33} \leqslant 0.18.
\end{aligned}$$

The optimal portfolio equals (0.8, 0.2) with rates of return 6.2% under Scenarios 2 or 3, and −5.8% under Scenario 1.

The performance tests were based on the randomly generated problems (39) with varying number q of securities (decision variables) and number m of scenarios. The generation procedure worked as follows. First, for each security j the maximum rate of return r_j was generated as a random number uniformly distributed in the interval [0.05, 0.15]. Next, this value was used to generate specific outcomes c_{ij} (the rate of return under scenarios i) as random variables uniformly distributed in the interval $[-0.75r_j, r_j]$. Further, strictly increasing and positive weights w_k were generated. The weights were not normalized which allowed us to define them by the corresponding increments $\delta_k = w_k - w_{k-1}$. The latter were generated as uniformly distributed random values in the range of 1.0 to 2.0, except from a few (5 on average) possibly larger increments ranged from 1.0 to $n/3$. Importance weights p_i were generated according to the exponential smoothing scheme, which assigns exponentially decreasing weights to older or subjectively less probable scenarios: $p_i = \alpha(1-\alpha)^{i-1}$ for $i = 1, 2, \ldots, m$ and the parameter α is chosen for each test problem size separately to keep the value of p_m around 0.001.

The basic tests were performed for the standard WOWA model with $n = m$. However, we also analyzed the case of larger n for more detailed preferences modeling, as well as the case of smaller n thus representing a rough preferences model. For each number of securities q and number of criteria (scenarios) m we solved 10 randomly generated problems (39). All computations were performed on a PC with the Athlon 64, 1.8 GHz processor employing the CPLEX 9.1 package. The 600 s time limit was used in all the computations.

In Tables 2 and 3 we show the solution times for the primal (27)–(31) and the dual (32) forms of the computational model, being the averages of 50 randomly generated problems. Upper index in front of the time value indicates the number of tests among 10 that exceeded the time limit. The empty cell (minus sign) shows that this occurred for all 10 instances. Both forms were solved by the CPLEX code without taking advantages of the constraints structure specificity. The dual form of the model performs much better in each tested problem size. It behaves very well with increasing number of securities if the number of scenarios does not exceed 100. Similarly, the model performs very well with increasing number of scenarios if only the number of securities does not exceed 50.

Table 4 presents solution times for different numbers of the preferential weights. The number of securities equals 50. It can be noticed that increasing the number of preferential weights and thus the number of breakpoints in the interpolation function induce moderate increase in the computational complexity. On the other hand, the computational efficiency can be significantly improved by reducing the number of preferential weights to a few which can be reasonable in non-automated decision making support systems.

The portfolio selection models equivalent to optimization of the generalized WOWA aggregation based on a few preferential weights attached to an irregular grid of breakpoints provided very good results on real-life market data [13]. Therefore, we have also tested computational efficiency of the following generalized WOWA maximization problem.

Table 2
Solution times [s] for the primal model.

Number of scenarios (m)	Number of securities (q)							
	10	20	50	100	150	200	300	400
10	0.00	0.00	0.00	0.02	0.02	0.00	0.02	0.00
20	0.02	0.04	0.40	0.02	0.02	0.06	0.04	0.08
50	0.74	1.00	1.40	1.60	1.56	1.58	1.58	1.62
100	21.76	28.40	32.66	43.98	49.94	65.02	86.78	95.16
150	182.52	244.46	312.28	354.30	404.90	456.12	^{11}514.32	^{22}556.06

Table 3
Solution times [s] for the dual model (32).

Number of scenarios (m)	Number of securities (q)							
	10	20	50	100	150	200	300	400
10	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.02
20	0.00	0.02	0.02	0.02	0.02	0.02	0.02	0.02
50	0.04	0.06	0.22	0.30	0.34	0.38	0.50	0.56
100	0.38	0.54	1.52	6.66	7.82	9.66	11.42	12.52
150	1.20	1.88	3.42	26.16	44.76	54.50	61.96	63.76
200	2.86	4.18	7.94	59.20	138.76	207.38	238.24	236.56
300	9.26	15.10	30.00	^{2}215.00	^{19}466.62	–	–	–
400	23.64	34.42	82.10	^{9}237.30	^{31}521.60	–	–	–

Table 4
Solution times [s] for different numbers of preferential weights ($q = 50$).

Number of scenarios (m)	Number of preferential weights (n)									
	3	5	10	20	50	100	150	200	300	400
10	0.00	0.00	0.00	0.02	0.02	0.02	0.02	0.02	0.06	0.08
20	0.00	0.00	0.00	0.00	0.02	0.04	0.10	0.10	0.26	0.38
50	0.02	0.00	0.02	0.06	0.24	0.24	0.36	0.56	1.02	1.46
100	0.00	0.02	0.08	0.22	1.10	1.54	1.44	2.06	3.54	5.20
150	0.02	0.06	0.18	0.58	3.56	4.78	3.42	4.66	7.54	11.26
200	0.06	0.10	0.32	1.10	7.36	11.22	6.00	7.92	14.08	23.28
300	0.10	0.22	0.74	3.26	17.58	25.92	11.84	16.40	29.96	40.18
400	0.18	0.44	1.54	6.36	34.00	49.26	20.48	28.60	48.82	82.84

Table 5
Solution times [s] for different numbers of preferential weights ($q = 50$).

Number of scenarios (m)	Number of weights (n)			
	3	6	11	15
10	0.00	0.00	0.00	0.00
20	0.00	0.02	0.02	0.02
50	0.02	0.02	0.04	0.08
100	0.02	0.08	0.18	0.20
150	0.04	0.12	0.30	0.36
200	0.08	0.20	0.44	0.54
300	0.10	0.30	0.78	0.98
400	0.16	0.46	1.18	1.54

Table 6
Solution times [s] for the large scale ($m = 50,000$) generalized WOWA problems.

Number of securities (q)	Number of weights (n)		
	2	4	6
50	18.7	123.2	296.2
100	52.1	327.4	867.1

$$\max_{\mathbf{x}} \quad \left\{ A_{\mathbf{w},\beta,\mathbf{p}}(\mathbf{Cx}) : \sum_{j=1}^{q} x_j = 1,\ x_j \geqslant 0 \text{ for } j = 1,\ldots,q \right\}. \tag{40}$$

Exactly, there was tested the corresponding LP dual model (36) with some irregular grids of 3–15 breakpoints and corresponding weights defined according to [13]. The results presented in Table 5 show exceptionally good performance in the considered range of scenario numbers.

Further, the generalized WOWA models have been also tested for a large number of scenarios. We have run computational test on 10 randomly generated test instances developed by Lim et al. [9]. They were originally generated from a multivariate normal distribution for 50 or 100 securities with the number of scenarios 50,000 just providing an adequate approximation to the underlying unknown continuous price distribution. Table 6 presents average computation times of the dual model (36) for $n = 4$ with breakpoints $\beta_1 = 0.5, \beta_2 = 0.75, \beta_3 = 0.9, \beta_4 = 1$, thus representing the parameters leading to good results on real life data [13], as well as for $n = 6$ with uniformly distributed tolerance levels $\beta_1 = 0.5, \beta_2 = 0.6, \beta_3 = 0.7,\ \beta_4 = 0.8, \beta_5 = 0.9, \beta_6 = 1$. Additionally, for $n = 2$ we have actually tested the (dual) generalized WOWA model (38) representing the CVaR_β measure optimization with the most commonly considered breakpoint (tolerance level) $\beta = 0.1$. The latter have been solved in less than a minute while the larger problem may require computation times up to 15 min for 100 securities.

6. Concluding remarks

The problem of averaging outcomes under several scenarios to form overall objective functions is of considerable importance in decision support under uncertainty. The WOWA aggregation [29] represents such a universal tool allowing one to take into account both the risk aversion preferences depicted with the preferential weights allocated to ordered outcomes as well as the scenarios importance expressed with weights allocated to several scenarios. The ordering operator used to define the WOWA aggregation is, in general, hard to implement. We have shown that the WOWA aggregations with the increasing weights can be modeled by introducing auxiliary linear constraints. Hence, an LP decision under risk problem with the risk averse WOWA aggregation of outcomes under several scenarios can be formed as a standard linear program and it can be further simplified by taking advantages of the LP duality. This model can also be applied to the generalized WOWA aggregations with preferential weights allocated to an arbitrary grid of breakpoints.

Our computational experiments show that the LP formulation enables to solve effectively medium size WOWA problems. Actually, the number of 100 scenarios covered by the dual approach to the LP model in less a minute seems to be quite enough for most applications to decisions under risk. Moreover, the large scale problems of 50,000 scenarios have been effectively solved with the generalized WOWA criterion built on a few breakpoints. Such a criterion turned out to provide very good optimization results in the real-life optimization problems [13].

The problems have been solved directly by a general purpose LP code. Taking advantages of the constraints structure specificity may remarkably extend the solution capabilities.

Acknowledgements

The research was supported by the Polish Ministry of Science and Higher Education under Grant N N516 4307 33.

The authors are indebted to Professor Churlzu Lim from the University of North Carolina at Charlotte for providing the test data of 50,000 scenarios portfolio selection problem.

References

[1] B.S. Ahn, Preference relation approach for obtaining OWA operators weights, Int. J. Approx. Reason. 47 (2008) 166–178.
[2] G.R. Amin, Notes on properties of the OWA weights determination model, Comput. Ind. Eng. 52 (2007) 533–538.
[3] G.R. Amin, A. Emrouznejad, An extended minimax disparity to determine the OWA operator weights, Comput. Ind. Eng. 50 (2006) 312–316.
[4] F. Andersson, H. Mausser, D. Rosen, S. Uryasev, Credit risk optimization with conditional value-at-risk criterion, Math. Program. 89 (2001) 273–291.
[5] J.L. Gastwirth, A general definition of the Lorenz curve, Econometrica 39 (1971) 1037–1039.
[6] F. Glover, D. Klingman, The simplex SON method for LP/embedded network problems, Math. Program. Study 15 (1981) 148–176.
[7] M. Grabisch, S.A. Orlovski, R.R. Yager, Fuzzy Aggregation of Numerical Preferences, Fuzzy Sets in Decision Analysis, Operations Research and Statistics, Kluwer AP, Dordrecht, 1999.
[8] H.L. Larsen, Importance weighted OWA aggregation of multicriteria queries, Proc. North Am. Fuzzy Info. Proc. Soc. Conf. (NAFIPS'99) (1999) 740–744.
[9] C. Lim, H.D. Sherali, S. Uryasev, Portfolio optimization by minimizing conditional value-at-risk via nondifferentiable optimization, Comput. Optim. Appl., forthcoming, doi:10.1007/s10589-008-9196-3.
[10] X. Liu, Some properties of the weighted OWA operator, IEEE Trans. Syst., Man Cyb. B 368 (2006) 118–127.
[11] X. Liu, On the properties of equidifferent OWA operator, Int. J. Approx. Reason. 43 (2006) 90–107.
[12] M.O. Lorenz, Methods of measuring the concentration of wealth, Publ. Am. Stat. Assoc. 9 (1905) 209–219.
[13] R. Mansini, W. Ogryczak, M.G. Speranza, Conditional value at risk and related linear programming models for portfolio optimization, Ann. Oper. Res. 152 (2007) 227–256.
[14] A. Müller, D. Stoyan, Comparison Methods for Stochastic Models and Risks, Wiley, 2002.
[15] D. Nettleton, J. Muniz, Processing and representation of meta-data for sleep apnea diagnosis with an artificial intelligence approach, Med. Inform. 63 (2001) 77–89.
[16] W. Ogryczak, Multiple criteria linear programming model for portfolio selection, Ann. Oper. Res. 97 (2000) 143–162.
[17] W. Ogryczak, Multiple criteria optimization and decisions under risk, Control Cyber. 31 (2002) 975–1003.
[18] W. Ogryczak, WOWA enhancement of the preference modeling in the reference point method, in: MDAI 2008, LNAI, vol. 5285, 2008, pp. 38–49.
[19] W. Ogryczak, A. Ruszczyński, Dual stochastic dominance and related mean-risk models, SIAM J. Optim. 13 (2002) 60–78.
[20] W. Ogryczak, T. Śliwiński, On equitable approaches to resource allocation problems: the conditional minimax solution, J. Telecom. Info. Technol. 3/02 (2002) 40–48.
[21] W. Ogryczak, T. Śliwiński, On solving linear programs with the ordered weighted averaging objective, Eur. J. Opnl. Res. 148 (2003) 80–91.
[22] W. Ogryczak, T. Śliwiński, On optimization of the importance weighted OWA aggregation of multiple criteria, in: ICCSA 2007, LNCS, vol. 4705, 2007, pp. 804–817.
[23] W. Ogryczak, T. Śliwiński, On decision support under risk by the WOWA optimization, in: ECSQARU 2007, LNAI, vol. 4724, 2007, pp. 779–790.
[24] W. Ogryczak, A. Tamir, Minimizing the sum of the k largest functions in linear time, Inform. Proc. Let. 85 (2003) 117–122.
[25] W. Ogryczak, M. Zawadzki, Conditional median — a parametric solution concept for location problems, Ann. Oper. Res. 110 (2002) 167–181.
[26] J. Quiggin, A theory of anticipated utility, J. Econ. Behav. Organ. 3 (1982) 323–343.
[27] J. Quiggin, Generalized Expected Utility Theory, The Rank-Dependent Model, Kluwer Academic, Dordrecht, 1993.
[28] A. Roell, Risk aversion in Quiggin and Yaari's rank-order model of choice under uncertainty, Econ. J. 97 (1987) 143–159.
[29] V. Torra, The weighted OWA operator, Int. J. Intell. Syst. 12 (1997) 153–166.
[30] V. Torra, On some relationship between the WOWA operator and the Choquet integral, in: Proceedings of the IPMU 1998 Conference, Paris, France, pp. 818–824.
[31] V. Torra, The WOWA operator and the interpolation function W^*: Chen and Otto's interpolation method revisited, Fuzzy Sets Syst. 113 (2000) 389–396.
[32] V. Torra, Y. Narukawa, Modeling Decisions Information Fusion and Aggregation Operators, Springer-Verlag, Berlin, 2007.
[33] A. Valls, V. Torra, Using classification as an aggregation tool for MCDM, Fuzzy Sets Syst. 115 (2000) 159–168.
[34] J. von Neumann, O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, 1944.
[35] Y.M. Wang, C. Parkan, Minimax disparity approach for obtaining OWA operator weights, Info. Sci. 175 (2005) 20–29.
[36] M.E. Yaari, The dual theory of choice under risk, Econometrica 55 (1987) 95–115.
[37] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Syst., Man Cyb. 18 (1988) 183–190.
[38] R.R. Yager, Constrained OWA aggregation, Fuzzy Sets Syst. 81 (1996) 89–101.
[39] R.R. Yager, Quantifier guided aggregation using OWA operators, Int. J. Intell. Syst. 11 (1996) 49–73.
[40] R.R. Yager, Including importances in OWA aggregations using fuzzy systems modeling, IEEE Trans. Fuzzy Syst. 6 (1998) 286–294.
[41] R.R. Yager, D.P. Filev, Essentials of Fuzzy Modeling and Control, Wiley, New York, 1994.
[42] R.R. Yager, J. Kacprzyk, The Ordered Weighted Averaging Operators: Theory and Applications, Kluwer AP, Dordrecht, 1997.