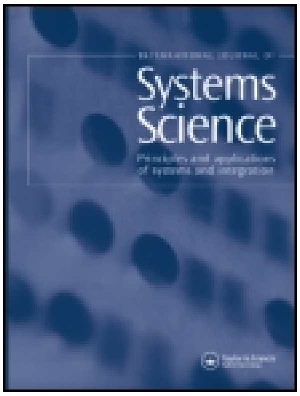


International Journal of Systems Science
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tsys20

# Tail mean and related robust solution concepts

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Published online: 23 Mar 2012.

*International Journal of Systems Science*, 2014
Vol. 45, No. 1, 29–38, http://dx.doi.org/10.1080/00207721.2012.669868



# Tail mean and related robust solution concepts

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(*Received 20 December 2010; final version received 7 January 2012*)

Robust optimisation might be viewed as a multicriteria optimisation problem where objectives correspond to the scenarios although their probabilities are unknown or imprecise. The simplest robust solution concept represents a conservative approach focused on the worst-case scenario results optimisation. A softer concept allows one to optimise the tail mean thus combining performances under multiple worst scenarios. We show that while considering robust models allowing the probabilities to vary only within given intervals, the tail mean represents the robust solution for only upper bounded probabilities. For any arbitrary intervals of probabilities the corresponding robust solution may be expressed by the optimisation of appropriately combined mean and tail mean criteria thus remaining easily implementable with auxiliary linear inequalities. Moreover, we use the tail mean concept to develope linear programming implementable robust solution concepts related to risk averse optimisation criteria.

**Keywords:** decisions under uncertainty; robust optimisation; tail means; linear programming; multiple criteria



## 1. Introduction

Several approaches have been developed to deal with uncertain or imprecise data in optimisation problems. The approaches focused on the quality of the solution for some data domains (bounded regions) are considered robust (Bertsimas and Thiele 2006; Liesiö, Mild, and Salo 2007; Ben-Tal, El Ghaoui, and Nemirovski 2009). The notion of robustness applied to decision problems was first introduced by Gupta and Rosenhead (1968). Practical importance of performance sensitivity against data uncertainty and errors has later attracted considerable attention to the search for robust solutions. Robust approaches are also widely applied in systems analysis and control (Liua, Fenga, and Maa 2012; Harib and Moustafa in press; Zhai, Zhang, and Liu in press). Actually, as suggested by Roy (1998), the concept of robustness should be applied not only to solutions, but also more generally to various assertions and recommendations generated within a decision support process. The precise concept of robustness depends on the way uncertain data domains and the quality or stability characteristics are introduced. Typically, in robust analysis one does not attribute any probability distribution to represent uncertainties. Data uncertainty is rather represented by nonattributed scenarios. Since one wishes to optimise results under each scenario, robust optimisation might be in some sense viewed as a multiobjective optimisation problem where objectives correspond to the scenarios. However, despite of many similarities of such robust optimisation concepts to multiobjective models, there are also some significant differences (Hites, De Smet, Risse, Salazar-Neumann, and Vincke 2006). Actually, robust optimisation is a problem of optimal distribution of objective values under several scenarios rather than a standard multiobjective optimisation model (Ogryczak 2002).

A conservative notion of robustness focusing on worst-case scenario results is widely accepted and the max-min optimisation is commonly used to seek robust solutions. The worst-case scenario analysis can be applied either to the absolute values of objectives (the absolute robustness) or to the regret values (the deviational robustness) (Kouvelis and Yu 1997). The latter, when considered from the multiobjective perspective, represents a simplified reference point approach with the utopian (ideal) objective values for all the scenarios used as aspiration levels. Recently, a more advanced concept of ordered weighted averaging was introduced into robust optimisation (Perny, Spanjaard, and Storme 2006), thus allowing to optimise combined performances under the worst-case scenario together with the performances under the second worst scenario, the third worst and so on. Such an approach better exploits the entire distribution of objective vectors in search for robust solutions and,

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more importantly, it introduces some tools for modelling robust preferences. Actually, while more sophisticated concepts of robust optimisation are considered within the area of discrete programming models, only the absolute robustness is usually applied to the majority of decision and design problems.

In this article we focus on robust approaches where the probabilities are unknown or imprecise. Having assumed that the probabilities may vary within given intervals, we optimise the worst-case expected outcome with respect to the probabilities perturbation set. For the case of unlimited perturbations the worst-case expectation becomes the worst outcome (max-min solution). In the general case, the worst-case expectation is a generalisation of the tail mean. Nevertheless, it can be effectively reformulated as a linear programming (LP) expansion of the original problem.

This article is organised as follows. In Section 2 we recall the tail mean (conditional value-at-risk, CVaR) solution concept providing a new proof of the LP computational model which remains applicable for more general problems related to robust solution concepts. Section 3 contains the main results. We show that the robust solution for only upper bounded probabilities is a tail $\beta$-mean solution for an appropriate $\beta$ value. For proportional upper and lower limits on probability perturbation the robust solution may be expressed as optimisation of appropriately combined mean and tail mean criteria. Finally, a general robust solution for any arbitrary intervals of probabilities or probabilities perturbations can be expressed with an optimisation problem very similar to that for tail $\beta$-mean and thereby easily implementable with auxiliary linear inequalities. In Section 4, we analyse robust approaches to risk functions optimisation. It turns out that the robust form of tail $\beta$-mean is also a tail mean with tightened tolerance level $\beta$ while the robust form of the mean absolute deviation (MAD) model also can be represented by a modified tail mean thus preserving the LP computability.

## 2. The solution concepts

Consider a decision problem under uncertainty where the decision is based on the maximisation of a scalar (real valued) outcome. The simplest representation of uncertainty depends on a finite set $I$ ($|I| = m$) of predefined scenarios. The final outcome is uncertain and only its realisations under various scenarios $i \in I$ are known. Exactly, for each scenario $i$ the corresponding outcome realisation is given as a function of the decision variables $y_i = f_i(\mathbf{x})$ where $\mathbf{x}$ denotes a vector of decision variables to be selected from the feasible set $Q \subset R^n$ of constraints under consideration. Let us define the set of attainable outcomes $A = \{\mathbf{y} = (y)_{i \in I}: \; y_i = f_i(\mathbf{x}) \; \forall i \in I, \; \mathbf{x} \in Q\}$. We are interested in larger outcomes under each scenario. Hence, the problem of decision under uncertainty can be considered a multiple criteria optimisation problem (Haimes 1993; Ogryczak 2002)

$$\max\{(y_1, y_2, \ldots, y_m) : \mathbf{y} \in A\}. \tag{1}$$

From the perspective of decision making under uncertainty, model (1) only specifies that we are interested in maximisation of outcomes under all scenarios $i \in I$. In order to make the multiple objective model operational for a decision support process, one needs to assume some solution concept well-adjusted to the decision maker's preferences.

A conservative notion of robustness is focused on the worst-case scenario results defined by maximisation of the objective function representing the *minimum* (worst) outcome

$$M(\mathbf{y}) = \min_{i \in I} y_i$$

and it is not affected by the scenario importance weights at all. It is widely accepted and max-min optimisation is commonly used to seek robust solutions.

Within decision problems under risk it is assumed that the exact values of the underlying scenario probabilities $p_i$ ($i \in I$) are given or can be estimated (Ruszczyński and Shapiro 2003). This is a basis for the stochastic programming approaches where the solution concept depends on the maximisation of the expected value (the mean outcome)

$$\mu(\mathbf{y}) = \sum_{i \in I} y_i p_i \tag{2}$$

or alternatively some risk function. In particular, the second-order quantile risk measures are recently used as such criteria. They have been introduced in different ways by many authors (Embrechts, Klüppelberg, and Mikosch 1997; Artzner, Delbaen, Eber, and Heath 1999; Ogryczak 1999; Rockafellar and Uryasev 2000). They generally represent the (worst) tail mean defined as the mean within the specified tolerance level (quantile) of the worst outcomes.

For any probabilities $p_i$ and tolerance level $\beta$ the corresponding tail mean can be mathematically formalised as follows (Ogryczak 2002; Ogryczak and Ruszczyński 2002). First, we introduce the right-continuous cumulative distribution function (cdf):

$$F_{\mathbf{y}}(\eta) = \sum_{i \in I} p_i \kappa_i(\eta) \quad \text{where} \quad \kappa_i(\eta) = \begin{cases} 1 & \text{if } y_i \le \eta, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

which for any real (outcome) value $\eta$ provides the measure of outcomes smaller or equal to $\eta$. Next, we introduce the quantile function $F_{\mathbf{y}}^{(-1)}$ as the left-continuous inverse of the cumulative distribution function $F_{\mathbf{y}}$:

$$F_{\mathbf{y}}^{(-1)}(\beta) = \inf\{\eta\colon F_{\mathbf{y}}(\eta) \geq \beta\} \quad \text{for } 0 < \beta \leq 1.$$

By integrating $F_{\mathbf{y}}^{(-1)}$ one gets the (worst) tail mean

$$\mu_\beta(\mathbf{y}) = \frac{1}{\beta}\int_0^\beta F_{\mathbf{y}}^{(-1)}(\alpha)\mathrm{d}\alpha \quad \text{for } 0 < \beta \leq 1, \tag{4}$$

the point value of the absolute Lorenz curve (Ogryczak 2000). The latter makes the tail means directly related to the dual theory of choice under risk (Quiggin 1982; Roell 1987; Yaari 1987).

Within the decision under risk literature the tail mean quantity is usually called CVaR representing quantile value $F_{\mathbf{y}}^{(-1)}(\beta)$ (Pflug 2000). Indeed, if $F_{\mathbf{y}}(F_{\mathbf{y}}^{(-1)}(\beta)) = \beta$, then

$$\mu_\beta(\mathbf{y}) = \mu\big(\mathbf{y}|\mathbf{y} \leq F_{\mathbf{y}}^{(-1)}(\beta)\big) = \frac{1}{\beta}\sum_{i\in I: y_i \leq F_{\mathbf{y}}^{(-1)}(\beta)} p_i y_i, \tag{5}$$

which may be interpreted as the expectation to fall below the VaR value (given that the outcomes on the level of $F_{\mathbf{y}}^{(-1)}(\beta)$ or smaller are considered). The latter has been introduced as expected shortfall (Embrechts et al. 1997), tail conditional expectation (Artzner et al. 1999) and finally CVaR (Rockafellar and Uryasev 2000). Although valid for many continuous distributions, in general, relation (5) cannot serve as a definition of tail $\beta$-mean because $\eta$ such that $F_{\mathbf{y}}(\eta) = \beta$ need not exist, see Ogryczak and Ruszczyński (2002) for a wider discussion of relations of tail mean to the classically understood CVaR measures. Nevertheless the name CVaR, after Rockafellar and Uryasev (2002), is now the most commonly used designation for tail $\beta$-mean in decision under risk, and especially related to financial applications. However, since we will consider the measure with respect to distributions without a formally defined probabilistic space we will refer to it as tail mean.

The tail mean maximisation is consistent with the second-degree stochastic dominance (Ogryczak and Ruszczyński 2002) and meets the requirements of coherent risk measurement (Pflug 2000). Apart from the area of decisions under risk tail means have been already applied to location problems (Ogryczak and Zawadzki 2002), to fair resource allocations in networks (Ogryczak and Śliwiński 2002; Ogryczak, Wierzbicki, and Milewski 2008) as well as to planning aperture modulation for radiation therapy treatment (Romeijn, Ahuja, Dempsey, and Kumar 2005) and to statistical learning problems (Takeda and Kanamori 2009).

Maximisation of tail $\beta$-mean

$$\max_{\mathbf{y}\in A} \mu_\beta(\mathbf{y}) \tag{6}$$

defines the tail $\beta$-mean solution concept. When parameter $\beta$ approaches 0, the tail $\beta$-mean tends to the smallest outcome ($\lim_{\beta\to 0_+}\mu_\beta(\mathbf{y}) = M(\mathbf{y})$). On the other hand, for $\beta=1$ the corresponding tail mean becomes the standard mean ($\mu_1(\mathbf{y}) = \mu(\mathbf{y})$). Tail mean represents a generalisation of the importance of weighted ordered weighted average (WOWA) (Ogryczak and Śliwiński 2009).

Note that due to the finite number of scenarios tail $\beta$-mean is well-defined by the following optimisation:

$$\mu_\beta(\mathbf{y}) = \min_{u_i}\left\{\frac{1}{\beta}\sum_{i\in I} y_i u_i\colon \sum_{i\in I} u_i = \beta,\ 0 \leq u_i \leq p_i\ \forall i \in I\right\}. \tag{7}$$

Problem (7) is a linear program for a given outcome vector $\mathbf{y}$ while it becomes nonlinear for $\mathbf{y}$ being a vector of variables as in the case of tail $\beta$-mean optimisation problem (6). It turns out that this difficulty can be overcome by an equivalent LP formulation of tail $\beta$-mean that allows one to implement tail $\beta$-mean optimisation problem (6) with auxiliary linear inequalities. Namely, the following theorem recalls the Rockafellar and Uryasev (2000) LP model for continuous distributions which remains valid for a general distribution (Ogryczak and Ruszczyński 2002). We introduce a new proof which can be further generalised for a family of robust solution concepts we consider.

**Theorem 2.1:** *For any outcome vector* $\mathbf{y}$ *with the corresponding probabilities* $p_i$, *and for any real value* $0 < \beta \leq 1$, *the tail* $\beta$*-mean outcome is given by the following linear program*:

$$\mu_\beta(\mathbf{y}) = \max_{t,d_i}\left\{t - \frac{1}{\beta}\sum_{i\in I} p_i d_i\colon y_i \geq t - d_i,\ d_i \geq 0\ \forall i \in I\right\}. \tag{8}$$

**Proof:** The theorem can be proved by taking advantage of the LP dual to (7). Introducing dual variable $t$ corresponding to the equation $\sum_{i\in I} u_i = \beta$ and variables $d_i$ corresponding to upper bounds on $u_i$ one gets the LP dual (8). Due the duality theory, for any given vector $\mathbf{y}$ the tail $\beta$-mean $\mu_\beta(\mathbf{y})$ can be found as the optimal value of LP problem (8). □

Frequently, scenario probabilities are unknown or imprecise. Uncertainty is then represented by limits (intervals) on possible values of probabilities varying independently (Dupacova 1987; Jaffray 1989; Yager and Kreinovich 1999; Guo and Tanaka 2010).

We focus on such representation to define a robust solution concept. Generally, we consider the case of unknown probabilities belonging to the hypercube:

$$\mathbf{u} \in U = \Big\{(u_1, u_2, \ldots, u_m) : \sum_{i\in I} u_i = 1,\ \Delta_i^l \le u_i \le \Delta_i^u\ \forall i \in I\Big\}, \tag{9}$$

where obviously $\sum_{i\in I}\Delta_i^l \le 1 \le \sum_{i\in I}\Delta_i^u$. Certainly, such a case covers also the situation when there are known probabilities $\bar{p}_i$ but imprecisely. They may be affected by perturbations varying within given intervals $[-\delta_i^-, \delta_i^+]$. It is indeed a special case of $U$ with $\Delta_i^l = \bar{p}_i - \delta_i^-$ and $\Delta_i^u = \bar{p}_i + \delta_i^+$ for all $i \in I$. However, we will distinguish the specific case of given probabilities $\bar{\mathbf{p}}$ with possible perturbations bounded proportionally $\Delta_i^l = (1-\delta^-)\bar{p}_i$ and $\Delta_i^u = (1+\delta^+)\bar{p}_i$ for all $i \in I$ for given $\delta^+ \ge 0$ and $0 \le \delta^- \le 1$. Thus, the probabilities belonging to the hypercube:

$$\mathbf{u} \in U(\bar{\mathbf{p}}) = \Big\{(u_1, u_2, \ldots, u_m) : \sum_{i\in I} u_i = 1,\ \bar{p}_i(1-\delta^-) \le u_i \le \bar{p}_i(1+\delta^+)\ \forall i \in I\Big\}.$$

Focusing on the mean outcome as the primary system efficiency measure to be optimised we get the robust mean solution concept

$$\max_{\mathbf{y}} \min_{\mathbf{u}} \Big\{ \sum_{i\in I} u_i y_i : \mathbf{u} \in U,\ \mathbf{y} \in A \Big\}. \tag{10}$$

Further, taking into account that all the constraints of attainable set $A$ remain unchanged while the probabilities are perturbed, the robust mean solution can be rewritten as

$$\max_{\mathbf{y}\in A} \min_{\mathbf{u}\in U} \sum_{i\in I} u_i y_i = \max_{\mathbf{y}\in A} \Big\{ \min_{\mathbf{u}\in U} \sum_{i\in I} u_i y_i \Big\} = \max_{\mathbf{y}\in A} \mu^U(\mathbf{y}), \tag{11}$$

where

$$\mu^U(\mathbf{y}) = \min_{\mathbf{u}\in U} \sum_{i\in I} u_i y_i = \min_{u_i} \Big\{ \sum_{i\in I} y_i u_i : \sum_{i\in I} u_i = 1,\ \Delta_i^l \le u_i \le \Delta_i^u\ \forall i \in I \Big\} \tag{12}$$

or respectively

$$\mu^U(\mathbf{y}) = \min_{\mathbf{u}\in U(\mathbf{p})} \sum_{i\in I} u_i y_i = \min_{u_i} \Big\{ \sum_{i\in I} y_i u_i : \sum_{i\in I} u_i = 1,\ \bar{p}_i - \delta_i^- \le u_i \le \bar{p}_i + \delta_i^+\ \forall i \Big\} \tag{13}$$

represent the worst-case mean outcomes for a given outcome vector $\mathbf{y} \in A$ with respect to the probabilities set $U$. Similar robust solution concepts can be built for various risk functions used instead of the mean.

## 3. Tail mean and robust optimisation

Let us consider first the robust mean solution (11) in the case of unlimited probability perturbations ($\Delta_i^l = 0$ and $\Delta_i^u = 1$). One may easily notice that the worst-case mean outcome (12) becomes the worst outcome

$$\mu^U(\mathbf{y}) = \min_{u_i} \Big\{ \sum_{i\in I} y_i u_i : \sum_{i\in I} u_i = 1,\ 0 \le u_i \le 1\ \forall i \in I \Big\} = \min_{i\in I} y_i,$$

thus, leading to the conservative robust solution concept represented by the max-min approach.

For the case of probabilities lying in a given box with relaxed lower limits ($\Delta_i^l = 0\ \forall i \in I$) the robust solution (11) may also be represented as the tail $\beta$-mean with respect to appropriately rescaled probabilities $p_i = \Delta_i^u / \sum_{i\in I}\Delta_i^u$ and tolerance level $\beta = 1/\sum_{i\in I}\Delta_i^u$.

**Theorem 3.1:** *The robust solution* (12) *with relaxed lower bounds may be represented as the tail $\beta$-mean with respect to probabilities $p_i = \Delta_i^u/\sum_{i\in I}\Delta_i^u$ and $\beta = 1/\sum_{i\in I}\Delta_i^u$, and it can be found by simple expansion of the optimisation problem with auxiliary linear constraints and variables to the following*:

$$\max_{\mathbf{y},\mathbf{d},t} \Big\{ t - \sum_{i\in I} \Delta_i^u d_i : \mathbf{y} \in A;\ y_i \ge t - d_i,\ d_i \ge 0\ \forall i \in I \Big\}. \tag{14}$$

**Proof:** Note that by simple rescaling of variables with $s^u = \sum_{i\in I}\Delta_i^u$ one gets

$$\mu^U(\mathbf{y}) = \min_{u_i} \Big\{ \sum_{i\in I} y_i u_i : \sum_{i\in I} u_i = 1,\ 0 \le u_i \le \Delta_i^u\ \forall i \in I \Big\} = s^u \min_{u'_i} \Big\{ \sum_{i\in I} y_i u'_i : \sum_{i\in I} u'_i = \frac{1}{s^u},\ 0 \le u'_i \le \frac{\Delta_i^u}{s^u}\ \forall i \in I \Big\}.$$

Hence, the robust solution may be represented as the tail $(1/s^u)$-mean with respect to probabilities $p_i = \Delta_i^u/s^u$. Following Theorem 2.1, it can be searched by solving (14). □

Note that with $\Delta_i^u = 1$ for $i \in I$ we represent the robust solution (12) as the tail $\beta$-mean with $p_i = 1/m$ and $\beta = 1/m$, thus representing the max-min model. In the case of $\Delta_i^u = k/m$ for $i \in I$ we get $p_i = 1/m$ and $\beta = 1/k$.

For the specific case of given probabilities $\bar{\mathbf{p}}$ with possible perturbations bounded proportionally it is possible to express the corresponding robust solution (11) as tail mean based on the original probabilities. Indeed, in the case of $\Delta_i^u = (1+\delta^+)\bar{p}_i$ we get $p_i = \Delta_i^u / \sum_{i\in I} \Delta_i^u = \bar{p}_i$. Hence, the following corollary is valid.

**Corollary 3.2:** *The tail β-mean represents a concept of robust mean solution* (13) *for proportionally upper bounded perturbations* $\Delta_i^u = (1+\delta^+)\bar{p}_i$ *with* $\delta^+ = (1-\beta)/\beta$ *and relaxed the lower bounds* $\Delta_i^l = 0$ *for all* $i \in I$, *and it can be found by simple expansion of the optimisation problem with auxiliary linear constraints and variables to the following*:

$$\max_{\mathbf{y},\mathbf{d},t}\left\{t-(1+\delta^+)\sum_{i\in I}\bar{p}_i d_i : \mathbf{y}\in A;\, y_i \ge t-d_i,\ d_i\ge 0\ \forall i\right\}. \tag{15}$$

**Proof:** For proportionally bounded upper perturbations $\Delta_i^u = (1+\delta^+)\bar{p}_i$ and relaxed lower bounds $\Delta_i^l = 0$ the corresponding worst-case mean outcome (13) can be expressed as follows:

$$\begin{aligned}\mu^U(\mathbf{y}) &= \min_{u_i}\left\{\sum_{i\in I} y_i u_i : \sum_{i\in I} u_i = 1,\ 0\le u_i \le \bar{p}_i(1+\delta^+)\ \forall i\right\}\\ &= (1+\delta^+)\min_{u_i'}\left\{\sum_{i\in I} y_i u_i' : \sum_{i\in I} u_i' = \frac{1}{1+\delta^+},\right.\\ &\qquad \left. 0\le u_i' \le \bar{p}_i\ \forall i\right\}\\ &= (1+\delta^+)\mu_{\frac{1}{1+\delta^+}}(\mathbf{y}).\end{aligned}$$

Due to $\delta^+ = (1-\beta)/\beta$, one gets $(1+\delta^+) = 1/\beta$ and $\mu^U(\mathbf{y}) = \mu_\beta(\mathbf{y})/\beta$ where, following Theorem 2.1, $\mu_\beta(\mathbf{y})$ may be optimised by the LP model (15). □

In the general case of possible lower limits, the robust mean solution concept cannot be directly expressed as an appropriate tail β-mean. It turns out, however, that it can be expressed by the optimisation with combined tail β-mean and mean criteria.

**Theorem 3.3:** *The robust mean solution concept* (11) *is equivalent to the convex combination of the mean and the tail β-mean criteria maximisation*

$$\max_{\mathbf{y}\in A}\mu^U(\mathbf{y}) = \max_{\mathbf{y}\in A}[\lambda\mu(\mathbf{y}) + (1-\lambda)\mu_\beta(\mathbf{y})] \tag{16}$$

*with*

$$\lambda = \sum_{i\in I}\Delta_i^l \quad \textit{and} \quad \beta = \left(1-\sum_{i\in I}\Delta_i^l\right)\Big/\sum_{i\in I}(\Delta_i^u - \Delta_i^l),$$

*where the tail mean* $\mu_\beta(\mathbf{y})$ *is defined according to probabilities* $p_i'$ *while the mean* $\mu(\mathbf{y})$ *is considered with respect to probabilities* $p_i''$:

$$p_i' = (\Delta_i^u - \Delta_i^l)\Big/\sum_{i\in I}(\Delta_i^u - \Delta_i^l) \quad \textit{and}$$

$$p_i'' = \Delta_i^l\Big/\sum_{i\in I}\Delta_i^l \quad \textit{for } i\in I.$$

**Proof:** When introducing scaling factors $s^u = \sum_{i\in I}\Delta_i^u$ and $s^l = \sum_{i\in I}\Delta_i^l$ one gets $\lambda = s^l$ and $\beta = (1-s^l)/(s^u - s^l)$. The worst-case mean outcome (12) can be expressed as follows:

$$\begin{aligned}\mu^U(\mathbf{y}) &= \min_{u_i}\left\{\sum_{i\in I} y_i u_i : \sum_{i\in I} u_i = 1,\ \Delta_i^l \le u_i \le \Delta_i^u\ \forall i\in I\right\}\\ &= \min_{u_i'}\left\{\sum_{i\in I} y_i u_i' : \sum_{i\in I} u_i' = 1-s^l,\right.\\ &\qquad \left. 0\le u_i' \le \Delta_i^u - \Delta_i^l\ \forall i\in I\right\} + \sum_{i\in I} y_i\Delta_i^l\\ &= (s^u - s^l)\min_{u_i''}\left\{\sum_{i\in I} y_i u_i'' : \sum_{i\in I} u_i'' = \frac{1-s^l}{s^u - s^l},\right.\\ &\qquad \left. 0\le u_i'' \le \frac{\Delta_i^u - \Delta_i^l}{s^u - s^l}\ \forall i\in I\right\} + s^l\sum_{i\in I} y_i\frac{\Delta_i^l}{s^l}\\ &= (1-s^l)\min_{u_i''}\left\{\frac{1}{\beta}\sum_{i\in I} y_i u_i'' : \sum_{i\in I} u_i'' = \beta,\right.\\ &\qquad \left. 0\le u_i'' \le p_i'\ \forall i\in I\right\} + s^l\sum_{i\in I} y_i p_i''\\ &= (1-s^l)\mu_\beta(\mathbf{y}) + s^l\mu(\mathbf{y}) = (1-\lambda)\mu_\beta(\mathbf{y}) + \lambda\mu(\mathbf{y}),\end{aligned}$$

which completes the proof. □

**Corollary 3.4:** *The robust mean solution concept* (11) *for the specific case of given probabilities* $\bar{\mathbf{p}}$ *with possible perturbations bounded proportionally* $\Delta_i^l = (1-\delta^-)\bar{p}_i$ *and* $\Delta_i^u = (1+\delta^+)\bar{p}_i$ *for all* $i\in I$ *is equivalent to the convex combination of the mean and tail β-mean criteria maximisation*

$$\max_{\mathbf{y}\in A}\mu^U(\mathbf{y}) = \max_{\mathbf{y}\in A}[\delta^-\mu_\beta(\mathbf{y}) + (1-\delta^-)\mu(\mathbf{y})] \tag{17}$$

*with* $\beta = \delta^-/(\delta^+ + \delta^-)$ *where both the mean* $\mu(\mathbf{y})$ *and the tail mean* $\mu_\beta(\mathbf{y})$ *are calculated with respect to the original probabilities* $\bar{p}_i$.

**Proof:** Following Theorem 3.3, for proportionally bounded perturbations $\Delta_i^l = (1-\delta^-)\bar{p}_i$ and $\Delta_i^u = (1+\delta^+)\bar{p}_i$ Equation (16) is fulfilled with

$$\beta = \frac{1-\sum_{i\in I}\Delta_i^l}{\sum_{i\in I}(\Delta_i^u - \Delta_i^l)} = \frac{\delta^-}{\delta^+ + \delta^-} \quad \text{and} \quad \lambda = \sum_{i\in I}\Delta_i^l = 1-\delta^-.$$

Further, the tail mean is defined according to probabilities

$$p'_i = \frac{\Delta^u_i - \Delta^l_i}{\sum_{i\in I}(\Delta^u_i - \Delta^l_i)} = \frac{(\delta^+ + \delta^-)\bar{p}_i}{\delta^+ \sum_{i\in I}\bar{p}_i + \delta^- \sum_{i\in I}\bar{p}_i} = \bar{p}_i$$

as well as the mean is also considered with respect to probabilities

$$p''_i = \frac{\Delta^l_i}{\sum_{i\in I}\Delta^l_i} = \frac{(1-\delta^l_i)\bar{p}_i}{(1-\delta^l_i)\sum_{i\in I}\bar{p}_i} = \bar{p}_i,$$

which completes the proof. □

Following Theorems 2.1 and 3.3, the robust mean solution concept (11) can be expressed as an LP expansion of the original mean problem.

**Corollary 3.5:** *The robust mean solution concept* (11) *can be found by simple expansion of the mean problem with auxiliary linear constraints and variables to the following*:

$$\max_{\mathbf{y},\mathbf{d},t}\left\{\sum_{i\in I}\Delta^l_i y_i + \left(1 - \sum_{i\in I}\Delta^l_i\right)t - \sum_{i\in I}(\Delta^u_i - \Delta^l_i)d_i : \mathbf{y}\in A;\ y_i \ge t - d_i,\ d_i \ge 0\ \forall i\right\}. \quad (18)$$

**Proof:** Following formula (16) of Theorem 3.3

$$\max_{\mathbf{y}\in A}\mu^U(\mathbf{y}) = \max_{\mathbf{y}\in A}\left[\sum_{i\in I}\Delta^l_i y_i + \left(1 - \sum_{i\in I}\Delta^l_i\right)\mu_\beta(\mathbf{y})\right],$$

where the tail mean $\mu_\beta(\mathbf{y})$ is defined with $\beta = (1 - \sum_{i\in I}\Delta^l_i)/\sum_{i\in I}(\Delta^u_i - \Delta^l_i)$ according to probabilities $p_i = (\Delta^u_i - \Delta^l_i)/\sum_{i\in I}(\Delta^u_i - \Delta^l_i)$. Hence, applying Theorem 2.1 to express $\mu_\beta(\mathbf{y})$ as a linear program (8) one gets formula (18). □

For the specific case of given probabilities $\bar{\mathbf{p}}$ with possible perturbations bounded proportionally formula (18) simplifies accordingly leading to the following expression of the robust mean solution concept (13) as an LP expansion of the original mean problem.

**Corollary 3.6:** *For the specific case of given probabilities $\bar{\mathbf{p}}$ with possible perturbations bounded proportionally $\Delta^l_i = (1-\delta^-)\bar{p}_i$ and $\Delta^u_i = (1+\delta^+)\bar{p}_i$ for all $i \in I$, the robust mean solution concept* (11) *can be found by simple expansion of the mean problem with auxiliary linear constraints and variables to the following*:

$$\max_{\mathbf{y},\mathbf{d},t}\left\{(1-\delta^-)\sum_{i\in I}\bar{p}_i y_i + \delta^- t - (\delta^+ + \delta^-)\sum_{i\in I}\bar{p}_i d_i : \mathbf{y}\in A;\ y_i \ge t - d_i,\ d_i \ge 0\ \forall i\right\}. \quad (19)$$

Alternatively to formula (17) of Corollary 3.4, one may build directly an LP model taking advantage of the fact that the structure of optimisation problem (12) is very similar to that of the tail $\beta$-mean (7). Problem (12) is an LP for a given outcome vector $\mathbf{y}$ while it becomes nonlinear for $\mathbf{y}$ being a vector of variables. This difficulty can be overcome similar to Theorem 2.1 for the tail $\beta$-mean by switching to the corresponding dual LP problem.

**Theorem 3.7:** *For any arbitrary intervals $[\Delta^l_i, \Delta^u_i]$ (for all $i \in I$) of probabilities, the corresponding robust mean solution* (11) *can be given by the following optimisation problem*:

$$\max_{\mathbf{y},t,d^u_i,d^l_i}\left\{t - \sum_{i\in I}\Delta^u_i d^u_i + \sum_{i\in I}\Delta^l_i d^l_i : \mathbf{y}\in A;\ y_i = t - d^u_i + d^l_i,\ d^u_i, d^l_i \ge 0\ \forall i \in I\right\}. \quad (20)$$

**Proof:** The theorem can be proved by taking advantages of the LP dual to (12). Introducing dual variable $t$ corresponding to the equation $\sum_{i\in I}u_i = 1$ and variables $d^u_i$ and $d^l_i$ corresponding to upper and lower bounds on $u_i$, respectively, one gets the following LP dual to problem (12):

$$\mu^U(\mathbf{y}) = \max_{t,d^u_i,d^l_i}\left\{t - \sum_{i\in I}\Delta^u_i d^u_i + \sum_{i\in I}\Delta^l_i d^l_i : y_i = t - d^u_i + d^l_i,\ d^u_i, d^l_i \ge 0\ \forall i \in I\right\}, \quad (21)$$

which completes the proof. □

Note that formulation (20) of Theorem 3.7 is equivalent to model (17) of Corollary 3.4. Indeed, eliminating from formulation (20) variables $d^l_i$ with substitution $d^l_i = y_i - t + d^u_i$ and renaming simply $d^u_i$ with $d_i$ one gets model (17) of Corollary 3.4. On the other hand, model (20) of Theorem 3.7 can be further relaxed to the form:

$$\max_{\mathbf{y},t,d^u_i,d^l_i}\left\{t - \sum_{i\in I}\Delta^u_i d^u_i + \sum_{i\in I}\Delta^l_i d^l_i : \mathbf{y}\in A;\ y_i \ge t - d^u_i + d^l_i,\ d^u_i, d^l_i \ge 0\ \forall i \in I\right\}. \quad (22)$$

It follows from the fact, that while building the LP dual to (12) with dual variable $t$ corresponding to the equation $\sum_{i\in I}u_i = 1$ and variables $d^u_i$ and $d^l_i$ corresponding to upper and lower bounds on $u_i$, respectively, one may take into account that variables $u_i$ in (12) are actually non-negative, due to non-negative lower bounds $\Delta^l_i$. This leads us to inequality dual constraint $y_i \ge t - d^u_i + d^l_i$ replacing the corresponding equation in (21).

**Corollary 3.8:** *For the specific case of given probabilities $\bar{\mathbf{p}}$ with possible perturbations bounded proportionally $\Delta_i^l = (1-\delta^-)\bar{p}_i$ and $\Delta_i^u = (1+\delta^+)\bar{p}_i$ for all $i \in I$, the robust mean solution concept* (11) *can be found by simple expansion of the mean problem with auxiliary linear constraints and variables to the following*:

$$\max_{\mathbf{y},t,d_i^u,d_i^l}\left\{t-(1+\delta^+)\sum_{i\in I}\bar{p}_i d_i^u+(1-\delta^-)\sum_{i\in I}\bar{p}_i d_i^l : \mathbf{y}\in A;\right.$$
$$\left. y_i \ge t-d_i^u+d_i^l,\ d_i^u,d_i^l\ge 0\ \forall i\in I\right\}.$$

## 4. Robust optimisation of risk functions

Robust solution concepts can be built for various risk functions instead of the mean. While considering the tail mean as the basic optimisation criterion (CVaR optimisation), in order to allow for imprecise probabilities we have to deal with the robust tail mean solution concepts:

$$\max_{\mathbf{y}\in A}\mu_\beta^U(\mathbf{y}), \tag{23}$$

where

$$\mu_\beta^U(\mathbf{y})=\min_{\mathbf{u}\in U}\min_{u_i'}\left\{\frac{1}{\beta}\sum_{i\in I}y_iu_i' : \sum_{i\in I}u_i'=\beta,\right.$$
$$\left. 0\le u_i'\le u_i\ \forall i\in I\right\}. \tag{24}$$

It turns out that this robust solution concept for any arbitrary perturbation set $U$ (9) may be expressed as the standard tail mean with appropriately defined tolerance level and probabilities.

**Theorem 4.1:** *The robust tail $\beta$-mean solution* (24) *with arbitrary $U$ set* (9) *may be represented as the tail $\beta'$-mean with respect to probabilities $p_i' = \Delta_i^u / \sum_{i\in I}\Delta_i^u$ and $\beta' = \beta/\sum_{i\in I}\Delta_i^u$.*

**Proof:** Indeed, one may easily notice that

$$\mu_\beta^U(\mathbf{y})=\min_{\mathbf{u}\in U}\min_{u_i'}\left\{\frac{1}{\beta}\sum_{i\in I}y_iu_i' : \sum_{i\in I}u_i'=\beta,\right.$$
$$\left. 0\le u_i'\le u_i\ \forall i\in I\right\}$$
$$=\min_{u_i'}\left\{\frac{1}{\beta}\sum_{i\in I}y_iu_i' : \sum_{i\in I}u_i'=\beta,\ 0\le u_i'\le \Delta_i^u\ \forall i\in I\right\}$$
$$=\frac{\sum_{i\in I}\Delta_i^u}{\beta}\min_{u_i''}\left\{\sum_{i\in I}y_iu_i'' : \sum_{i\in I}u_i''=\frac{\beta}{\sum_{i\in I}\Delta_i^u},\right.$$
$$\left. 0\le u_i''\le \frac{\Delta_i^u}{\sum_{i\in I}\Delta_i^u}\ \forall i\in I\right\}. \tag{25}$$

Hence, the robust tail $\beta$-mean solution (24) may be represented as the standard tail mean with respect to probabilities $p_i' = \Delta_i^u/\sum_{i\in I}\Delta_i^u$ and the tolerance level $\beta/\sum_{i\in I}\Delta_i^u$. □

**Corollary 4.2:** *The robust tail $\beta$-mean solution* (24) *with arbitrary $U$ set* (9) *can be found by simple expansion of the optimisation problem with auxiliary linear constraints and variables to the following*:

$$\max_{\mathbf{y},\mathbf{d},t}\left\{t-\frac{1}{\beta}\sum_{i\in I}\Delta_i^u d_i : \mathbf{y}\in A;\ y_i\ge t-d_i,\ d_i\ge 0\ \forall i\in I\right\}. \tag{26}$$

**Corollary 4.3:** *The robust tail $\beta$-mean solution concept* (24) *for the specific case of given probabilities $\bar{\mathbf{p}}$ with possible perturbations upper bounded proportionally $\Delta_i^u = (1+\delta^+)\bar{p}_i$ and arbitrary lower bounded* (*any $\Delta_i^l \le \bar{p}_i$*) *for all $i \in I$ is equivalent to the tail $\beta'$-mean with respect to probabilities $\bar{\mathbf{p}}$ and $\beta' = \beta/(1+\delta^+)$, and it can be found by simple expansion of the optimisation problem with auxiliary linear constraints and variables to the following*:

$$\max_{\mathbf{y},\mathbf{d},t}\left\{t-\frac{1+\delta^+}{\beta}\sum_{i\in I}\bar{p}_i d_i : \mathbf{y}\in A;\right.$$
$$\left. y_i\ge t-d_i,\ d_i\ge 0\ \forall i\in I\right\}. \tag{27}$$

Let us consider the MAD related risk function based on the downside mean semideviation (Mansini, Ogryczak, and Speranza 2003)

$$\mu_d(\mathbf{y})=\mu(\mathbf{y})-\sum_{i\in I}\max\{\mu(\mathbf{y})-y_ip_i,0\}$$
$$=\sum_{i\in I}\min\{\mu(\mathbf{y}),y_i\}p_i, \tag{28}$$

which is consistent with the second degree stochastic dominance (Ogryczak and Ruszczyński 2001) and thereby coherent (Artzner et al. 1999). The corresponding robust MAD solution can be expressed as

$$\max_{\mathbf{y}\in A}\mu_d^U(\mathbf{y}), \tag{29}$$

where $\mu_d^U(\mathbf{y})$ is the robust downside mean

$$\mu_d^U(\mathbf{y})=\min_{\mathbf{u}\in U}\sum_{i\in I}\min\left\{\sum_{i\in I}u_iy_i,y_i\right\}u_i. \tag{30}$$

**Theorem 4.4:** *The robust downside mean* (30) *with arbitrary $U$ set* (9) *may be represented as the standard robust solution of the worst mean truncated distribution of outcomes*

$$\mu_d^U(\mathbf{y})=\mu^U(\mathbf{y}^U)\quad \textit{where}$$
$$\mathbf{y}^U=(\min\{y_1,\mu^U(\mathbf{y})\},\ldots,\min\{y_m,\mu^U(\mathbf{y})\}).$$

**Proof:** Let $\bar{\mathbf{u}} = \arg\min_{\mathbf{u}\in U} \sum_{i\in I} u_i y_i$, i.e. $\sum_{i\in I} \bar{u}_i y_i = \mu^U(\mathbf{y})$. Note that $\bar{\mathbf{u}}$ minimises also $\sum_{i\in I} u_i \min\{\alpha, y_i\}$ for any real $\alpha$. Moreover, $\alpha' < \alpha''$ implies $\sum_{i\in I} u_i \min\{\alpha', y_i\} \leq \sum_{i\in I} u_i \min\{\alpha'', y_i\}$ for any $\mathbf{u}\in U$. Hence,

$$\begin{aligned}\sum_{i\in I}\min\left\{\sum_{i\in I}\bar{u}_i y_i, y_i\right\}\bar{u}_i &= \min_{\mathbf{u}''\in U}\sum_{i\in I}\min\left\{\sum_{i\in I}\bar{u}_i y_i, y_i\right\}u''_i \\ &= \min_{\mathbf{u}'\in U}\min_{\mathbf{u}''\in U}\sum_{i\in I}\min\left\{\sum_{i\in I}u'_i y_i, y_i\right\}u''_i \\ &= \min_{\mathbf{u}\in U}\sum_{i\in I}\min\left\{\sum_{i\in I}u_i y_i, y_i\right\}u_i \\ &= \mu^U_d(\mathbf{y}).\end{aligned}$$

Thus, $\bar{\mathbf{u}}$ is the robust downside mean minimiser. Therefore,

$$\begin{aligned}\mu^U_d(\mathbf{y}) &= \sum_{i\in I}\min\left\{\sum_{i\in I}\bar{u}_i y_i, y_i\right\}\bar{u}_i \\ &= \min_{\mathbf{u}\in U}\sum_{i\in I}\min\left\{\sum_{i\in I}\bar{u}_i y_i, y_i\right\}u_i \\ &= \min_{\mathbf{u}\in U}\sum_{i\in I}\min\{\mu^U(\mathbf{y}), y_i\}u_i = \mu^U(\mathbf{y}^U),\end{aligned}$$

which completes the proof. □

Following Theorem 4.4, the robust downside mean (30) is a downside extension of the standard robust solution concept, similar to extended downside risk measures (Krzemienowski and Ogryczak 2005). Therefore, in the case of relaxed lower bounds the robust MAD solution (29) may be represented as tail $\beta$-mean of the tail $\beta$-mean truncated distribution of outcomes and thereby it is LP implementable.

**Corollary 4.5:** *The robust MAD solution* (29) *with relaxed lower bounds may be represented as the tail $\beta$-mean of the tail $\beta$-mean truncated distribution of outcomes*

$$\mu^U_d(\mathbf{y}) = \mu_\beta((\min\{y_1, \mu_\beta(\mathbf{y})\}, \ldots, \min\{y_m, \mu_\beta(\mathbf{y})\})) \tag{31}$$

*with respect to probabilities $p_i = \Delta^u_i / \sum_{i\in I}\Delta^u_i$ and $\beta = 1/\sum_{i\in I}\Delta^u_i$, and it can be found by simple expansion of the optimisation problem with auxiliary linear constraints and variables to the following*:

$$\begin{aligned}\max_{\mathbf{y}\in A}\mu^U_d(\mathbf{y}) = \max_{\mathbf{y},\mathbf{d},\mathbf{d}',t,t'}\Bigg\{ & t - \sum_{i\in I}\Delta^u_i d_i : \mathbf{y}\in A; \\ & y_i \geq t - d_i,\ d_i \geq 0\ \forall i\in I \\ & t' - \sum_{j\in I}\Delta^u_j d'_j \geq t - d_i\ \forall i\in I \\ & y_i \geq t' - d'_i,\ d'_i \geq 0\ \forall i\in I\Bigg\}.\end{aligned} \tag{32}$$

**Proof:** Following Theorem 3.1 $\mu^U(\mathbf{y})$ for relaxed lower bounds may be represented as tail $\beta$-mean with respect to probabilities $p_i = \Delta^u_i/\sum_{i\in I}\Delta^u_i$ and $\beta = 1/\sum_{i\in I}\Delta^u_i$. Thus, Equation (31) is valid.

Moreover, following Theorem 3.1

$$\max_{\mathbf{y}\in A}\mu_\beta(\mathbf{y}) = \max_{\mathbf{y},\mathbf{d},t}\left\{t - \sum_{i\in I}\Delta^u_i d_i : \mathbf{y}\in A;\ y_i \geq t - d_i,\ d_i \geq 0\ \forall i\in I\right\}.$$

Therefore,

$$\begin{aligned}\max_{\mathbf{y}\in A}\mu^U_d(\mathbf{y}) &= \max_{\mathbf{y}\in A}\mu_\beta(\mathbf{y}^U) \\ &= \max_{\mathbf{y}^U,\mathbf{d},t}\left\{t - \sum_{i\in I}\Delta^u_i d_i : y^U_i \geq t - d_i,\ d_i \geq 0\ \forall i\in I\right\},\end{aligned}$$

where

$$y^U_i = \min\left\{y_i, \max_{\mathbf{d}',t'}\left\{t' - \sum_{j\in I}\Delta^u_j d'_j :\ y_j \geq t' - d'_j,\ d'_j \geq 0\ \forall j\in I\right\}\right\} \quad \text{for } \mathbf{y}\in A.$$

Hence,

$$\begin{aligned}\max_{\mathbf{y}\in A}\mu^U_d(\mathbf{y}) = \max_{\mathbf{y},\mathbf{d},t}\Bigg\{ & t - \sum_{i\in I}\Delta^u_i d_i : \mathbf{y}\in A; \\ & y_i \geq t - d_i,\ d_i \geq 0\ \forall i\in I \\ & \max_{\mathbf{d}',t'}\left\{t' - \sum_{j\in I}\Delta^u_j d'_j : y_j \geq t' - d'_j,\ d'_j \geq 0\ \forall j\in I\right\} \geq t - d_i\ \forall i\in I\Bigg\},\end{aligned}$$

thus leading us to LP formulation (32), which completes the proof. □

Actually, as for arbitrary upper and lower limits on the probabilities the robust mean solution may be represented by optimisation of combined mean and tail mean criteria (Theorem 3.3), following Theorem 4.4 the robust MAD solution (29) may also be sought by the LP optimisation.

**Corollary 4.6:** *The robust downside mean* (30) *with arbitrary $U$ set* (9) *may be found by the following expansion of the optimisation problem with auxiliary linear constraints and variables*:

$$\max_{\mathbf{y},\mathbf{y}^U,\mathbf{d},\mathbf{d}',t,t'}\Bigg\{(1-s^l)\left(t - \frac{1}{\beta}\sum_{i\in I}p'_i d_i\right) + s^l\sum_{i\in I}p''_i y^U_i : \\ \mathbf{y}\in A;\ y^U_i \geq t - d_i,\ d_i \geq 0\ \forall i\in I$$

$$
\left.\begin{aligned}
&y_i^U \le y_i,\ y_i^U \le (1-s^l)\left(t' - \frac{1}{\beta}\sum_{j\in I} p'_j d'_j\right) \\
&\qquad + s^l \sum_{j\in I} p''_j y_j \ \forall i \in I \\
&y_i \ge t' - d'_i,\ d'_i \ge 0 \ \forall i \in I
\end{aligned}\right\},
$$

*where* $s^u = \sum_{i\in I} \Delta_i^u$, $s^l = \sum_{i\in I} \Delta_i^l$, *while* $\beta$, $p'_i$ *and* $p''_i$ *are defined according to Theorem* 3.3, *i.e.* $\beta = (1-s^l)/(s^u - s^l)$, $p'_i = (\Delta_i^u - \Delta_i^l)/(s^u - s^l)$ *and* $p''_i = \Delta_i^l / s^l$ *for* $i \in I$.

## 5. Conclusions

We have analysed the robust mean solution concept where uncertainty is represented by limits (intervals) on possible values of scenario probabilities varying independently. Such an approach, in general, leads to complex optimisation models with variable coefficients (probabilities). We have shown, however, that the robust mean solution concepts can be expressed with auxiliary linear inequalities, similar to the tail $\beta$-mean solution concept based on maximisation of mean in $\beta$ portion of the worst outcomes. Actually, the robust mean solution for upper limits on probabilities turns out to be tail $\beta$-mean for an appropriate $\beta$ value. In the case of specified both upper and lower limits the robust mean solution may be sought by optimisation of appropriately combined mean and tail mean criteria. Thus, a general robust mean solution for any arbitrary intervals probabilities can be expressed with an optimisation problem very similar to that for tail $\beta$-mean and thereby easily implementable with auxiliary linear inequalities.

Our analysis has shown that the robust mean solution concept is closely related to tail mean which is the basic equitable solution concept (Kostreva, Ogryczak, and Wierzbicki 2004). It corresponds to recent approaches to robust optimisation based on the use of equitable optimisation (Perny et al. 2006; Miettinen et al. 2008; Takeda and Kanamori 2009). Further study on equitable solution concepts and their relations to robust solutions seems to be a promising research direction. In particular, more complex robust preferences can be modelled by combining with various weights the corresponding tail means for larger and smaller perturbations, thus leading to combinations of multiple CVaR measures (Mansini, Ogryczak, and Speranza 2007) or to the generalised importance of WOWA (Ogryczak and Śliwiński 2009).

## Acknowledgements

This research was partially supported by the Polish National Budget Funds 2010–2013 for science under the grant N N514 044438. The author is indebted to anonymous referees for their helpful comments.

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