WOWA Enhancement of the Preference Modeling in the Reference Point Method*

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Abstract. The Reference Point Method (RPM) is an interactive technique formalizing the so-called quasi-satisficing approach to multiple criteria optimization. The DM's preferences are there specified in terms of reference (target) levels for several criteria. The reference levels are further used to build the scalarizing achievement function which generates an efficient solution when optimized. Typical RPM scalarizing functions are based on the augmented min-max aggregation where the worst individual achievement minimization process is additionally regularized with the average achievement. The regularization by the average achievement is easily implementable but it may disturb the basic min-max model. We show that the OWA regularization allows one to overcome this flaw since taking into account differences among all ordered achievement values. Further, allowing to define importance weights we introduce the WOWA enhanced RPM. Both the theoretical and implementation issues of the WOWA enhanced method are analyzed. Linear Programming implementation model is developed and proven.

1 Introduction

Consider a decision problem defined as an optimization problem with m criteria (objective functions). In this paper, without loss of generality, it is assumed that all the criteria are minimized. Hence, we consider the following Multiple Criteria Optimization (MCO) problem:

$$\min \{ \ (f_1(\mathbf{x}), \dots, f_m(\mathbf{x})) \ : \ \mathbf{x} \in Q \ \} \tag{1}$$

where $\mathbf{x}$ denotes a vector of decision variables to be selected within the feasible set $Q \subset R^n$, and $\mathbf{f}(x) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x}))$ is a vector function that maps the feasible set Q into the criterion space R^m. Note that neither any specific form of the feasible set Q is assumed nor any special form of criteria $f_i(\mathbf{x})$ is required. We refer to the elements of the criterion space as outcome vectors. An outcome vector $\mathbf{y}$ is attainable if it expresses outcomes of a feasible solution, i.e. $\mathbf{y} = \mathbf{f}(\mathbf{x})$ for some $\mathbf{x} \in Q$.

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Model (1) only specifies that we are interested in minimization of all objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. Thus it allows only to identify (to eliminate) obviously inefficient solutions leading to dominated outcome vectors, while still leaving the entire efficient set to look for a satisfactory compromise solution. In order to make the multiple criteria model operational for the decision support process, one needs assume some solution concept well adjusted to the DM preferences. This can be achieved with the so-called quasi-satisficing approach to multiple criteria decision problems. The best formalization of the quasi-satisficing approach to multiple criteria optimization was proposed and developed mainly by Wierzbicki [15] as the Reference Point Method (RPM). The reference point method was later extended to permit additional information from the DM and, eventually, led to efficient implementations of the so-called Aspiration/Reservation Based Decision Support (ARBDS) approach with many successful applications [2,16].

The RPM is an interactive technique. The basic concept of the interactive scheme is as follows. The DM specifies requirements in terms of reference levels, i.e., by introducing reference (target) values for several individual outcomes. The reference levels are used to build the scalarizing achievement function which generates an efficient solution when minimized. The computed efficient solution is presented to the DM as the current solution allowing comparison with previous solutions and modifications of the aspiration levels if necessary. In building the function it is assumed that the DM prefers outcomes that satisfy all the reference levels to any outcome that does not reach one or more of the reference levels.

The scalarizing achievement function can be viewed as two-stage transformation of the original outcomes. First, the strictly monotonic partial achievement functions are built to measure individual performance with respect to given reference levels. Having all the outcomes transformed into a uniform scale of individual achievements they are aggregated at the second stage to form a unique scalarization. The RPM is based on the so-called augmented (or regularized) min-max aggregation. Thus, the worst individual achievement is essentially minimized but the optimization process is additionally regularized with the term representing the average achievement. The min-max aggregation is crucial for allowing the RPM to generate all efficient solutions even for nonconvex (and particularly discrete) problems. On the other hand, the regularization is necessary to guarantee that only efficient solution are generated. The regularization by the average achievement is easily implementable but it may disturb the basic min-max model. Actually, the only consequent regularization of the min-max aggregation is the lex-min order or more practical the OWA aggregation with monotonic weights. The latter combines all the partial achievements allocating the largest weight to the worst achievement, the second largest weight to the second worst achievement, the third largest weight to the third worst achievement, and so on. The recent progress in optimization methods for ordered averages [8,11] allows one to implement the OWA RPM quite effectively. Further, following the concept of Weighted OWA [13,14] the importance weighting of several achievements may be incorporated into the RPM. Such a WOWA enhancement of the ARBDS uses importance weights to affect achievement importance by rescaling accordingly its measure within the distribution of achievements rather than straightforward rescaling of achievement values against those defined by the reference levels [12].

The paper is organized as follows. In the next section we formalize the scalarization achievement functions of the RPM with the detailed formulas for the ARBDS technique. In Section 3 we introduce the OWA and WOWA extensions of the RPM. We show that the WOWA enhanced RPM always generates an efficient solution to the original MCO problem complying simultaneously with the ARBDS preference model assumptions. Further, in Section 4 we develop and prove the Linear Programming implementation model for the method.

2 Scalarizations of the RPM

In the RPM method, depending on the specified reference levels, a special scalarizing achievement function is built which, when minimized, generates an efficient solution to the problem. While building the scalarizing achievement function the following properties of the preference model are assumed. First of all, each solution generated by the scalarizing function optimization must be an efficient solution of the original MCO problem. To meet this requirement the function must be strictly increasing with respect to each outcome. Second, a solution with all individual outcomes satisfying the corresponding reference levels is preferred to any solution with at least one individual outcome worse (greater) than its reference level. That means, the scalarizing achievement function minimization must enforce reaching the reference levels prior to further improving of criteria. Thus, similar to the goal programming approaches, the reference levels are treated as the targets but following the quasi-satisficing approach they are interpreted consistently with basic concepts of efficiency in the sense that the optimization is continued even when the target point has been reached already.

The generic scalarizing achievement function takes the following form [15]:

$$S(\mathbf{a}) = \max_{1\le i\le m}\{a_i\} + \frac{\varepsilon}{m}\sum_{i=1}^{m} a_i \tag{2}$$

where ε is an arbitrary small positive number and $a_i = s_i(f_i(\mathbf{x}))$, for $i = 1, 2, \ldots, m$, are the partial achievement measuring actual performances of the individual outcomes with partial achievement functions $s_i : R \to R$ with respect to the corresponding reference levels. Let $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ represent the achievement vector. The scalarizing achievement function (2) is, essentially, defined by the worst partial (individual) achievement but additionally regularized with the sum of all partial achievements. The regularization term is introduced only to guarantee the solution efficiency in the case when the minimization of the main term (the worst partial achievement) results in a non-unique optimal solution. Due to combining two terms with arbitrarily small parameter ε, formula (2) is easily implementable and it provides a direct interpretation of the scalarizing achievement function as expressing (dis)utility.

Various functions s_i provide a wide modeling environment for measuring partial achievements [16]. The basic RPM model is based on a single vector of the reference levels, the aspiration vector $\mathbf{r}^a$ and the piecewise linear functions s_i. Real-life applications of the RPM methodology usually deal with more complex partial achievement functions defined with more than one reference point [1,16] which enriches the preference models and simplifies the interactive analysis. In particular, the ARBDS models taking advantages of two reference vectors: vector of aspiration levels $\mathbf{r}^a$ and vector of reservation levels $\mathbf{r}^r$ [2] are used, thus allowing the DM to specify requirements by introducing acceptable and required values for several outcomes. The partial achievement function s_i can be interpreted then as a measure of the DM's satisfaction with the current value of outcome of the ith criterion. It is a strictly increasing function of outcome with value $a_i = 0$ if $f_i(\mathbf{x}) = r_i^a$, and $a_i = 1$ for $f_i(\mathbf{x}) = r_i^r$. Thus the partial achievement functions map the outcomes values onto a normalized scale of the DM's satisfaction. Various functions can be built meeting those requirements [16]. The simplest for implementation is convex piece-wise linear partial achievement function introduced in the ARBDS system for the multiple criteria transshipment problems with facility location [7]:

$$a_i = s_i(f_i(\mathbf{x})) = \begin{cases} \alpha(f_i(\mathbf{x}) - r_i^a)/(r_i^r - r_i^a), & f_i(\mathbf{x}) \le r_i^a \\ (f_i(\mathbf{x}) - r_i^a)/(r_i^r - r_i^a), & r_i^a < f_i(\mathbf{x}) < r_i^r \\ \gamma(f_i(\mathbf{x}) - r_i^r)/(r_i^r - r_i^a) + 1, & f_i(\mathbf{x}) \ge r_i^r \end{cases} \tag{3}$$

where α and γ are arbitrarily defined parameters satisfying $0 < \alpha < 1 < \gamma$. Parameter α represents additional increase of the DM's satisfaction (negative dissatisfaction values) when a criterion generates outcomes better than the corresponding aspiration level. On the other hand, parameter $\gamma > 1$ represents dissatisfaction connected with outcomes worse than the reservation level.

When accepting the loss of a direct utility interpretation, one may consider more powerful lexicographic preference modeling [4,5] based on linear partial achievement $a_i = (f_i(\mathbf{x}) - r_i^a)/(r_i^r - r_i^a)$ but splitted into separate preemptive multilevel interval achievement measures: the reservation level underachievement a_i^r, the aspiration level underachievement a_i^a and the aspiration level overachievement a_i^o defined by the following formula:

$$\begin{array}{ll} a_i^r = s_i^r(f_i(\mathbf{x})) = (f_i(\mathbf{x}) - r_i^r)_+/(r_i^r - r_i^a) & \forall\, i \in I \\ a_i^a = s_i^a(f_i(\mathbf{x})) = \min\{(f_i(\mathbf{x}) - r_i^a)_+/(r_i^r - r_i^a), 1\} & \forall\, i \in I \\ a_i^o = s_i^o(f_i(\mathbf{x})) = (r_i^a - f_i(\mathbf{x}))_+/(r_i^r - r_i^a) & \forall\, i \in I \end{array} \tag{4}$$

Minimization of the scalarizing achievement function (2)–(3) is then replaced with the lexicographic optimization of the multilevel aggregations:

$$\operatorname{lex}\min_{\mathbf{x}} \{(S(\mathbf{a}^r), S(\mathbf{a}^a), S(-\mathbf{a}^o)) : \text{Eq. (4)},\ \mathbf{x} \in Q\} \tag{5}$$

Note that instead of (4), the interval achievements may be defined with the goal programming modeling techniques [6]:

$$f_i(\mathbf{x})/(r_i^r - r_i^a) + a_i^o - a_i^a - a_i^r = r_i^a,\ a_i^o \ge 0,\ 0 \le a_i^a \le 1,\ a_i^r \ge 0 \quad \forall\, i \in I \tag{6}$$

3 WOWA Extension of the RPM

The crucial properties of the RPM are related to the min-max aggregation of partial achievements while the regularization is only introduced to guarantee the aggregation monotonicity. Unfortunately, the distribution of achievements may make the min-max criterion partially passive when one specific achievement is relatively very small for all the solutions. Minimization of the worst achievement may then leave all other achievements unoptimized. Nevertheless, the selection is then made according to linear aggregation of the regularization term instead of the min-max aggregation, thus destroying the preference model of the RPM. This can be illustrated with an example of a simple discrete problem of 7 alternative feasible solutions to be selected according to 6 criteria. Table 1 presents six partial achievements for all the solutions where all the outcome values were within the corresponding intervals between the aspiration and the reservation levels. Thus the partial achievements may be viewed as a_i^a defined according to formula (4) (with $a_r^r = 0$ and $a_i^o = 0$) as well as the a_i defined according to formula (3). All the solutions are efficient. Solutions S1 to S5 reach the aspiration levels (achievement values 0.0) for four of the first five criteria while being quite far from one of them and the aspiration level for the sixth criterion as well (achievement values 0.9). Solution S6 is close to the aspiration levels (achievement values 0.2) for the first five criteria while being far only to the aspiration level for the sixth criterion (achievement values 0.9). All the solutions generate the same worst achievement value 0.9. Therefore, while using the standard augmented min-max aggregation (2) the final selection of a solution depends on the total achievement (regularization term). Actually, one of solutions S1 to S5 will be selected as better than S6.

In order to avoid inconsistencies caused by the regularization in the aggregation (2), the min-max solution may be regularized according to the ordered averaging rules [17]. This is mathematically formalized as follows. Within the space of achievement vectors we introduce map $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$ which orders the coordinates of achievements vectors in a nonincreasing order, i.e., $\Theta(a_1, \ldots, a_m) = (\theta_1(\mathbf{a}), \ldots, \theta_m(\mathbf{a}))$ iff there exists a permutation τ such that $\theta_i(\mathbf{a}) = a_{\tau(i)}$ for all i and $\theta_1(\mathbf{a}) \geq \theta_2(\mathbf{a}) \geq \ldots \geq \theta_m(\mathbf{a})$. The standard min-max aggregation depends on minimization of $\theta_1(\mathbf{a})$ and it ignores values of $\theta_i(\mathbf{a})$ for $i \geq 2$. In order to take into account all the achievement values, one needs to maximize the weighted combination of the ordered achievements thus representing the so-called Ordered Weighted Averaging (OWA) aggregation [17]. Note that the weights are then assigned to the specific positions within the ordered achievements rather than to the partial achievements themselves. With the OWA aggregation one gets the following RPM model:

$$\min_{\mathbf{x}} \{ \sum_{i=1}^{m} w_i \theta_i(\mathbf{a}) \ : \ a_i = s_i(f_i(\mathbf{x})) \ \forall \ i, \ \mathbf{x} \in Q \ \} \tag{7}$$

where $w_1 > w_2 > \ldots > w_m$ are positive and strictly decreasing weights. Actually, they should be significantly decreasing to represent regularization of

Table 1. Sample achievements with passive min-max criterion

									$\mathbf{w}$	0.5	0.25	0.15	0.05	0.03	0.02	
Sol.	a_1	a_2	a_3	a_4	a_5	a_6	max	$\sum$		θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	$A_{\mathbf{w}}$
S1	0.9	0.0	0.0	0.0	0.0	0.9	0.9	1.8		0.9	0.9	0.0	0.0	0.0	0.0	0.675
S2	0.0	0.9	0.0	0.0	0.0	0.9	0.9	1.8		0.9	0.9	0.0	0.0	0.0	0.0	0.675
S3	0.0	0.0	0.9	0.0	0.0	0.9	0.9	1.8		0.9	0.9	0.0	0.0	0.0	0.0	0.675
S4	0.0	0.0	0.0	0.9	0.0	0.9	0.9	1.8		0.9	0.9	0.0	0.0	0.0	0.0	0.675
S5	0.0	0.0	0.0	0.0	0.9	0.9	0.9	1.8		0.9	0.9	0.0	0.0	0.0	0.0	0.675
S6	0.2	0.2	0.2	0.2	0.2	0.9	0.9	1.9		0.9	0.2	0.2	0.2	0.2	0.2	0.550
S7	0.9	0.9	0.9	0.2	0.6	0.2	0.9	3.7		0.9	0.9	0.9	0.6	0.2	0.2	0.895

the min-max order. Note that the standard RPM model with the scalarizing achievement function (2) can be expressed as the OWA model (7) with weights $w_2 \ldots = w_m = \varepsilon/m$ and $w_1 = 1 + \varepsilon/m$ thus strictly decreasing in the case of $m = 2$. Unfortunately, for $m > 2$ it abandons the differences in weighting of the second largest achievement, the third largest one etc ($w_2 = \ldots = w_m = \varepsilon/m$). The OWA RPM model (7) allows one to differentiate all the weights by introducing decreasing series (e.g. geometric ones). One may notice that application of decreasing weights $\mathbf{w} = (0.5, 0.25, 0.15, 0.05, 0.03, 0.02)$ within the OWA RPM enables selection of solution S6 from Table 1.

Typical RPM model allows weighting of several achievements only by straightforward rescaling of the achievement values [12]. The OWA RPM model enables one to introduce importance weights to affect achievement importance by rescaling accordingly its measure within the distribution of achievements as defined in the so-called Weighted OWA (WOWA) aggregation [13]. Let $\mathbf{w} = (w_1, \ldots, w_m)$ be a vector of preferential (OWA) weights and let $\mathbf{p} = (p_1, \ldots, p_m)$ denote the vector of importance weights ($p_i \geq 0$ for $i = 1, 2, \ldots, m$ as well as $\sum_{i=1}^m p_i = 1$). The corresponding Weighted OWA aggregation of achievements $\mathbf{a} = (a_1, \ldots, a_m)$ is defined as follows:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{i=1}^{m} \omega_i \theta_i(\mathbf{a}), \qquad \omega_i = w^*(\sum_{k \leq i} p_{\tau(k)}) - w^*(\sum_{k<i} p_{\tau(k)}) \tag{8}$$

where w^* is a monotone increasing function that interpolates points $(\frac{i}{m}, \sum_{k \leq i} w_k)$ together with the point (0.0) and τ representing the ordering permutation for $\mathbf{a}$ (i.e. $a_{\tau(i)} = \theta_i(\mathbf{a})$). We focus on the linear interpolation. The WOWA may be expressed with more direct formula where preferential (OWA) weights w_i are applied to averages of the corresponding portions of ordered achievements (quantile intervals) according to the distribution defined by importance weights p_i [9,10]:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{i=1}^{m} w_i m \int_{\frac{i-1}{m}}^{\frac{i}{m}} \overline{F}_{\mathbf{a}}^{(-1)}(\xi)\, d\xi \tag{9}$$

Table 2. WOWA selection with $\mathbf{p} = (\frac{4}{12}, \frac{3}{12}, \frac{2}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$

$\mathbf{w}$	0.5		0.25		0.15		0.05		0.03		0.02		$A_{\mathbf{w},\mathbf{p}}(\mathbf{a})$
S1	0.9	0.9	0.9	0.9	0.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.7425
S2	0.9	0.9	0.9	0.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.675
S3	0.9	0.9	0.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5625
S4	0.9	0.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.45
S5	0.9	0.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.45
S6	0.9	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.375
S7	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.6	0.2	0.2	0.8815

Table 3. WOWA selection with $\mathbf{p} = (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{7}{12})$

$\mathbf{w}$	0.5		0.25		0.15		0.05		0.03		0.02		$A_{\mathbf{w},\mathbf{p}}(\mathbf{a})$
S1	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.0	0.0	0.0	0.0	0.855
S2	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.0	0.0	0.0	0.0	0.855
S3	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.0	0.0	0.0	0.0	0.855
S4	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.0	0.0	0.0	0.0	0.855
S5	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.0	0.0	0.0	0.0	0.855
S6	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.2	0.2	0.2	0.2	0.2	0.8475
S7	0.9	0.9	0.9	0.6	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.6875

where $\overline{F}_{\mathbf{y}}^{(-1)}$ is the stepwise function $\overline{F}_{\mathbf{y}}^{(-1)}(\xi) = \theta_i(\mathbf{y})$ for $\beta_{i-1} < \xi \leq \beta_i$. It can also be mathematically formalized as follows. First, we introduce the left-continuous right tail cumulative distribution function (cdf) defined as:

$$\overline{F}_{\mathbf{y}}(d) = \sum_{i \in I} p_i \delta_i(d) \quad \text{where} \quad \delta_i(d) = \begin{cases} 1 & \text{if } y_i \geq d \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

which for any real (outcome) value d provides the measure of outcomes greater or equal to d. Next, we introduce the quantile function $\overline{F}_{\mathbf{y}}^{(-1)}$ as the right-continuous inverse of the cumulative distribution function $\overline{F}_{\mathbf{y}}$:

$$\overline{F}_{\mathbf{y}}^{(-1)}(\xi) = \sup \{\eta : \overline{F}_{\mathbf{y}}(\eta) \geq \xi\} \quad \text{for} \quad 0 < \xi \leq 1.$$

For instance applying importance weighting $\mathbf{p} = (\frac{4}{12}, \frac{3}{12}, \frac{2}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$ to solution achievements from Table 1 and using them together with given there OWA weights $\mathbf{w}$ one gets the WOWA aggregations from Table 2. The corresponding RPM method selects then solution S6, similarly to the case of equal importance weights. On the other hand, when increasing the importance of the last outcome achievements with $\mathbf{p} = (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{7}{12})$ one gets the WOWA values from Table 3.

The WOWA enhanced ARBDS can be formulated as based on the following lexicographic optimization problem:

$$\operatorname{lex}\min_{\mathbf{x}} \{(A_{\mathbf{w},\mathbf{p}}(\mathbf{a}^r), A_{\mathbf{w},\mathbf{p}}(\mathbf{a}^a), A_{\mathbf{w},\mathbf{p}}(-\mathbf{a}^o)) : \text{Eq. (4)},\ \mathbf{x} \in Q\} \tag{11}$$

used to generate current solutions according to the specified preferences. We will show that problem (11) always generates an efficient solution to the original MCO problem complying simultaneously with the ARBDS preference model assumptions.

Theorem 1. *For any reference levels $r_i^a < r_i^r$, any positive weights $\mathbf{w}$ and $\mathbf{p}$, if $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ is an optimal solution of the problem (11), then $\bar{\mathbf{x}}$ is an efficient solution of the corresponding multiple criteria problem (1).*

Proof. Let $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ be an optimal solution of the problem (11) with some positive weighting vectors $\mathbf{w}$ and $\mathbf{p}$. Suppose that $\bar{x}$ is not efficient to the multiple criteria problem (1). This means, there exists a decision vector $x \in Q$ such that $f_i(\mathbf{x}) \leq f_i(\bar{\mathbf{x}})$ for all $i \in I$ and $f_{i_o}(\mathbf{x}) < f_{i_o}(\bar{\mathbf{x}})$ for some outcome index $i_o \in I$. Let us define a_i^r, a_i^a and a_i^o according to formula (4). The quadruple $(\mathbf{x}, \mathbf{a}^r, \mathbf{a}^a, \mathbf{a}^o)$ is then a feasible solution of problem (11). Moreover, $a_i^r \leq \bar{a}_i^r$, $a_i^a \leq \bar{a}_i^a$ and $a_i^o \geq \bar{a}_i^o$ for all $i \in I$ where at least one of strict inequalities $a_{i_0}^r < \bar{a}_{i_0}^r$ or $a_{i_0}^a < \bar{a}_{i_0}^a$ or $a_{i_0}^o > \bar{a}_{i_0}^o$ holds. Hence, due to strict monotonicity of the WOWA aggregation with positive weighting vectors, one gets $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}^r) \leq A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}}^r)$, $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}^a) \leq A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}}^a)$ and $A_{\mathbf{w},\mathbf{p}}(-\mathbf{a}^o) \leq A_{\mathbf{w},\mathbf{p}}(-\bar{\mathbf{a}}^o)$ with at least one inequality strict. The latest assertion contradicts the lexicographic optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ for problem (11), which completes the proof.

Theorem 2. *For any reference levels $r_i^a < r_i^r$, any positive weights $\mathbf{w}$ and $\mathbf{p}$, if $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ is an optimal solution of the problem (11), then all the reservation level underachievements $\bar{a}_i^r$ are equal 0 whenever there exists a feasible solution $\mathbf{x} \in Q$ such that $f_i(\mathbf{x}) \leq r_i^r$ for all $i \in I$.*

Proof. Let $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ be an optimal solution of the problem (11) with some positive weighting vectors $\mathbf{w}$ and $\mathbf{p}$. Suppose that $\bar{a}_{i_0}^r > 0$ for some $i_0 \in I$ and there exists a feasible solution $\mathbf{x} \in Q$ such that $f_i(\mathbf{x}) \leq r_i^r$ for all $i \in I$. Let us define a_i^r, a_i^a and a_i^o according to formula (4) and note that $a_i^r = 0$ for all $i \in I$. The quadruple $(\mathbf{x}, \mathbf{a}^r, \mathbf{a}^a, \mathbf{a}^o)$ is then a feasible solution of problem (11) and, due to positive weights, $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}^r) = 0 < A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}}^r)$ thus contradicting the lexicographic optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$.

Theorem 3. *For any reference levels $r_i^a < r_i^r$, any positive weights $\mathbf{w}$ and $\mathbf{p}$, if $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ is an optimal solution of the problem (11), then all the aspiration level underachievements $\bar{a}_i^a$ are equal 0 whenever there exists a feasible solution $\mathbf{x} \in Q$ such that $f_i(\mathbf{x}) \leq r_i^a$ for all $i \in I$.*

Proof. Let $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ be an optimal solution of the problem (11) with some positive weighting vectors $\mathbf{w}$ and $\mathbf{p}$. Suppose that $\bar{a}_{i_0}^a > 0$ for some $i_0 \in I$ and there exists a feasible solution $\mathbf{x} \in Q$ such that $f_i(\mathbf{x}) \leq r_i^a$ for all $i \in I$. Let us define a_i^r, a_i^a and a_i^o according to formula (4) and note that $a_i^a = a_i^r = 0$ for all $i \in I$. The quadruple $(\mathbf{x}, \mathbf{a}^r, \mathbf{a}^a, \mathbf{a}^o)$ is then a feasible solution of problem (11) and, due to positive weights, $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}^a) = 0 < A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}}^a)$ thus contradicting the lexicographic optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$.

In order to show that the WOWA ARBDS model provides us with a complete parameterization of the efficient set, we will prove in the following theorem that for each efficient solution $\bar{\mathbf{x}}$ there exist aspiration and reservation vectors for which $\bar{\mathbf{x}}$ with the corresponding values of the multilevel achievements is an optimal solution of problem (11).

Theorem 4. *If $\bar{\mathbf{x}}$ is an efficient solution of the multiple criteria problem (1), then there exist aspirations levels r_i^a such that the quadruple $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ is an optimal solution of the corresponding problem (11), for any reservation levels $r_i^r > r_i^a$ and positive weighting vectors $\mathbf{w}$ and $\mathbf{p}$.*

Proof. Let us set the aspiration levels as $r_i^a = f_i(\bar{x})$ for $i \in I$. For any reservation levels $r_i^r > r_i^a$, all the corresponding multilevel achievements defined according to formula (4) take the zero values: $\bar{\mathbf{a}}^r = 0$, $\bar{\mathbf{a}}^a = 0$ and $\bar{\mathbf{a}}^o = 0$. Suppose that for some weights the quadruple $(\bar{\mathbf{x}}, 0, 0, 0)$ is not an optimal solution of the corresponding problem (11). This means there exists a vector $\mathbf{x} \in Q$ such that $\mathbf{a}^r = 0$, $\mathbf{a}^a = 0$, $\mathbf{a}^o \geq 0$ and $A_{\mathbf{w},\mathbf{p}}(-\mathbf{a}^o) < A_{\mathbf{w},\mathbf{p}}(-\bar{\mathbf{a}}^o)$. Hence, $f_i(\mathbf{x}) \leq f_i(\bar{\mathbf{x}})\ \forall\, i \in I$ and $f_{i_o}(\mathbf{x}) < f_{i_o}(\bar{\mathbf{x}})$ for some index $i_o \in I$. The latest assertion contradicts the efficiency of $\bar{\mathbf{x}}$ to (1), which completes the proof.

In the proof of Theorem 4 we have used one set of preferential parameters leading to the given solution. Obviously, there are many alternative parameter settings allowing to reach this goal. For instance, one may set the reservation levels as $r_i^r = f_i(\bar{x})$ for $i \in I$ while taking any aspiration levels $r_i^a < r_i^r$.

4 Linear Programming Implementation

An important advantage of the RPM depends on its easy implementation as an expansion of the original MCO problem. Actually, even complicated partial achievement functions of the form (3) are strictly increasing and convex, thus allowing for implementation of the entire RPM model (2) by an LP expansion [7]. The same applies to the WOWA enhanced ARBDS.

Recall that formula (9) defines the WOWA value applying preferential weights w_i to importance weighted averages within quantile intervals. It may be reformulated to use the tail averages

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{k=1}^{m} w'_k m L(\mathbf{a}, \mathbf{p}, \frac{k}{m}), \qquad L(\mathbf{y}, \mathbf{p}, \xi) = \int_0^{\xi} \overline{F}_{\mathbf{y}}^{(-1)}(\alpha) d\alpha \tag{12}$$

where weights $w'_k = w_k - w_{k+1}$ for $k = 1, \ldots, m-1$ and $w'_m = w_m$ and $L(\mathbf{y}, \mathbf{p}, \xi)$ is defined by left-tail integrating of $\overline{F}_{\mathbf{y}}^{(-1)}$.

Values $L(\mathbf{a}, \mathbf{p}, \xi)$ for any $0 \leq \xi \leq 1$ can be given by optimization:

$$L(\mathbf{a}, \mathbf{p}, \xi) = \max_{s_i} \{\sum_{i=1}^{m} a_i s_i : \sum_{i=1}^{m} s_i = \xi, \quad 0 \leq s_i \leq p_i \quad \forall\, i \in I\} \tag{13}$$

Introducing dual variable t corresponding to the equation $\sum_{i=1}^{m} s_i = \xi$ and variables d_i corresponding to upper bounds on s_i one gets the following LP dual expression for $L(\mathbf{a}, \mathbf{p}, \xi)$

$$L(\mathbf{a}, \mathbf{p}, \xi) = \min_{t,d_i} \{\xi t + \sum_{i=1}^{m} p_i d_i : t + d_i \geq a_i,\ d_i \geq 0 \quad \forall\, i \in I\} \tag{14}$$

Following (12) and (14) one gets finally the following model for the WOWA enhanced ARBDS:

$$\begin{array}{lll}
\operatorname{lex}\min\ [\sum_{k=1}^{m} w'_k z^r_k, \sum_{k=1}^{m} w'_k z^a_k, \sum_{k=1}^{m} w'_k z^o_k] & & \\
\quad \text{s.t.}\ \ \mathbf{x} \in Q & & \\
\qquad f_i(\mathbf{x})/(r^r_i - r^a_i) + a^o_i - a^a_i - a^r_i = r^a_i & \forall\, i \in I & \\
\qquad a^o_i \geq 0, \quad 0 \leq a^a_i \leq 1, \quad a^r_i \geq 0 & \forall\, i \in I & \\
\qquad z^r_k = k t^r_k + m \sum_{i=1}^{m} p_i d^r_{ik},\ a^r_i \leq t^r_k + d^r_{ik},\ d^r_{ik} \geq 0 & \forall\, i,k \in I & (15) \\
\qquad z^a_k = k t^a_k + m \sum_{i=1}^{m} p_i d^a_{ik},\ a^a_i \leq t^a_k + d^a_{ik},\ d^a_{ik} \geq 0 & \forall\, i,k \in I & \\
\qquad z^o_k = k t^o_k + m \sum_{i=1}^{m} p_i d^o_{ik},\ -a^o_i \leq t^o_k + d^o_{ik},\ d^o_{ik} \geq 0 & \forall\, i,k \in I &
\end{array}$$

thus allowing for implementation as an LP expansion of the original problem. The following theorem justifies model (15) as an implementation of the WOWA ARBDS approach (11) thus preserving its preference model properties.

Theorem 5. *For any reference levels $r^a_i < r^r_i$, any positive importance weights p_i and positive strictly decreasing weights w_i, if $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ is an optimal solution of the problem (15), then it is an optimal solution of the corresponding problem (11).*

Proof. Let $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ be an optimal solution of the problem (15) with some positive weighting vectors $\mathbf{w}$ and $\mathbf{p}$. Following the WOWA formulas (12) and (14) one may notice that the problem (15) is equivalent to the following lexicographic optimization:

$$\operatorname{lex}\min_{\mathbf{x}} \{(A_{\mathbf{w},\mathbf{p}}(\mathbf{a}^r), A_{\mathbf{w},\mathbf{p}}(\mathbf{a}^a), A_{\mathbf{w},\mathbf{p}}(-\mathbf{a}^o)) : \text{Eq. (6)},\ \mathbf{x} \in Q\} \tag{16}$$

Hence, if $\bar{a}^r_i$, $\bar{a}^a_i$ and $\bar{a}^o_i$ fulfill formula (4) for $\bar{\mathbf{x}}$, then the quadruple $\bar{\mathbf{x}}$ is an optimal solution of the corresponding problem (11). In order to prove that formula (4) is satisfied it is enough to show that $\bar{a}^o_i \bar{a}^a_i = 0$ and $(1 - \bar{a}^a_i)\bar{a}^r_i = 0$.

Suppose that $\bar{a}^o_{i_0} \bar{a}^a_{i_0} > 0$ for some index $i_0 \in I$. One may decrease then values of both variables $\bar{a}^o_{i_0}$ and $\bar{a}^a_{i_0}$ by the same small positive number. This means, for sufficiently small positive number δ the quadruple $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^o - \delta \mathbf{e}_{i_0}, \bar{\mathbf{a}}^a - \delta \mathbf{e}_{i_0}, \bar{\mathbf{a}}^r)$, where $\mathbf{e}_{i_0}$ denotes the unit vector corresponding to index i_0, is feasible to problem (16). Due to positive weights w_i and p_i, one gets $(A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}}^r)$,

$A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}}^a - \delta \mathbf{e}_{i_0}), A_{\mathbf{w},\mathbf{p}}(-\bar{\mathbf{a}}^o + \delta \mathbf{e}_{i_0})) <_{lex} (A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}}^r), A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}}^a), A_{\mathbf{w},\mathbf{p}}(-\bar{\mathbf{a}}^o))$ which contradicts optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ to problem (16) and thereby (15).

Further, suppose that $(1 - \bar{a}^a_{i_0})\bar{a}^r_{i_0} > 0$ for some index $i_0 \in I$. One may decrease then value of variable $\bar{a}^r_{i_0}$ and simultaneously increase $\bar{a}^a_{i_0}$ by the same small positive number. This means, for sufficiently small positive number δ the quadruple $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^-, \bar{\mathbf{a}}^a + \delta \mathbf{e}_{i_0}, \bar{\mathbf{a}}^r - \delta \mathbf{e}_{i_0})$ is feasible to problem (16). Due to positive weights w_i and p_i, one gets $(A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}}^r - \delta \mathbf{e}_{i_0}), A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}}^a + \delta \mathbf{e}_{i_0}), A_{\mathbf{w},\mathbf{p}}(-\bar{\mathbf{a}}^o)) <_{lex} (A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}}^r), A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}}^a), A_{\mathbf{w},\mathbf{p}}(-\bar{\mathbf{a}}^o))$ which contradicts optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ to problem (16) and thereby (15).

Thus $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^o, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^r)$ fulfills formula (4) and therefore it is an optimal solution of the corresponding problem (11).

Corollary 1. *For any reference levels $r^a_i < r^r_i$ any positive importance weights p_i and positive strictly decreasing weights w_i, if $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ is an optimal solution of the problem (15), then $\bar{\mathbf{x}}$ is an efficient solution of the corresponding multi-criteria problem (1).*

Corollary 2. *If $\bar{\mathbf{x}}$ is an efficient solution of the multiple criteria problem (1), then there exist aspirations levels $r^a_i = f_i(\mathbf{x})$ such that $(\bar{\mathbf{x}}, \bar{\mathbf{a}}^r, \bar{\mathbf{a}}^a, \bar{\mathbf{a}}^o)$ is an optimal solution of the corresponding problem (15), for any reservation levels $r^r_i > r^a_i$, any positive importance weights p_i and positive strictly decreasing weights w_i.*

5 Conclusions

The reference point method is a very convenient technique for interactive analysis of the multiple criteria optimization problems. It provides the DM with a tool for an open analysis of the efficient frontier. The interactive analysis is navigated with the commonly accepted control parameters expressing reference levels for the individual objective functions. The partial achievement functions quantify the DM satisfaction from the individual outcomes with respect to the given reference levels. The final scalarizing function is built as the augmented min-max aggregation of partial achievements which means that the worst individual achievement is essentially maximized but the optimization process is additionally regularized with the term representing the average achievement. The regularization by the average achievement is easily implementable but it may disturb the basic max-min aggregation. In order to avoid inconsistencies caused by the regularization, the max-min solution may be regularized with the OWA aggregation combining all the partial achievements by allocating the largest weight to the worst achievement, the second largest weight to the second worst achievement, the third largest weight to the third worst achievement, and so on. Further following the concept of the Weighted OWA [13] the importance weighting of several achievements may be incorporated into the RPM. Such a WOWA enhancement of the RPM uses importance weights to affect achievement importance by rescaling accordingly its measure within the distribution of achievements rather than straightforward rescaling of achievement values [12].

The ordered regularizations are more complicated in implementation due to the requirement of pointwise ordering of partial achievements. However, the recent progress in optimization methods for ordered averages [8] allows one to implement the OWA RPM quite effectively by taking advantages of piecewise linear expression of the cumulated ordered achievements. Similar, model can be achieved for the WOWA enhanced ARBDS. Actually, the resulting formulation extends the original constraints and criteria with simple linear inequalities thus allowing for a quite efficient implementation.

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