Bicriteria Models for Fair and Efficient Resource Allocation*

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Abstract. Resource allocation problems are concerned with the allocation of limited resources among competing agents so as to achieve the best system performances. In systems which serve many users, like in networking, there is a need to respect some fairness rules while looking for the overall efficiency. The so-called Max-Min Fairness is widely used to meet these goals. However, allocating the resource to optimize the worst performance may cause a dramatic worsening of the overall system efficiency. Therefore, several other fair allocation schemes are searched and analyzed. In this paper we show how the scalar inequality measures can be consistently used in bicriteria models to search for fair and efficient allocations while taking into account importance weighting of the agents.

1 Introduction

Resource allocation problems are concerned with the allocation of limited resources among competing activities [12]. In this paper, we focus on approaches that, while allocating resources to maximize the system efficiency, they also attempt to provide a fair treatment of all the competing agents (activities) [19,29]. The problems of efficient and fair resource allocation arise in various systems which serve many users, like in telecommunication systems among others. In networking a central issue is how to allocate bandwidth to flows efficiently and fairly [3,6,9,15,32,31,34]. In location analysis of public services, the decisions often concern the placement of a service center or another facility in a position so that the users are treated fairly in an equitable way, relative to certain criteria [25].

The generic resource allocation problem may be stated as follows. Each activity is measured by an individual performance function that depends on the corresponding resource level assigned to that activity. A larger function value is considered better, like the performance measured in terms of quality level, capacity, service amount available, etc. Hence, it may be viewed as a multiagent optimization problem. It covers various complex resource allocation problem like network dimensioning as well as general many to many multiagent assignment

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problems with the special case of the Santa Claus problem [2] representing fair allocation of indivisible goods. Such problems occur in many contexts like paper assignment problem [11,18], social meeting on the web or transportation problems [24], among others.

Models with an (aggregated) objective function that maximizes the mean (or simply the sum) of individual performances are widely used to formulate resource allocation problems, thus defining the so-called mean solution concept. This solution concept is primarily concerned with the overall system efficiency. As based on averaging, it often provides solution where some smaller services are discriminated in terms of allocated resources. An alternative approach depends on the so-called Max-Min solution concept, where the worst performance is maximized. The Max-Min approach is consistent with Rawlsian [36] theory of justice, especially when additionally regularized with the lexicographic order. The latter is called the Max-Min Fairness (MMF) and commonly used in networking [34]. Allocating the resources to optimize the worst performances may cause, however, a large worsening of the overall (mean) performances. Moreover, the MMF approach does not allow us to reflect any importance weighting of agents. Therefore, there is a need to seek a compromise between the two extreme approaches discussed above.

Fairness is, essentially, an abstract socio-political concept that implies impartiality, justice and equity [35,41], Nevertheless, fairness was frequently quantified with the so-called inequality measures to be minimized [1,37,39]. Unfortunately, direct minimization of typical inequality measures contradicts the maximization of individual outcomes and it may lead to inferior decisions. The concept of fairness has been studied in various areas beginning from political economics problems of fair allocation of consumption bundles [8,33,35] to abstract mathematical formulation [40]. In order to ensure fairness in a system, all system entities have to be equally well provided with the system's services. This leads to concepts of fairness expressed by the equitable efficiency [16,31]. The concept of equitably efficient solution is a specific refinement of the Pareto-optimality taking into account the inequality minimization according to the Pigou-Dalton approach. In this paper the use of scalar inequality measures in bicriteria models to search for fair and efficient allocations is analyzed. There is shown that properties of convexity and positive homogeneity together with some boundedness condition are sufficient for a typical inequality measure to guarantee that it can be used consistently with the equitable optimization rules.

The paper is organized as follows. In the next section we introduce the fairness notion based on the equitable optimization with the preference structure that complies with both the efficiency (Pareto-optimality) principle and with the Pigou-Dalton principle of transfers. It is additionally extended in Section 3 to the problems with importance weighted agents. In Section 4 the basic inequality measures are discussed and the fair consistency concepts based on the underachievement criteria are introduced. Further, in Section 5, the equitable consistency of the underachievement criteria is analyzed and sufficient conditions for the inequality measures to keep this consistency property are introduced. We verify the properties for the basic inequality measures.

2 Equity and Fairness

The generic resource allocation problem may be stated as follows. There is a system dealing with a set I of m services. There is given a measure of services realization within a system. In applications we consider, the measure usually expresses the service quality. In general, outcomes can be measured (modeled) as service time, service costs, service delays as well as in a more subjective way. There is also given a set Q of allocation patterns (allocation decisions). For each service $i \in I$ a function $f_i(\mathbf{x})$ of the allocation pattern $\mathbf{x} \in Q$ has been defined. This function, called the individual objective function, measures the outcome (effect) $y_i = f_i(\mathbf{x})$ of allocation $\mathbf{x}$ pattern for service i. In typical formulations a larger value of the outcome means a better effect (higher service quality or client satisfaction). Otherwise, the outcomes can be replaced with their complements to some large number. Therefore, without loss of generality, we can assume that each individual outcome y_i is to be maximized which allows us to view the generic resource allocation problem as a vector maximization model:

$$\max \{\mathbf{f}(\mathbf{x}) \ : \ \mathbf{x} \in Q\} \tag{1}$$

where $\mathbf{f}(\mathbf{x})$ is a vector-function that maps the decision space $X = R^n$ into the criterion space $Y = R^m$, and $Q \subset X$ denotes the feasible set. We do not assume any special form of the feasible set Q while analyzing properties of the solution concepts. We rather allow the feasible set to be a general, possibly discrete (nonconvex), set. Hence, the problem cover various complex resource allocation problem like network dimensioning as well as general many to many multiagent assignment problems with the special case of the Santa Claus problem [2] representing fair allocation of indivisible goods. Although we allow the feasible set to contain more complex relations than the basic assignment constraints, like in problems of network resource allocation [34].

Model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. In order to make it operational, one needs to assume some solution concept specifying what it means to maximize multiple objective functions. The solution concepts may be defined by properties of the corresponding preference model. The preference model is completely characterized by the relation of weak preference, denoted hereafter with $\succeq$. The corresponding relations of strict preference $\succ$ and indifference $\cong$ are defined then by the following formulas:

$$\mathbf{y}' \succ \mathbf{y}'' \Leftrightarrow (\mathbf{y}' \succeq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \not\succeq \mathbf{y}'),$$
$$\mathbf{y}' \cong \mathbf{y}'' \Leftrightarrow (\mathbf{y}' \succeq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \succeq \mathbf{y}').$$

The standard preference model related to the Pareto-optimal (efficient) solution concept assumes that the preference relation $\succeq$ is *reflexive*:

$$\mathbf{y} \succeq \mathbf{y}, \tag{2}$$

transitive:

$$(\mathbf{y}' \succeq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \succeq \mathbf{y}''') \quad \Rightarrow \quad \mathbf{y}' \succeq \mathbf{y}''', \tag{3}$$

and *strictly monotonic*:

$$\mathbf{y} + \varepsilon \mathbf{e}_i \succ \mathbf{y} \quad \text{for } \varepsilon > 0;\ i = 1, \ldots, m, \tag{4}$$

where $\mathbf{e}_i$ denotes the i–th unit vector in the criterion space. The last assumption expresses that for each individual objective function more is better (maximization). The preference relations satisfying axioms (2)–(4) are called hereafter *rational preference relations*. The rational preference relations allow us to formalize the Pareto-optimality (efficiency) concept with the following definitions. We say that outcome vector $\mathbf{y}'$ rationally dominates $\mathbf{y}''$ ($\mathbf{y}' \succ_r \mathbf{y}''$), iff $\mathbf{y}' \succ \mathbf{y}''$ for all rational preference relations $\succeq$. We say that feasible solution $\mathbf{x} \in Q$ is a *Pareto-optimal (efficient)* solution of the multiple criteria problem (1), iff $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is rationally nondominated.

Simple solution concepts for multiple criteria problems are defined by aggregation (or utility) functions $g : Y \to R$ to be maximized. Thus the multiple criteria problem (1) is replaced with the maximization problem

$$\max \{g(\mathbf{f}(\mathbf{x})) \ : \ \mathbf{x} \in Q\} \tag{5}$$

In order to guarantee the consistency of the aggregated problem (5) with the maximization of all individual objective functions in the original multiple criteria problem (or Pareto-optimality of the solution), the aggregation function must be strictly increasing with respect to every coordinate.

The simplest aggregation functions commonly used for the multiple criteria problem (1) are defined as the mean (average) outcome

$$\mu(\mathbf{y}) = \frac{1}{m} \sum_{i=1}^{m} y_i \tag{6}$$

or the worst outcome

$$M(\mathbf{y}) = \min_{i=1,\ldots,m} y_i. \tag{7}$$

The mean (6) is a strictly increasing function while the minimum (7) is only nondecreasing. Therefore, the aggregation (5) using the sum of outcomes always generates a Pareto-optimal solution while the maximization of the worst outcome may need some additional refinement. The mean outcome maximization is primarily concerned with the overall system efficiency. As based on averaging, it often provides a solution where some services are discriminated in terms of performances. On the other hand, the worst outcome maximization, ie, the so-called Max-Min solution concept is regarded as maintaining equity. Indeed, in the case of a simplified resource allocation problem with the knapsack constraints, the Max-Min solution meets the perfect equity requirement. In the general case, with possibly more complex feasible set structure, this property is not fulfilled. Nevertheless, if the perfectly equilibrated outcome vector $\bar{y}_1 = \bar{y}_2 = \ldots = \bar{y}_m$

is nondominated, then it is the unique optimal solution of the corresponding Max-Min optimization problem [26]. In other words, the perfectly equilibrated outcome vector is a unique optimal solution of the Max-Min problem if one cannot find any (possibly not equilibrated) vector with improved at least one individual outcome without worsening any others. Unfortunately, it is not a common case and, in general, the optimal set to the Max-Min aggregation may contain numerous alternative solutions including dominated ones. The Max-Min solution may be then regularized according to the Rawlsian principle of justice [36] which leads us to the lexicographic Max-Min concepts or the so-called Max-Min Fairness [13,20,30,4]. Although they are possible alternative refinements of the Max-Min ordering [10].

In order to ensure fairness in a system, all system entities have to be equally well provided with the system's services. This leads to concepts of fairness expressed by the equitable rational preferences [16]. First of all, the fairness requires impartiality of evaluation, thus focusing on the distribution of outcome values while ignoring their ordering. That means, in the multiple criteria problem (1) we are interested in a set of outcome values without taking into account which outcome is taking a specific value. Hence, we assume that the preference model is impartial (anonymous, symmetric). In terms of the preference relation it may be written as the following axiom

$$(y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(m)}) \cong (y_1, y_2, \ldots, y_m) \quad \text{for any permutation } \pi \text{ of } I \tag{8}$$

which means that any permuted outcome vector is indifferent in terms of the preference relation. Further, fairness requires equitability of outcomes which causes that the preference model should satisfy the (Pigou–Dalton) principle of transfers. The principle of transfers states that a transfer of any small amount from an outcome to any other relatively worse–off outcome results in a more preferred outcome vector. As a property of the preference relation, the principle of transfers takes the form of the following axiom

$$y_{i'} > y_{i''} \quad \Rightarrow \quad \mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''} \succ \mathbf{y} \quad \text{for } 0 < \varepsilon \leq (y_{i'} - y_{i''})/2 \tag{9}$$

The rational preference relations satisfying additionally axioms (8) and (9) are called hereafter *fair (equitable) rational preference relations*. We say that outcome vector $\mathbf{y}'$ *fairly (equitably) dominates* $\mathbf{y}''$ ($\mathbf{y}' \succ_e \mathbf{y}''$), iff $\mathbf{y}' \succ \mathbf{y}''$ for all fair rational preference relations $\succeq$. In other words, $\mathbf{y}'$ fairly dominates $\mathbf{y}''$, if there exists a finite sequence of vectors $\mathbf{y}^j$ ($j = 1, 2, \ldots, s$) such that $\mathbf{y}^1 = \mathbf{y}''$, $\mathbf{y}^s = \mathbf{y}'$ and $\mathbf{y}^j$ is constructed from $\mathbf{y}^{j-1}$ by application of either permutation of coordinates, equitable transfer, or increase of a coordinate. An allocation pattern $\mathbf{x} \in Q$ is called *fairly (equitably) efficient* or simply *fair* if $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is fairly nondominated. Note that each fairly efficient solution is also Pareto-optimal, but not vice verse.

In order to guarantee fairness of the solution concept (5), additional requirements on the class of aggregation (utility) functions must be introduced. In particular, the aggregation function must be additionally symmetric (impartial), i.e. for any permutation π of I,

$$g(y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(m)}) = g(y_1, y_2, \ldots, y_m) \tag{10}$$

as well as be equitable (to satisfy the principle of transfers)

$$g(y_1, \ldots, y_{i'} - \varepsilon, \ldots, y_{i''} + \varepsilon, \ldots, y_m) > g(y_1, y_2, \ldots, y_m) \tag{11}$$

for any $0 < \varepsilon \leq (y_{i'} - y_{i''})/2$. In the case of a strictly increasing function satisfying both the requirements (10) and (11), we call the corresponding problem (5) a *fair (equitable) aggregation* of problem (1). Every optimal solution to the fair aggregation (5) of a multiple criteria problem (1) defines some fair (equitable) solution.

Note that both the simplest aggregation functions, the sum (6) and the minimum (7), are symmetric although they do not satisfy the equitability requirement (11). To guarantee the fairness of solutions, some enforcement of concave properties is required. For any strictly concave, increasing utility function $u : R \rightarrow R$, the function $g(\mathbf{y}) = \sum_{i=1}^{m} u(y_i)$ is a strictly monotonic and equitable thus defining a family of the fair aggregations. Various concave utility functions u can be used to define such fair solution concepts. In the case of the outcomes restricted to positive values, one may use logarithmic function thus resulting in the *Proportional Fairness* (PF) solution concept [14]. Actually, it corresponds to the so-called Nash criterion [23] which maximizes the product of additional utilities compared to the status quo. For a common case of upper bounded outcomes $y_i \leq y^*$ one may maximize power functions $-\sum_{i=1}^{m} (y^* - y_i)^p$ for $1 < p < \infty$ which corresponds to the minimization of the corresponding p-norm distances from the common upper bound y^* [17].

Fig. 1 presents the structure of fair dominance for two-dimensional outcome vectors. For any outcome vector $\bar{\mathbf{y}}$, the fair dominance relation distinguishes set $D(\bar{\mathbf{y}})$ of dominated outcomes (obviously worse for all fair rational preferences) and set $S(\bar{\mathbf{y}})$ of dominating outcomes (obviously better for all fair rational preferences). However, some outcome vectors are left (in white areas) and they can be differently classified by various specific fair rational preferences. The MMF fairness assigns the entire interior of the inner white triangle to the set of preferred outcomes while classifying the interior of the external open triangles as worse outcomes. Isolines of various utility functions $u(\mathbf{y}) = u(\bar{\mathbf{y}})$ may split the white areas in different ways. For instance, there is no fair dominance between vectors $(0.01, 1)$ and $(0.02, 0.02)$ and the MMF considers the latter as better while the proportional fairness points out the former. On the other hand, vector $(0.02, 0.99)$ fairly dominates $(0.01, 1)$ and all fairness models (including MMF and PF) prefers the former. One may notice that the set $D(\bar{\mathbf{y}})$ of directions leading to outcome vectors being dominated by a given $\bar{\mathbf{y}}$ is, in general, not a cone and it is not convex. Although, when we consider the set $S(\bar{\mathbf{y}})$ of directions leading to outcome vectors dominating given $\bar{\mathbf{y}}$ we get a convex set.

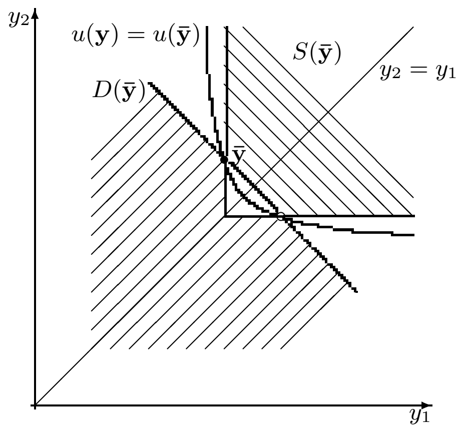

Fig. 1. Structure of the fair dominance: $D(\bar{\mathbf{y}})$ – the set fairly dominated by $\bar{\mathbf{y}}$, $S(\bar{\mathbf{y}})$ – the set of outcomes fairly dominating $\bar{\mathbf{y}}$

3 Fairness with Importance Weighted Agents

Frequently, one may be interested in putting into allocation models some additional agent weights $v_i > 0$. Typically the model of distribution weights is introduced to represent the agent importance thus defining distribution of outcomes $y_i = f_i(\mathbf{x})$ according to measures defined by the weights v_i for $i = 1, \ldots, m$. Note that such distribution weights allow us for a clear interpretation of weights as the agent repetitions [7]. Splitting an agent into two agents does not cause any change of the final distribution of outcomes. For theoretical considerations one may assume that the problem is transformed (disaggregated) to the unweighted one (that means all the agent weights are equal to 1). Note that such a disaggregation is possible for integer as well as rational agent weights, but it usually dramatically increases the problem size. Therefore, we are interested in solution concepts which can be applied directly to the weighted problem.

As mentioned, for some theoretical considerations it might be convenient to disaggregate the weighted problems into the unweighted one. Therefore, to simplify the analysis we will assume integer weights v_i, although while discussing solution concepts we will use the normalized agent weights $\bar{v}_i = v_i / \sum_{i=1}^{m} v_i$ for $i = 1, \ldots, m$, rather than the original quantities v_i. Note that, in the case of unweighted problem (all $v_i = 1$), all the normalized weights are given as $\bar{v}_i = 1/m$. Furthermore, to avoid possible misunderstandings between the weighted outcomes and the corresponding unweighted form of outcomes we will use the following notation. Index set I will always denote unweighted agents (with possible repetitions if originally weighted) and vector $\mathbf{y} = (y_i)_{i \in I} = (y_1, y_2, \ldots, y_m)$ will denote the unweighted outcomes. While directly dealing with the weighted problem (without its disaggregation to the unweighted one) we will use I_v to denote the set of agents and the corresponding outcomes will be represented by vector $\mathbf{y} = (y_{v_i})_{i \in I_v}$. We illustrate this with the following small example.

Let us consider a weighted resource allocation problem with two agents A1 and A2 having assigned demand weights $v_1 = 1$ and $v_2 = 9$, respectively. Their outcomes relate to two potential allocation patterns P1 and P2 are given as follows:

	A1	A2
P1	10	0
P2	0	0

Hence, $I_v = \{1, 2\}$ and the potential resource allocations generate two outcome vectors $\mathbf{y}' = (10_1, 0_9)$ and $\mathbf{y}'' = (0_1, 0_9)$, respectively. The demand weights are understood as agents repetitions. Thus, the problem is understood as equivalent to the unweighted problem with 10 agents ($I = \{1, 2, \ldots, 10\}$) where the first one corresponds to A1 and the further nine unweighted agents correspond to single agent A2. In this disaggregated form, the outcome vectors generated by allocation patterns P1 and P2 are given as $\mathbf{y}' = (10, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ and $\mathbf{y}'' = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$, respectively. Note that outcome vector $\mathbf{y}''$ with all the coordinates 0 is obviously worse than unequal vector $\mathbf{y}'$ with one distance reduced to 0. Actually, $\mathbf{y}'$ Pareto dominates $\mathbf{y}''$.

The classical solution concepts of mean and Max-Min are well defined for aggregated models using importance weights $v_i > 0$. Exactly, the Max-Min solution concept is defined by maximization of the minimum outcome

$$M(\mathbf{y}) = \max_{i \in I} y_i = \max_{i \in I_v} y_{v_i}, \tag{12}$$

thus not affected by the importance weights at all. The same applies to its lexicographic regularization expressed as the MMF concept.

The solution concept of the mean outcome (6) can easily accommodate the importance weights as

$$\mu(\mathbf{y}) = \frac{1}{m} \sum_{i \in I} y_i = \sum_{i \in I_v} \bar{v}_i y_{v_i}. \tag{13}$$

Similarly, for any utility function $u : R \rightarrow R$ we get

$$\mu(u(\mathbf{y})) = \frac{1}{m} \sum_{i \in I} u(y_i) = \sum_{i \in I_v} \bar{v}_i u(y_{v_i}). \tag{14}$$

The fair dominance for general weighted problems can be derived by their disaggregation to the unweighted ones. It can be mathematically formalized as follows. First, we introduce the right-continuous cumulative distribution function (cdf):

$$F_{\mathbf{y}}(d) = \sum_{i \in I_v} \bar{v}_i \delta_i(d), \quad \delta_i(d) = \begin{cases} 1 & \text{if } y_{v_i} \leq d \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

which for any real (outcome) value d provides the measure of outcomes smaller or equal to d. Next, we introduce the quantile function $F_{\mathbf{y}}^{(-1)}$ as the left-continuous inverse of the cumulative distribution function $F_{\mathbf{y}}$:

$$F_{\mathbf{y}}^{(-1)}(\beta) = \inf \{\eta : F_{\mathbf{y}}(\eta) \geq \beta\} \quad \text{for } 0 < \beta \leq 1.$$

By integrating $F_{\mathbf{y}}^{(-1)}$ one gets $F_{\mathbf{y}}^{(-2)}(0) = 0$ and

$$F_{\mathbf{y}}^{(-2)}(\beta) = \int_0^\beta F_{\mathbf{y}}^{(-1)}(\alpha) d\alpha \quad \forall\ 0 < \beta \leq 1, \tag{16}$$

where $F_{\mathbf{y}}^{(-2)}(1) = \mu(\mathbf{y})$. The graph of function $F_{\mathbf{y}}^{(-2)}(\beta)$ (with respect to β) take the form of concave curves. It is called Absolute Lorenz Curve (ALC) [28], due to its relation to the classical Lorenz curve used in income economics as a cumulative population versus income curve to compare equity of income distributions. Indeed, the Lorenz curve may be viewed as function $LC(\xi) = \frac{1}{\mu(\mathbf{y})} \int_0^\xi F_{\mathbf{y}}^{(-1)}(\alpha) d\alpha$ thus equivalent to function $F_{\mathbf{y}}^{(-2)}(\beta)$ normalized by the distribution average. Therefore, the classical Lorenz model is focused on equity while ignoring the average result and any perfectly equal distribution of income has the diagonal line as the Lorenz curve (the same independently from the income value). Within the ALC model both equity and values of outcomes are represented. The ALC defines the relation (partial order) equivalent to the equitable dominance. Exactly, outcome vector $\mathbf{y}'$ equitably dominates $\mathbf{y}''$, if and only if $F_{\mathbf{y}'}^{(-2)}(\beta) \geq F_{\mathbf{y}''}^{(-2)}(\beta)$ for all $\beta \in (0, 1]$ where at least one strict inequality holds. Note that for the expanded form to the unweighted outcomes, the ALC is completely defined by the values of the (cumulated) ordered outcomes. Hence, $\bar{\theta}_i(\mathbf{y}) = mF_{\mathbf{y}}^{(-2)}(i/m)$ for $i = 1, \ldots, m$, and pointwise comparison of cumulated ordered outcomes is enough to justify equitable dominance.

Finally, the impartiality of the allocation process (8) is considered in terms that two allocation schemes leading to the same distribution (cdf) of outcomes are indifferent

$$F_{\mathbf{y}'} = F_{\mathbf{y}''} \quad \Rightarrow \quad \mathbf{y}' \cong \mathbf{y}''. \tag{17}$$

The principle of transfers (9) is considered for single units of demand. Although it can can be applied directly to the outcomes of importance weighted agents in the following form [27]:

$$y_{v_{i'}} > y_{v_{i''}} \quad \Rightarrow \quad \mathbf{y} - \varepsilon \bar{v}_{i''} \mathbf{e}_{v_{i'}} + \varepsilon \bar{v}_{i'} \mathbf{e}_{v_{i''}} \succ \mathbf{y} \quad \text{for } 0 < \varepsilon \leq \frac{y_{i'} - y_{i''}}{\bar{v}_{i'} + \bar{v}_{i''}}. \tag{18}$$

Alternatively, the fair dominance can be expressed on the cumulative distribution functions. Having introduced the right-continuous cumulative distribution function one may further integrate the cdf (15) to get the second order cumulative distribution function $F_{\mathbf{y}}^{(2)}(\eta) = \int_{-\infty}^{\eta} F_{\mathbf{y}}(\xi) d\xi$ for $\eta \in R$, representing average shortage to any real target η. By the theory of convex conjugate functions, the pointwise comparison of the second order cumulative distribution functions provides an alternative characterization of the equitable dominance relation [28].

Exactly, $\mathbf{y}'$ fairly dominates $\mathbf{y}''$, if and only if $F^{(2)}_{\mathbf{y}'}(\eta) \leq F^{(2)}_{\mathbf{y}''}(\eta)$ for all η where at least one strict inequality holds.

Furthermore, the classical results of majorization theory [21] allow us to refer the equitable dominance to the mean utility. For any convex, increasing utility function $u : R \to R$, if outcome vector $\mathbf{y}'$ fairly dominates $\mathbf{y}''$, then

$$\sum_{i=1}^{m} \frac{u(y'_i)}{m} = \sum_{i \in I_v} \bar{v}_i u(y'_{v_i}) \geq \sum_{i=1}^{m} \frac{u(y''_i)}{m} = \sum_{i \in I_v} \bar{v}_i u(y''_{v_i}).$$

Finally, there are three alternative analytical characterizations of the relation of fair dominance as specified in the following theorem. Note that according to condition (iii), the fair dominance is actually the so-called increasing convex order which is more commonly known as the second degree stochastic dominance (SSD) [28].

Theorem 1. *For any outcome vectors $\mathbf{y}', \mathbf{y}'' \in A$ each of the three following conditions is equivalent to the (weak) equitable dominance $\mathbf{y}' \succeq_e \mathbf{y}''$:*

(i) $F^{(-2)}_{\mathbf{y}'}(\beta) \geq F^{(-2)}_{\mathbf{y}''}(\beta)$ *for all* $\beta \in (0, 1]$;

(ii) $F^{(2)}_{\mathbf{y}'}(\eta) \leq F^{(2)}_{\mathbf{y}''}(\eta)$ *for all real* η;

(iii) $\sum_{i \in I_v} \bar{v}_i u(y'_i) \geq \sum_{i \in I_v} \bar{v}_i u(y''_i)$ *for any concave, increasing function* u.

Following Theorem 1, the importance weighted fair preference models are mathematically equivalent to the risk averse preference models for the decisions under risk, where the scenarios correspond to the agents and the importance weights define their probabilities while the agent outcomes represent realizations of a return under various scenarios.

4 Inequality Measures and Fair Consistency

Inequality measures were primarily studied in economics [39] while recently they become very popular tools in Operations Research. Typical inequality measures are some deviation type dispersion characteristics. They are *translation invariant* in the sense that $\varrho(\mathbf{y} + a\mathbf{e}) = \varrho(\mathbf{y})$ for any outcome vector $\mathbf{y}$ and real number a (where $\mathbf{e}$ vector of units $(1, \ldots, 1)$), thus being not affected by any shift of the outcome scale.

Moreover, the inequality measures are also *inequality relevant* which means that they are equal to 0 in the case of perfectly equal outcomes while taking positive values for unequal ones.

The simplest inequality measures are based on the absolute measurement of the spread of outcomes, like the *maximum (absolute) difference*

$$d(\mathbf{y}) = \max_{i,j \in I} |y_i - y_j| = \max_{i,j \in I_v} |y_{v_i} - y_{v_j}| \tag{19}$$

or the *mean absolute difference* also called the Gini's mean difference

$$\Gamma(\mathbf{y}) = \frac{1}{2m^2} \sum_{i \in I} \sum_{j \in I} |y_i - y_j| = \frac{1}{2} \sum_{i \in I_v} \sum_{j \in I_v} |y_{v_i} - y_{v_j}| \bar{v}_i \bar{v}_j. \tag{20}$$

In most application frameworks better intuitive appeal may have inequality measures related to deviations from the mean outcome like the *maximum (absolute) deviation*

$$R(\mathbf{y}) = \max_{i \in I} |y_i - \mu(\mathbf{y})| = \max_{i \in I_v} |y_{v_i} - \mu(\mathbf{y})| \tag{21}$$

or the *mean (absolute) deviation*

$$\delta(\mathbf{y}) = \frac{1}{m} \sum_{i \in I} |y_i - \mu(\mathbf{y})| = \sum_{i \in I_v} |y_{v_i} - \mu(\mathbf{y})| \bar{v}_i. \tag{22}$$

Note that the *standard deviation* σ (or the *variance* σ^2) represents both the deviations and the spread measurement as

$$\begin{aligned} \sigma(\mathbf{y}) &= \sqrt{\frac{1}{m} \sum_{i \in I} (y_i - \mu(\mathbf{y}))^2} = \sqrt{\frac{1}{2m^2} \sum_{i \in I} \sum_{j \in I} (y_i - y_j)^2} \\ &= \sqrt{\sum_{i \in I_v} (y_{v_i} - \mu(\mathbf{y}))^2 \bar{v}_i} = \sqrt{\frac{1}{2} \sum_{i \in I_v} \sum_{j \in I_v} (y_{v_i} - y_{v_j})^2 \bar{v}_i \bar{v}_j}. \end{aligned} \tag{23}$$

Deviational measures may be focused on the downside semideviations as related to worsening of outcome while ignoring downside semideviations related to improvement of outcome. One may define the *maximum (downside) semideviation*

$$\Delta(\mathbf{y}) = \max_{i \in I} (\mu(\mathbf{y}) - y_i) = \max_{i \in I_v} (\mu(\mathbf{y}) - y_{v_i}), \tag{24}$$

and the *mean (downside) semideviation*

$$\bar{\delta}(\mathbf{y}) = \frac{1}{m} \sum_{i \in I} (\mu(\mathbf{y}) - y_i)_+ = \sum_{i \in I_v} (\mu(\mathbf{y}) - y_{v_i})_+ \bar{v}_i, \tag{25}$$

where $(.)_+$ denotes the nonnegative part of a number. Similarly, the *standard (downside) semideviation* is given as

$$\bar{\sigma}(\mathbf{y}) = \sqrt{\frac{1}{m} \sum_{i \in I} (\mu(\mathbf{y}) - y_i)_+^2} = \sqrt{\sum_{i \in I_v} (\mu(\mathbf{y}) - y_{v_i})_+^2 \bar{v}_i}. \tag{26}$$

Due to the mean definition, the mean absolute semideviation is always equal to half of the mean absolute deviation ($\bar{\delta}(\mathbf{y}) = \frac{1}{2}\delta(\mathbf{y})$) but similar symmetry property in general does not apply to the maximum semideviation or the standard semideviation.

One can easily notice that direct minimization of typical inequality measures may contradict the optimization of individual outcomes resulting in equal but very low outcomes. As some resolution one may consider a bicriteria mean-equity model:

$$\max \{(\mu(\mathbf{f}(\mathbf{x})), -\varrho(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\} \tag{27}$$

which takes into account both the efficiency with optimization of the mean outcome $\mu(\mathbf{y})$ and the equity with minimization of an inequality measure $\varrho(\mathbf{y})$.

For typical inequality measures bicriteria model (27) is computationally very attractive since both the criteria are concave and LP implementable for many measures. Unfortunately, for any dispersion type inequality measures the bicriteria mean-equity model is not consistent with the outcomes maximization, and therefore is not consistent with the fair dominance. When considering a simple discrete problem with two allocation patterns P1 and P2 generating outcome vectors $\mathbf{y}' = (0, 0)$ and $\mathbf{y}'' = (2, 8)$, respectively, for any dispersion type inequality measure one gets $\varrho(\mathbf{y}'') > 0 = \varrho(\mathbf{y}')$ while $\mu(\mathbf{y}'') = 5 > 0 = \mu(\mathbf{y}')$. Hence, $\mathbf{y}''$ is not bicriteria dominated by $\mathbf{y}'$ and vice versa. Thus for any dispersion type inequality measure ϱ, allocation P1 with obviously worse outcome vector than that for allocation P2 is a Pareto-optimal solution in the corresponding bicriteria mean-equity model (27).

Note that the lack of consistency of the mean-equity model (27) with the outcomes maximization applies also to the case of the maximum semideviation $\Delta(\mathbf{y})$ (24) used as an inequality measure whereas subtracting this measure from the mean $\mu(\mathbf{y}) - \Delta(\mathbf{y}) = M(\mathbf{y})$ results in the worst outcome and thereby the first criterion of the MMF model. In other words, although a direct use of the maximum semideviation in the mean-equity model may contradict the outcome maximization, the measure can be used complementary to the mean leading us to the worst outcome criterion which does not contradict the outcome maximization. This construction can be generalized for various (dispersion type) inequality measures. For any inequality measure ϱ we introduce the corresponding underachievement function defined as the difference between the mean outcome and the inequality measure itself, i.e.

$$M_{\varrho}(\mathbf{y}) = \mu(\mathbf{y}) - \varrho(\mathbf{y}). \tag{28}$$

In the case of maximum semideviation the corresponding underachievement $M_{\Delta}(\mathbf{y})$ function represents the worst outcome $M(\mathbf{y})$. Similarly, in the case of mean semideviation one gets the underachievement function

$$M_{\bar{\delta}}(\mathbf{y}) = \mu(\mathbf{y}) - \bar{\delta}(\mathbf{y}) = \frac{1}{m}\sum_{i \in I} \min\{y_i, \mu(\mathbf{y})\} = \sum_{i \in I_v} \bar{v}_i \min\{y_{v_i}, \mu(\mathbf{y})\}$$

representing the mean underachievement. Further, due to $|y_i - y_j| = y_i + y_j - 2\min\{y_i, y_j\}$, one gets an alternative formula for the mean absolute difference

$$\Gamma(\mathbf{y}) = \mu(\mathbf{y}) - \frac{1}{m^2}\sum_{i \in I}\sum_{j \in I} \min\{y_i, y_j\} = \mu(\mathbf{y}) - \sum_{i \in I_v}\sum_{j \in I_v} \bar{v}_i \bar{v}_j \min\{y_{v_i}, y_{v_j}\} \tag{29}$$

and the corresponding underachievement function

$$M_{\Gamma}(\mathbf{y}) = \mu(\mathbf{y}) - \Gamma(\mathbf{y}) = \frac{1}{m^2}\sum_{i \in I}\sum_{j \in I} \min\{y_i, y_j\} = \sum_{i \in I_v}\sum_{j \in I_v} \bar{v}_i \bar{v}_j \min\{y_{v_i}, y_{v_j}\}$$

representing the mean pairwise worse outcome.

Note that one could consider a scaled $\varrho_{\alpha}(\mathbf{y}) = \alpha\varrho(\mathbf{y})$ as a different inequality measure. Therefore, in order to avoid creation of such redundant new inequality measures we allow the measures to be scaled with any positive factor $\alpha > 0$. For any inequality measure ϱ we introduce the corresponding underachievement function defined as the difference of the mean outcome and the (scaled) inequality measure itself, i.e.

$$M_{\alpha\varrho}(\mathbf{y}) = \mu(\mathbf{y}) - \alpha\varrho(\mathbf{y}). \tag{30}$$

This allows us to replace the original mean-equity bicriteria optimization (27) with the following bicriteria problem:

$$\max\{(\mu(\mathbf{f}(\mathbf{x})), \mu(\mathbf{f}(\mathbf{x})) - \alpha\varrho(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\} \tag{31}$$

where the second objective represents the corresponding underachievement measure $M_{\alpha\varrho}(\mathbf{y})$ (30). Note that for any inequality measure $\varrho(\mathbf{y}) \geq 0$ one gets $M_{\alpha\varrho}(\mathbf{y}) \leq \mu(\mathbf{y})$ thus really expressing underachievements (comparing to mean) from the perspective of outcomes being maximized.

We will say that an inequality measure ϱ is *fairly α-consistent* if

$$\mathbf{y}' \succeq_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') - \alpha\varrho(\mathbf{y}') \geq \mu(\mathbf{y}'') - \alpha\varrho(\mathbf{y}'') \tag{32}$$

The relation of fair α-consistency will be called *strong* if, in addition to (32), the following holds

$$\mathbf{y}' \succ_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') - \alpha\varrho(\mathbf{y}') > \mu(\mathbf{y}'') - \alpha\varrho(\mathbf{y}''). \tag{33}$$

Theorem 2. *If the inequality measure $\varrho(\mathbf{y})$ is fairly α-consistent (32), then except for outcomes with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$, every efficient solution of the bicriteria problem (31) is a fairly efficient allocation pattern. In the case of strong consistency (33), every allocation pattern $\mathbf{x} \in Q$ efficient to (31) is, unconditionally, fairly efficient.*

Proof. Let $\mathbf{x}^0 \in Q$ be an efficient solution of (31). Suppose that $\mathbf{x}^0$ is not fairly efficient. This means, there exists $\mathbf{x} \in Q$ such that $\mathbf{y} = \mathbf{f}(\mathbf{x}) \succ_e \mathbf{y}^0 = \mathbf{f}(\mathbf{x}^0)$. Then, it follows $\mu(\mathbf{y}) \geq \mu(\mathbf{y}^0)$, and simultaneously $\mu(\mathbf{y}) - \alpha\varrho(\mathbf{y}) \geq \mu(\mathbf{y}^0) - \alpha\varrho(\mathbf{y}^0)$, by virtue of the fair α-consistency (32). Since $\mathbf{x}^0$ is efficient to (31) no inequality can be strict, which implies $\mu(\mathbf{y}) = \mu(\mathbf{y}^0)$ and and $\varrho(\mathbf{y}) = \varrho(\mathbf{y}^0)$.

In the case of the strong fair α-consistency (33), the supposition $\mathbf{y} = \mathbf{f}(\mathbf{x}) \succ_e \mathbf{y}^0 = \mathbf{f}(\mathbf{x}^0)$ implies $\mu(\mathbf{y}) \geq \mu(\mathbf{y}^0)$ and $\mu(\mathbf{y}) - \alpha\varrho(\mathbf{y}) > \mu(\mathbf{y}^0) - \alpha\varrho(\mathbf{y}^0)$ which contradicts the efficiency of $\mathbf{x}^0$ with respect to (31). Hence, the allocation pattern $\mathbf{x}^0$ is fairly efficient.

5 Fair Consistency Conditions

Typical dispersion type inequality measures are directly defined for the weighted distributions of outcomes without any need of disaggregation. Actually, they depend only distribution of outcomes. Hence, they are impartial in the sense that

$$F_{\mathbf{y}'} = F_{\mathbf{y}''} \quad \Rightarrow \quad \varrho(\mathbf{y}') \cong \varrho(\mathbf{y}''). \tag{34}$$

as well as *clustering invariant* in the sense that any split of equal outcomes does not affect the measure. Moreover, typical inequality measures are convex, i.e. $\varrho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \leq \lambda\varrho(\mathbf{y}') + (1-\lambda)\varrho(\mathbf{y}'')$ for any $\mathbf{y}', \mathbf{y}''$ and $0 \leq \lambda \leq 1$. Certainly, the underachievement function $M_{\alpha\varrho}(\mathbf{y})$ must be also monotonic for the fair consistency which enforces more restrictions on the inequality measures. We will show further that convexity together with positive homogeneity and some boundedness of an inequality measure is sufficient to guarantee monotonicity of the corresponding underachievement measure and thereby to guarantee the fair α-consistency of inequality measure itself.

We say that (dispersion type) inequality measure $\varrho(\mathbf{y}) \geq 0$ is *Δ-bounded* if it is upper bounded by the maximum downside deviation, i.e.,

$$\varrho(\mathbf{y}) \leq \Delta(\mathbf{y}) \quad \forall \mathbf{y}. \tag{35}$$

Moreover, we say that $\varrho(\mathbf{y}) \geq 0$ is strictly Δ-bounded if inequality (35) is a strict bound, except from the case of perfectly equal outcomes, i.e., $\varrho(\mathbf{y}) < \Delta(\mathbf{y})$ for any $\mathbf{y}$ such that $\Delta(\mathbf{y}) > 0$.

Theorem 3. *Let $\varrho(\mathbf{y}) \geq 0$ be a convex, positively homogeneous, clustering invariant and translation invariant (dispersion type) inequality measure. If $\alpha\varrho(\mathbf{y})$ is Δ-bounded, then $\varrho(\mathbf{y})$ is fairly α-consistent in the sense of (32).*

Proof. The relation of fair dominance $\mathbf{y}' \succeq_e \mathbf{y}''$ denotes that there exists a finite sequence of vectors $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \ldots, \mathbf{y}^t$ such that $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'} + \varepsilon_k \mathbf{e}_{i''}$, $0 \leq \varepsilon_k \leq y^{k-1}_{i'} - y^{k-1}_{i''}$ for $k = 1, 2, \ldots, t$ and there exists a permutation π such that $y'_{\pi(i)} \geq y^t_i$ for all $i \in I$. Note that the underachievement function $M_{\alpha\varrho}(\mathbf{y})$, similar as $\varrho(\mathbf{y})$ depends only on the distribution of outcomes. Further, if $\mathbf{y}' \geq \mathbf{y}''$, then $\mathbf{y}' = \mathbf{y}'' + (\mathbf{y}' - \mathbf{y}'')$ and $\mathbf{y}' - \mathbf{y}'' \geq 0$. Hence, due to concavity and positive homogeneity, $M_{\alpha\varrho}(\mathbf{y}') \geq M_{\alpha\varrho}(\mathbf{y}'') + M_{\alpha\varrho}(\mathbf{y}' - \mathbf{y}'')$. Moreover, due to the bound (35), $M_{\alpha\varrho}(\mathbf{y}' - \mathbf{y}'') \geq \mu(\mathbf{y}' - \mathbf{y}'') - \Delta(\mathbf{y}' - \mathbf{y}'') \geq \mu(\mathbf{y}' - \mathbf{y}'') - \mu(\mathbf{y}' - \mathbf{y}'') = 0$. Thus, $M_{\alpha\varrho}(\mathbf{y})$ satisfies also the requirement of monotonicity. Hence, $M_{\alpha\varrho}(\mathbf{y}') \geq M_{\alpha\varrho}(\mathbf{y}^t)$. Further, let us notice that $\mathbf{y}^k = \lambda\bar{\mathbf{y}}^{k-1} + (1-\lambda)\mathbf{y}^{k-1}$ where $\bar{\mathbf{y}}^{k-1} = \mathbf{y}^{k-1} - (y_{i'} - y_{i''})\mathbf{e}_{i'} + (y_{i'} - y_{i''})\mathbf{e}_{i''}$ and $\lambda = \varepsilon/(y_{i'} - y_{i''})$. Vector $\bar{\mathbf{y}}^{k-1}$ has the same distribution of coefficients as $\mathbf{y}^{k-1}$ (actually it represents results of swapping $y_{i'}$ and $y_{i''}$). Hence, due to concavity of $M_{\alpha\varrho}(\mathbf{y})$, one gets $M_{\alpha\varrho}(\mathbf{y}^k) \geq \lambda M_{\alpha\varrho}(\bar{\mathbf{y}}^{k-1}) + (1-\lambda)M_{\alpha\varrho}(\mathbf{y}^{k-1}) = M_{\alpha\varrho}(\mathbf{y}^{k-1})$. Thus, $M_{\alpha\varrho}(\mathbf{y}') \geq M_{\alpha\varrho}(\mathbf{y}'')$ which justifies the fair α-consistency of $\varrho(\mathbf{y})$.

For strong fair α-consistency some strict monotonicity and concavity properties of the underachievement function are needed. Obviously, there does not exist any inequality measure which is positively homogeneous and simultaneously strictly convex. However, one may notice from the proof of Theorem 3 that only convexity properties on equally distributed outcome vectors are important for monotonous underachievement functions.

We say that inequality measure $\varrho(\mathbf{y}) \geq 0$ is *strictly convex on equally distributed outcome vectors*, if

$$\varrho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') < \lambda\varrho(\mathbf{y}') + (1-\lambda)\varrho(\mathbf{y}'')$$

for $0 < \lambda < 1$ and any two vectors $\mathbf{y}' \neq \mathbf{y}''$ representing the same outcomes distribution as some $\mathbf{y}$, i.e., $\mathbf{y}' = (y_{\pi'(1)}, \ldots, y_{\pi'(m)})$ π' and $\mathbf{y}'' = (y_{\pi''(1)}, \ldots, y_{\pi''(m)})$ for some permutations π' and π'', respectively.

Theorem 4. *Let $\varrho(\mathbf{y}) \geq 0$ be a convex, positively homogeneous, clustering invariant and translation invariant (dispersion type) inequality measure. If $\varrho(\mathbf{y})$ is also strictly convex on equally distributed outcomes and $\alpha\varrho(\mathbf{y})$ is strictly Δ-bounded, then the measure $\varrho(\mathbf{y})$ is fairly strongly α-consistent in the sense of (33).*

Proof. Due to the clustering invariance we may consider only the unweighted case. The relation of weak fair dominance $\mathbf{y}' \succeq_e \mathbf{y}''$ denotes that there exists a finite sequence of vectors $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \ldots, \mathbf{y}^t$ such that $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'} + \varepsilon_k \mathbf{e}_{i''}$, $0 \leq \varepsilon_k \leq y_{i'}^{k-1} - y_{i''}^{k-1}$ for $k = 1, 2, \ldots, t$ and there exists a permutation π such that $y'_{\pi(i)} \geq y_i^t$ for all $i \in I$. The strict fair dominance $\mathbf{y}' \succ_e \mathbf{y}''$ means that $y'_{\pi(i)} > y_i^t$ for some $i \in I$ or at least one ε_k is strictly positive. Note that the underachievement function $M_{\alpha\varrho}(\mathbf{y})$ is strictly monotonous and strictly convex on equally distributed outcome vectors. Hence, $M_{\alpha\varrho}(\mathbf{y}') > M_{\alpha\varrho}(\mathbf{y}'')$ which justifies the fair strong α-consistency of the measure $\varrho(\mathbf{y})$.

The specific case of fair 1-consistency is also called *the mean-complementary fair consistency.* Note that the fair $\bar{\alpha}$-consistency of measure $\varrho(\mathbf{y})$ actually guarantees the mean-complementary fair consistency of measure $\alpha\varrho(\mathbf{y})$ for all $0 < \alpha \leq \bar{\alpha}$, and the same remain valid for the strong consistency properties. It follows from a possible expression of $\mu(\mathbf{y}) - \alpha\varrho(\mathbf{y})$ as the convex combination of $\mu(\mathbf{y}) - \bar{\alpha}\varrho(\mathbf{y})$ and $\mu(\mathbf{y})$. Hence, for any $\mathbf{y}' \succeq_e \mathbf{y}''$, due to $\mu(\mathbf{y}') \geq \mu(\mathbf{y}'')$ one gets $\mu(\mathbf{y}') - \alpha\varrho(\mathbf{y}') \geq \mu(\mathbf{y}'') - \alpha\varrho(\mathbf{y}'')$ in the case of the fair $\bar{\alpha}$-consistency of measure $\varrho(\mathbf{y})$ (or respective strict inequality in the case of strong consistency). Therefore, while analyzing specific inequality measures we seek the largest values α guaranteeing the corresponding fair consistency.

As mentioned, typical inequality measures are convex and many of them are positively homogeneous. Moreover, the measures such as the mean absolute (downside) semideviation $\bar{\delta}(\mathbf{y})$ (25), the standard downside semideviation $\bar{\sigma}(\mathbf{y})$ (26), and the mean absolute difference $\Gamma(\mathbf{y})$ (20) are Δ-bounded. Indeed, one may easily notice that $\mu(\mathbf{y}) - y_i \leq \Delta(\mathbf{y})$ and therefore

$$\bar{\delta}(\mathbf{y}) \leq \sum_{i \in I_v} \Delta(\mathbf{y})\bar{v}_i = \Delta(\mathbf{y})$$

$$\bar{\sigma}(\mathbf{y}) \leq \sqrt{\sum_{i \in I_v} \Delta(\mathbf{y})^2 \bar{v}_i} = \Delta(\mathbf{y})$$

$$\Gamma(\mathbf{y}) = \sum_{i \in I_v} \sum_{j \in I_v} (\mu(\mathbf{y}) - \min\{y_i, y_j\})\bar{v}_i\bar{v}_j \leq \sum_{i \in I_v} \sum_{j \in I_v} \Delta(\mathbf{y})\bar{v}_i\bar{v}_j = \Delta(\mathbf{y})$$

where the last formula is due to (29). Actually, all those inequality measures are strictly Δ-bounded since for any unequal outcome vector at least one outcome must be below the mean thus leading to strict inequalities in the above bounds.

Obviously, Δ-bounded (but not strictly) is also the maximum absolute downside deviation $\Delta(\mathbf{y})$ itself. This allows us to justify the maximum downside deviation $\Delta(\mathbf{y})$ (24), the mean absolute (downside) semideviation $\bar{\delta}(\mathbf{y})$ (25), the standard downside semideviation $\bar{\sigma}(\mathbf{y})$ (26) and the mean absolute difference $\Gamma(\mathbf{y})$ (20) as fairly 1-consistent (mean-complementary fairly consistent) in the sense of (32). Recall that the mean absolute semideviation is twice the mean absolute (downside) semideviation which means that $\alpha\delta(\mathbf{y})$ is Δ-bounded for any $0 < \alpha \leq 0.5$.

We emphasize that, despite the standard semideviation is a fairly 1-consistent inequality measure, the consistency is not valid for variance, semivariance and even for the standard deviation. These measures, in general, do not satisfy the all assumptions of Theorem 3. Certainly, we have enumerated only the simplest inequality measures studied in the resource allocation context which satisfy the assumptions of Theorem 3 and thereby they are fairly 1-consistent. Theorem 3 allows one to show this property for many other measures. In particular, one may easily find out that any convex combination of fairly α-consistent inequality measures remains also fairly α-consistent. On the other hand, among typical inequality measures the mean absolute difference seems to be the only one meeting the stronger assumptions of Theorem 4 and thereby maintaining the strong consistency.

Table 1. Fair consistency results for the basic inequality measures

Measure			α–consistency	
Mean absolute semideviation	$\bar{\delta}(\mathbf{y})$	(25)	1	
Mean absolute deviation	$\delta(\mathbf{y})$	(22)	0.5	
Maximum semideviation	$\Delta(\mathbf{y})$	(24)	1	
Mean absolute difference	$\Gamma(\mathbf{y})$	(20)	1	strong
Standard semideviation	$\bar{\sigma}(\mathbf{y})$	(26)	1	

The fair consistency results for basic dispersion type inequality measures considered in resource allocation problems are summarized in Table 1 where α values for unweighted as well as weighted problems are given and the strong consistency is indicated. Table 1 points out how the inequality measures can be used in resource allocation models to guarantee their harmony both with outcome maximization (Pareto-optimality) and with inequalities minimization (Pigou-Dalton equity theory). Exactly, for each inequality measure applied with the corresponding value α from Table 1 (or smaller positive value), every efficient solution of the bicriteria problem (31), ie. $\max\{(\mu(\mathbf{f}(\mathbf{x})), \mu(\mathbf{f}(\mathbf{x})) - \alpha\varrho(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}$, is a fairly efficient allocation pattern, except for outcomes with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$. In the case of strong consistency (as for mean absolute difference), every solution $\mathbf{x} \in Q$ efficient to (31) is, unconditionally, fairly efficient.

As mentioned, the mean absolute semideviation is twice the mean absolute semideviation which means that $\alpha\delta(\mathbf{y})$ is Δ-bounded for any $0 < \alpha \leq 0.5$. The symmetry of mean absolute semideviations

Table 2. Marginal fair consistency results for symmetric inequality measures

Measure		α–consistency		
Maximum absolute deviation	$R(\mathbf{y})$ (21)	$\dfrac{\min_{i\in I_v} \bar{v}_i}{(1-\min_{i\in I_v} \bar{v}_i)}$	$\dfrac{1}{m-1}$	
Maximum absolute difference	$d(\mathbf{y})$ (19)	$\min_{i\in I_v} \bar{v}_i$	$\dfrac{1}{m}$	
Standard deviation	$\sigma(\mathbf{y})$ (23)	$\dfrac{\min_{i\in I_v} \bar{v}_i}{1-\min_{i\in I_v} \bar{v}_i}$	$\dfrac{1}{\sqrt{m-1}}$	strong

$$\bar{\delta}(\mathbf{y}) = \sum_{i\in I_v} (y_{v_i} - \mu(\mathbf{y}))_+ \bar{v}_i = \sum_{i\in I_v} (\mu(\mathbf{y}) - y_{v_i})_+ \bar{v}_i$$

can be also used to derive some marginal $\varDelta$-boundedness relations for other inequality measures. In particular, one may find out that any downside semideviation from the mean cannot be larger than $\kappa = (1 - \min_{i\in I_v} \bar{v}_i)/\min_{i\in I_v} \bar{v}_i$ downside semideviations, where $\kappa = m-1$ for the case of m unweighted agents. Hence, the maximum absolute deviation satisfies the inequality $\min_{i\in I_v} \bar{v}_i/(1-\min_{i\in I_v} \bar{v}_i) R(\mathbf{y}) \leq \varDelta(\mathbf{y})$, while the maximum absolute difference fulfills $\min_{i\in I_v} \bar{v}_i d(\mathbf{y}) \leq \varDelta(\mathbf{y})$. In the case of unweighted agents these bounds take the forms $\frac{1}{m-1} R(\mathbf{y}) \leq \varDelta(\mathbf{y})$ and $\frac{1}{m} d(\mathbf{y}) \leq \varDelta(\mathbf{y})$, respectively. Similarly, for the standard deviation one gets

$$\sqrt{\frac{\min_{i\in I_v} \bar{v}_i}{1-\min_{i\in I_v} \bar{v}_i}} \sigma(\mathbf{y}) \leq \varDelta(\mathbf{y}) \quad \text{or} \quad \frac{1}{\sqrt{m-1}} \sigma(\mathbf{y}) \leq \varDelta(\mathbf{y})$$

for weighted or unweighted problems, respectively. Hence, $\alpha\sigma(\mathbf{y})$ is strictly $\varDelta$-bounded for any $\sqrt{\frac{\min_{i\in I_v} \bar{v}_i}{1-\min_{i\in I_v} \bar{v}_i}}$, since for any unequal outcome vector at least one outcome must be below the mean thus leading to strict inequalities in the above bounds. These allow us to justify the maximum absolute deviation with $0 < \alpha \leq \min_{i\in I_v} \bar{v}_i/(1-\min_{i\in I_v} \bar{v}_i)$, the maximum absolute difference with $0 < \alpha \leq \min_{i\in I_v} \bar{v}_i$ and the standard deviation with $0 < \alpha \leq \frac{\min_{i\in I_v} \bar{v}_i}{1-\min_{i\in I_v} \bar{v}_i}$ as fairly α-consistent within the specified intervals of α. Moreover, the α-consistency of the standard deviation is strong. These marginal consistency results are summarized in Table 2 for weighted and unweighted agents, respectively.

6 Conclusions

The problems of efficient and fair resource allocation arise in various systems which serve many users. Fairness is, essentially, an abstract socio-political concept that implies impartiality, justice and equity. Nevertheless, in operations research it was quantified with various solution concepts. The equitable optimization with the preference structure that complies with both the efficiency (Pareto-optimality) and with the Pigou-Dalton principle of transfers may be used to formalize the fair solution concepts. Multiple criteria models equivalent to equitable optimization allows to generate a variety of fair and efficient resource allocation patterns [31].

In this paper we have analyzed how scalar inequality measures can be used to guarantee the fair consistency. It turns out that several inequality measures can be combined with the mean itself into the optimization criteria generalizing the concept of the worst outcome and generating fairly consistent underachievement measures. We have shown that properties of convexity and positive homogeneity together with being bounded by the maximum downside semideviation are sufficient for a typical inequality measure to guarantee the corresponding fair consistency. It allows us to identify various inequality measures which can be effectively used to incorporate fairness factors into various resource allocation problems while preserving the consistency with outcomes maximization. Among others the mean semideviation turns out to be such a consistent inequality measure while the mean absolute difference is strongly consistent. In multiagent allocation problems another way of defining a fair allocation as a bicriteria decision problem would be to take as second criterion a measure of envy-freeness [5]. This could lead to another class of significant questions on consistency for further research.

The considered fairness model is primarily well suited for the centralized resource allocation problems, like the bandwidth allocation problem [3,9,34]. Nevertheless, the classical unweighted fairness models are used as the basis for some distributed systems managements (c.f., [38]). The analyzed bicriteria fairness models may be considered as introduction of a compensation term into the utilitarian model. Hence, they may help to develop fair distributed mechanisms. It seems to be a promising direction for further research on possible implementations for specific environments.

Our analysis is related to the properties of solutions to resource allocation models. It has been shown how inequality measures can be included into the models avoiding contradiction to the maximization of outcomes. We do not analyze algorithmic issues for the specific resource allocation problems. Generally, the requirement of the measures convexity necessary for the fair consistency, guarantees that the corresponding optimization criteria belong to the class of convex optimization, not complicating the original resource allocation model with any additional discrete structure. Most of the inequality measures, we analyzed, can be implemented with auxiliary linear programming constraints thus offering reasonable optimization models for continuous as well as discrete problems [18]. Actually, among the measures of Table 1 only the standard semideviation leads to nonlinear optimization while the maximum semideviation, the mean absolute semideviation as well as the mean absolute difference are LP implementable. Nevertheless, further research on efficient computational algorithms for solving the specific models is necessary.

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