On Direct Methods for Lexicographic Min-Max Optimization*

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Abstract. The approach called the Lexicographic Min-Max (LMM) optimization depends on searching for solutions minimal according to the lex-max order on a multidimensional outcome space. LMM is a refinement of the standard Min-Max optimization, but in the former, in addition to the largest outcome, we minimize also the second largest outcome (provided that the largest one remains as small as possible), minimize the third largest (provided that the two largest remain as small as possible), and so on. The necessity of point-wise ordering of outcomes within the lexicographic optimization scheme causes that the LMM problem is hard to implement. For convex problems it is possible to use iterative algorithms solving a sequence of properly defined Min-Max problems by eliminating some blocked outcomes. In general, it may not exist any blocked outcome thus disabling possibility of iterative Min-Max processing. In this paper we analyze two alternative optimization models allowing to form lexicographic sequential procedures for various nonconvex (possibly discrete) LMM problems. Both the approaches are based on sequential optimization of directly defined artificial criteria. The criteria can be introduced into the original model with some auxiliary variables and linear inequalities thus the methods are easily implementable.

1 Lexicographic Min-Max

There are several multiple criteria decision problems where the Pareto-optimal solution concept is not powerful enough to resolve the problem since the equity or fairness among uniform individual outcomes is an important issue [10, 11, 17]. Uniform and equitable outcomes arise in many dynamic programs where individual objective functions represent the same outcome for various periods [9]. In the stochastic problems uniform objectives may represent various possible values of the same nondeterministic outcome ([15] and references therein). Moreover, many modeling techniques for decision problems first introduce some uniform objectives and next consider their impartial aggregations. The most direct models with uniform equitable criteria are related to the optimization of systems

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which serve many users. For instance, efficient and fair way of distribution of network resources among competing demands becomes a key issue in computer networks [5] and the telecommunication networks design, in general [20].

The generic decision problem we consider may be stated as follows. There is given a set I of m clients (users, services). There is also given a set Q of feasible decisions. For each service $j \in I$ a function $f_j(\mathbf{x})$ of the decision $\mathbf{x}$ is defined. This function, called the individual objective function, measures the outcome (effect) $y_j = f_j(\mathbf{x})$ of the decision for client j. An outcome can be measured (modeled) as service time, service costs, service delays as well as in a more subjective way as individual utility (or disutility) level. In typical formulations a smaller value of the outcome means a better effect (higher service quality or client satisfaction). Therefore, without loss of generality, we can assume that each individual outcome y_i is to be minimized.

The Min-Max solution concept depends on optimization of the worst outcome

$$\min_{\mathbf{x}}\{ \max_{j=1,\dots,m} f_j(\mathbf{x}) \ : \ \mathbf{x} \in Q \ \}$$

and it is regarded as maintaining equity. Indeed, for a simplified resource allocation problem $\min\{\max_j y_j \ : \ \sum_j y_j \leq b \ \}$ the Min-Max solution takes the form $\bar{y}_j = b/m$ for all $j \in I$ thus meeting the perfect equity. In the general case with possibly more complex feasible set structure this property is not fulfilled. Actually, the distribution of outcomes may make the Min-Max criterion partially passive when one specific outcome is relatively large for all the solutions. For instance, while allocating clients to service facilities, such a situation may be caused by existence of an isolated client located at a considerable distance from all facilities. Minimization of the maximum distance is then reduced to that single isolated client leaving other allocation decisions unoptimized. This is a clear case of inefficient solution where one may still improve other outcomes while maintaining fairness (equitability) by leaving at its best possible value the worst outcome.

The Min-Max solution may be lexicographically regularized according to the Rawlsian principle of justice [22]. Applying the Rawlsian approach, any two states should be ranked according to the accessibility levels of the least well–off individuals in those states; if the comparison yields a tie, the accessibility levels of the next–least well–off individuals should be considered, and so on. Formalization of this concept leads us to the lexicographic Min-Max optimization. Let $\langle \mathbf{a} \rangle = (a_{\langle 1 \rangle}, a_{\langle 2 \rangle}, \dots, a_{\langle m \rangle})$ denote the vector obtained from $\mathbf{a}$ by rearranging its components in the non-increasing order. That means $a_{\langle 1 \rangle} \geq a_{\langle 2 \rangle} \geq \dots \geq a_{\langle m \rangle}$ and there exists a permutation π of set I such that $a_{\langle i \rangle} = a_{\pi(i)}$ for $i \in I$. Comparing lexicographically such ordered vectors $\langle \mathbf{y} \rangle$ one gets the so-called lex-max order. The general problem we consider depends on searching for the solutions that are minimal according to the lex-max order:

$$\operatorname{lex}\min_{\mathbf{x}} \ \{(\theta_1(\mathbf{f}(\mathbf{x})), \dots, \theta_m(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\} \quad \text{ where } \theta_j(\mathbf{y}) = y_{\langle j \rangle} \tag{1}$$

The lexicographic Min-Max under consideration is related to the problems with outcomes being minimized. Similar consideration of the maximization problems leads to the lexicographic Max-Min solution concept. Obviously, all the results presented further for the lexicographic Min-Max can be adjusted to the lexicographic Max-Min while preserving assumption that the outcomes are ordered from the worst one to the best one.

The lexicographic Min-Max solution is known in game theory as the nucleolus of a matrix game. It originates from an idea [6] to select from the optimal strategy set those which allow one to exploit mistakes of the opponent optimally. It has been later refined to the formal nucleolus definition [21]. The concept was early considered in the Tschebyscheff approximation [23] as a refinement taking into account the second largest deviation, the third one and further to be hierarchically minimized. Similar refinement of the fuzzy set operations has been recently analyzed [7]. Within the telecommunications or network applications the lexicographic Max-Min approach has appeared already in [2] and now under the name Max-Min Fairness (MMF) is treated as one of the standard fairness concepts [16, 20]. The LMM approach has been used for general linear programming multiple criteria problems [1, 12], as well as for specialized problems related to (multiperiod) resource allocation [9, 11].

Note that the lexicographic minimization in the LMM is not applied to any specific order of the original criteria. Nevertheless, in the case of linear programming (LP) problems (or generally convex optimization), there exists a dominating objective function which is constant (blocked) on the entire optimal set of the Min-Max problem [12]. Hence, having solved the Min-Max problem, one may try to identify the blocked objective and eliminate it to formulate a new restricted Min-Max problem on the former optimal set. Therefore, the LMM solution to LP problems can be found by the sequential Min-Max optimization with elimination of the blocked outcomes.

The LMM approach has been considered also for various discrete optimization problems [3, 4, 8] including the location-allocation ones [14]. In discrete models, due to the lack of convexity there may not exist any blocked outcome [13] thus disabling possibility of the sequential Min-Max algorithm. In this paper we analyze capabilities of an effective use of earlier developed ordered cumulated outcomes methodology [17, 18, 19] to solve the LMM problem by sequential optimization of directly defined criteria. We develop and analyze two alternative approaches allowing to form lexicographic sequential procedures for various nonconvex (possibly discrete) LMM problems. Both the approaches are based on criteria directly introduced with some LP expansion of the original model.

2 Direct Models

2.1 Ordered Outcomes

The ordered outcomes $y_{\langle k \rangle}$ used in definition of the LMM solution concept can be expressed with a direct formula, although requiring the use of integer variables [24]. Namely, for any $k = 1, 2, \ldots, m$ the following formula is valid:

$$y_{\langle k \rangle} = \min_{t_k, z_{kj}} \{t_k : t_k - y_j \geq -M z_{kj},\ z_{kj} \in \{0,1\}\ \forall j, \quad \sum_{j=1}^{m} z_{kj} \leq k-1\}$$

where M is a sufficiently large constant (larger than the span of individual outcomes y_j) which allows one to enforce inequality $t_k \geq y_j$ for $z_{kj} = 0$ while ignoring it for $z_{kj} = 1$. Note that for $k = 1$ all binary variables z_{1j} are forced to 0 thus reducing the optimization in this case to the standard LP model. However, for any other $k > 1$ all m binary variables z_{kj} are an important part of the model. Nevertheless, with the use of auxiliary integer variables, any LMM problem (either convex or non-convex) can be formulated as the standard lexicographic minimization with directly given objective functions

$$\begin{aligned} \operatorname{lex} \min_{\mathbf{x}, t_k, z_{kj}} \ (t_1, t_2, \ldots, t_m) \ \text{ s.t. } \ \mathbf{x} \in Q, \quad & \sum_{j=1}^{m} z_{kj} \leq k - 1 \quad \forall\, k \\ & t_k - f_j(\mathbf{x}) \geq -M z_{kj}, \ z_{kj} \in \{0, 1\} \quad \forall\, j, k \end{aligned} \tag{2}$$

Let us consider cumulated criteria $\bar{\theta}_k(\mathbf{y}) = \sum_{i=1}^{k} y_{\langle i \rangle}$ expressing, respectively: the worst (largest) outcome, the total of the two worst outcomes, the total of the three worst outcomes, etc. When normalized by k the quantities $\mu_k(\mathbf{y}) = \bar{\theta}_k(\mathbf{y})/k$ can be interpreted as the worst conditional means [17]. Within the lexicographic optimization a cumulation of criteria does not affect the optimal solution. Hence, the LMM problem can be formulated as the standard lexicographic minimization with cumulated ordered outcomes as objective functions

$$\operatorname{lex} \min_{\mathbf{x}} \ \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}$$

where $\bar{\theta}_k(\mathbf{y}) = \sum_{i=1}^{k} y_{\langle i \rangle}$. This simplifies dramatically the optimization problem since quantities $\bar{\theta}_k(\mathbf{y})$ can be optimized without use of any integer variables. First, let us notice that for any given vector $\mathbf{y}$, the cumulated ordered value $\bar{\theta}_k(\mathbf{y})$ can be found as the optimal value of the following LP problem:

$$\bar{\theta}_k(\mathbf{y}) = \max_{u_{kj}} \ \{\sum_{j=1}^{m} y_j u_{kj} : \sum_{j=1}^{m} u_{kj} = k, \quad 0 \leq u_{kj} \leq 1 \quad \forall\, j \ \} \tag{3}$$

The above problem is an LP for a given outcome vector $\mathbf{y}$ while it becomes nonlinear for $\mathbf{y}$ being a vector of variables. This difficulty can be overcome by taking advantage of the LP dual to (3). Introducing dual variable t_k corresponding to the equation $\sum_{j=1}^{m} u_{kj} = k$ and variables d_{kj} corresponding to upper bounds on u_{kj} one gets the following LP dual of problem (3):

$$\bar{\theta}_k(\mathbf{y}) = \min_{t_k, d_{kj}} \ \{k t_k + \sum_{j=1}^{m} d_{kj} : t_k + d_{kj} \geq y_j, \ d_{kj} \geq 0 \quad \forall\, j\} \tag{4}$$

Due to the duality theory, for any given vector $\mathbf{y}$ the cumulated ordered coefficient $\bar{\theta}_k(\mathbf{y})$ can be found as the optimal value of the above LP problem.

It follows from (4) that $\bar{\theta}_k(\mathbf{f}(\mathbf{x})) = \min \ \{k t_k + \sum_{j=1}^{m} (f_j(\mathbf{x}) - t_k)_+ : \mathbf{x} \in Q \ \}$, where $(.)_+$ denotes the nonnegative part of a number and t_k is an auxiliary (unbounded) variable. This is equivalent to the computational formulation of the

k–centrum model introduced in [19]. Hence, formula (4) provides an alternative proof of that formulation.

Following formula (4), the following assertion is valid for any LMM problem.

Theorem 1. *Every optimal solution to the LMM problem (1) can be found as an optimal solution to a standard lexicographic optimization problem with predefined linear criteria:*

$$\begin{array}{c} \text{lex} \min_{\mathbf{x},t_k,d_{kj}} \; [t_1 + \sum_{j=1}^{m} d_{1j}, 2t_2 + \sum_{j=1}^{m} d_{2j}, \ldots, mt_m + \sum_{j=1}^{m} d_{mj}] \\ s.t. \; \mathbf{x} \in Q, \quad t_k + d_{kj} \geq f_j(\mathbf{x}), \; d_{kj} \geq 0 \quad \forall \, j,k \end{array} \tag{5}$$

This direct lexicographic formulation remains valid for nonconvex (e.g. discrete) feasible sets Q, where the standard sequential approaches [11, 12] are not applicable [14]. Note that model (5) does not use integer variables and it can be considered as an LP expansion of the original Min-Max problem. Thus, this model preserves the problem's convexity if the original problem is defined with a convex feasible set Q and a linear objective functions f_j. The size of the problem is quadratic with respect to the number of outcomes ($m^2 + m$ auxiliary variables and m^2 constraints).

2.2 Ordered Values

For some specific classes of discrete, or rather combinatorial, optimization problems, one may take advantage of the finiteness of the set of all possible values of functions f_j on the finite set of feasible solutions. The ordered outcome vectors may be treated as describing a distribution of outcomes generated by a given decision $\mathbf{x}$. In the case when there exists a finite set of all possible outcomes of the individual objective functions (or the set of outcome values can be restricted to its finite approximation, i.e. with fuzzy modeling), one can directly describe the distribution of outcomes with frequencies of outcomes. Let $V = \{v_1, v_2, \ldots, v_r\}$ (where $v_1 > v_2 > \ldots > v_r$) denote the set of all attainable outcomes (all possible values of the individual objective functions f_j for $\mathbf{x} \in Q$). We introduce integer functions $h_k(\mathbf{y})$ ($k = 1, 2, \ldots, r$) expressing the number of values v_k in the outcome vector $\mathbf{y}$. Having defined functions h_k we can introduce cumulative distribution functions $\bar{h}_k(\mathbf{y}) = \sum_{l=1}^{k} h_l(\mathbf{y})$ where $\bar{h}_r(\mathbf{y}) = m$ for any outcome vector. Function $\bar{h}_k$ expresses the number of outcomes larger or equal to v_k. Since we want to minimize all the outcomes, we are interested in the minimization of all functions $\bar{h}_k$ for $k = 1, 2, \ldots, r-1$. Indeed, the LMM solution concept can be expressed in terms of the standard lexicographic minimization problem with objectives $\bar{h}_k(\mathbf{f}(\mathbf{x}))$ [13].

Theorem 2. *In the case of finite outcome set $\mathbf{f}(Q) = V^m$, the LMM problem (1) is equivalent to the standard lexicographic optimization problem with $r-1$ criteria:*

$$\text{lex} \min_{\mathbf{x}} \; \{(\bar{h}_1(\mathbf{f}(\mathbf{x})), \ldots, \bar{h}_{r-1}(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\} \tag{6}$$

Unfortunately, for functions $\bar{h}_k$ there is no simple analytical formula allowing to minimize them without use of some auxiliary integer variables. This difficulty can be overcome by taking advantages of possible weighting and cumulating criteria in lexicographic optimization. Namely, for any positive weights w_i the lexicographic optimization

$$\text{lex}\min_{\mathbf{x}} \{(w_1\bar{h}_1(\mathbf{f}(\mathbf{x})), w_1\bar{h}_1(\mathbf{f}(\mathbf{x})) + w_2\bar{h}_2(\mathbf{f}(\mathbf{x})), \ldots, \sum_{i=1}^{r-1} w_l\bar{h}_i(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}$$

is equivalent to (6). Let us cumulate vector $\bar{\mathbf{h}}(\mathbf{y})$ weights $w_i = v_i - v_{i+1}$ to get

$$\hat{h}_k(\mathbf{y}) = \sum_{i=1}^{k-1}(v_i - v_{i+1})\bar{h}_i(\mathbf{y}) = \sum_{i=1}^{k-1}(v_i - v_k)h_i(\mathbf{y}) = \sum_{j=1}^{m}(y_j - v_k)_+$$

where quantities $\hat{h}_k(\mathbf{f}(\mathbf{y}))$ for $k = 2, 3, \ldots, r$ represent the total exceed of outcomes over the corresponding values v_k. Due to the use of positive weights $w_i > 0$, the lexicographic problem (6) is equivalent to the lexicographic minimization

$$\text{lex}\min_{\mathbf{x}} \{(\hat{h}_2(\mathbf{f}(\mathbf{x})), \ldots, \hat{h}_r(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}$$

Moreover, criteria defined this way are piecewise linear convex functions [13] which allow to compute them directly by the minimization:

$$\hat{h}_k(\mathbf{y}) = \min_{v_k, h_{kj}} \{\sum_{j=1}^{m} h_{kj} : h_{kj} \geq y_j - v_k,\ h_{kj} \geq 0\ \forall\ j\}$$

Therefore, the following assertion is valid for any LMM problem.

Theorem 3. *In the case of finite outcome set* $\mathbf{f}(Q) = V^m$*, every optimal solution to the LMM problem (1) can be found as an optimal solution to a standard lexicographic optimization problem with predefined linear criteria:*

$$\begin{aligned} \text{lex} \min_{\mathbf{x}, v_k, h_{kj}} \ & [\sum_{j=1}^{m} h_{2j}, \sum_{j=1}^{m} h_{3j}, \ldots, \sum_{j=1}^{m} h_{rj}] \\ s.t.\ & \mathbf{x} \in Q, \quad h_{kj} \geq f_j(\mathbf{x}) - v_k,\ h_{kj} \geq 0 \quad \forall\ j, k \end{aligned} \tag{7}$$

Formulation (7) does not use integer variables and can be considered as an LP expansion of the original problem. Thus, this model preserves the problem's convexity if the original problem is defined with a convex feasible set Q and objective functions f_j. The size of the problem depends on the number of different outcome values. For many models with not too large number of outcome values, the problem can easily be solved directly and even for convex problems such an approach may be more efficient than the sequential algorithms. Note that in many problems of systems optimization the objective functions express the quality of service and one can easily consider a limited finite scale (grid) of the corresponding outcome values (possibly fuzzy values).

3 Computational Experiments

We have run initial tests to analyze the computational performances of both direct models for the LMM problem. For this purpose we have solved randomly generated (discrete) location problems defined as follows. There is given a set of m clients. There is also given a set of n potential locations for the facilities, in particular, we considered all points representing the clients as valid potential locations ($n = m$). Further, the number (or the maximal number) p of facilities to be located is given ($p \leq n$). The main decisions to be made in the location problem can be described with the binary variables: x_i equal to 1 if location i is to be used and equal to 0 otherwise ($i = 1, 2, \ldots, n$). The allocation decisions are modeled with the additional allocation variables: x'_{ij} equal to 1 if location i is used to service client j and equal to 0 otherwise ($i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$).

$$\begin{array}{lll} \sum_{i=1}^{n} x_i = p, \quad \sum_{i=1}^{n} x'_{ij} = 1 & & j = 1, 2, \ldots, m \\ x'_{ij} \leq x_j, \quad x_j, x'_{ij} \in \{0, 1\} & & i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m \end{array} \tag{8}$$

For each client j a function $f_j(\mathbf{x})$ of the location pattern $\mathbf{x}$ has been defined to measure the outcome (effect) of the location pattern for client j. Individual objective functions f_j depend on effects of several allocations decisions. It means they depend on allocation effect coefficients $d_{ij} > 0$ ($i = 1, \ldots, m; j = 1, \ldots, n$), called hereafter simply distances as they usually express the distance (or travel time) between location i and client j. For the standard uncapacitated location problem it is assumed that all the potential facilities provide the same type of service and each client is serviced by the nearest located facility. With the explicit use of the allocation variables and the corresponding constraints the individual objective functions f_j can be written in the linear form: $f_j(\mathbf{x}) = \sum_{i=1}^{n} d_{ij} x'_{ij}$. There should be found the location pattern $\mathbf{x}$ lexicographically minimaximizing the vector of individual objective functions $(f_j(\mathbf{x}))_{j=1,\ldots,m}$.

For the tests we used two-dimensional discrete location problems. The locations of the clients were defined by coordinates being the multiple of 5 generated as random numbers uniformly distributed in the interval [0, 100]. In the computations we used rectilinear distances. We tested solution times for different size

Table 1. Computation times (in seconds) for the ordered outcomes approach

number of	number of facilities (p)						
clients (m)	1	2	3	5	7	10	15
2	0.1						
5	0.0	0.0	0.1				
10	0.7	1.5	1.4	0.9	0.4		
15	4.7	15.2	14.6	10.8	6.5	4.4	
20	13.2	54.0	118.6	84.7	60.9	26.9	11.0
25	36.6	–	–	–	–	–	67.4

parameters m and p. All the experiments were performed on the PC computer with Pentium 4, 1.7 Ghz processor employing the CPLEX 9.1 package.

Tables 1 and 2 present solution times for two approaches being analyzed. The times are the averages of 10 randomly generated problems. The empty cell (minus sign) shows that the timeout of 120 seconds occurred. One can see fast growing times for the ordered outcomes approach with increasing number of clients (criteria). The growth is faster than the corresponding growth of the

Table 2. Computation times (in seconds) for the ordered values approach

number of clients (m)	number of facilities (p) 1	2	3	5	7	10	15
2	0.0						
5	0.1	0.0	0.0				
10	0.2	0.1	0.1	0.0	0.0		
15	0.7	0.7	0.3	0.1	0.1	0.0	
20	1.7	2.5	2.6	1.2	0.3	0.1	0.1
25	3.4	8.8	2.7	0.8	1.7	0.4	0.2

Table 3. Computation times (in seconds) of algorithms steps ($m = 20$)

step number	ordered outcomes approach $p = 1$	$p = 3$	$p = 5$	ordered values approach $p = 1$	$p = 3$	$p = 5$
1	0.8	3.5	3.2	0.0	0.0	0.1
2	2.0	5.2	4.4	0.1	0.1	0.0
3	1.2	4.4	4.1	0.1	0.1	0.2
4	1.1	4.5	4.4	0.1	0.2	0.1
5	0.8	4.3	4.3	0.0	0.2	0.3
6	0.9	4.7	4.7	0.1	0.2	0.2
7	0.8	5.1	4.9	0.1	0.4	0.2
8	0.7	5.9	5.6	0.0	0.3	0.1
9	0.8	5.9	5.0	0.1	0.3	0.0
10	0.7	12.5	5.5	0.0	0.3	
11	0.9	7.7	6.5	0.1	0.3	
12	0.7	6.0	6.2	0.0	0.1	
13	0.4	6.8	5.7	0.1	0.1	
14	0.3	7.8	5.0	0.0	0.0	
15	0.3	8.1	4.4	0.0		
16	0.2	6.8	3.3	0.1		
17	0.2	6.6	2.8	0.0		
18	0.2	3.7	1.8	0.0		
19	0.1	6.3	1.5	0.1		
20	0.1	2.8	1.4	0.0		
21 – 30				0.6		
31 – 34				0.0		

problem sizes. Actually, it turns out that the solution times for the ordered outcomes model (5) are not significantly better (and in some instances even worse) that those for model (2), despite the latter uses auxiliary integer variables. On the other hand, the ordered values approach performs very well particularly with the number of the clients increasing. In fact, in this approach the number of steps depends not on the number of clients but on the number of different values of distances and this is constant. Moreover, the ordered values approach requires less steps for bigger number of facilities. This is due to the fact that the largest distance in the experiments does not exceed $200/p$.

Table 3 shows how introduction of the auxiliary constraints affects the performance in consecutive steps of the algorithms. One can notice that the ordered values technique generates MIP problems that are solved below 0.3 sec. (below 0.1 for $p = 1$) while the ordered outcomes problems require much longer computations. Despite a similar structure of auxiliary constraints in both approaches, the ordered values problems are much easier to solve. This property additionally contributes to the overall outperformance of the former and supports the attractiveness of the ordered values algorithm even for problems with the large number of different outcome values.

4 Concluding Remarks

The point-wise ordering of outcomes causes that the Lexicographic Min-Max optimization problem is, in general, hard to implement. We have analyzed optimization models allowing to form lexicographic sequential procedures for various nonconvex (possibly discrete) LMM optimization problems. Two approaches based on some LP expansion of the original model remain relatively simple for implementation independently of the problem structure. However, the ordered outcomes model performs similarly to the classical model with integer variables used to implement ordering and it is clearly outperfomed by the ordered values approach. Further work on specialized algorithms (including heuristics) for the ordered values approach to various classes of discrete optimization problems seems to be a very promising research direction.

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