

Determining OWA Operator Weights by Mean Absolute Deviation Minimization

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Abstract. The ordered weighted averaging (OWA) operator uses the weights assigned to the ordered values rather than to the specific criteria. This allows one to model various aggregation preferences, preserving simultaneously the impartiality (neutrality) with respect to the individual attributes. The determination of ordered weighted averaging (OWA) operator weights is a crucial issue of applying the OWA operator for decision making. This paper considers determining monotonic weights of the OWA operator by minimization the mean absolute deviation inequality measure. This leads to a linear programming model which can also be solved analytically.

1 Introduction

The problem of aggregating multiple numerical criteria to form overall objective functions is of considerable importance in many disciplines. The most commonly used aggregation is based on the weighted sum. The preference weights can be effectively introduced with the so-called Ordered Weighted Averaging (OWA) aggregation developed by Yager [17]. In the OWA aggregation the weights are assigned to the ordered values (i.e. to the smallest value, the second smallest and so on) rather than to the specific criteria. Since its introduction, the OWA aggregation has been successfully applied to many fields of decision making [8,12,21]. The OWA operator allows us to model various aggregation functions from the maximum through the arithmetic mean to the minimum. Thus, it enables modeling of various preferences from the optimistic to the pessimistic one.

Several approaches has been introduced for obtaining the OWA weights with a predefined degree of orness [2,16]. O'Hagan [7] proposed a maximum entropy approach, which involved a constrained nonlinear optimization problem with a predefined degree of orness as its constraint and the entropy as the objective function. Actually, the maximum entropy model can be transformed into a polynomial equation and then solved analytically [3]. A minimum variance approach to obtain the minimal variability OWA operator weights was also considered [4]. The minimax disparity approach proposed by Wang and Parkan [14] was the first method of finding OWA operator using Linear Programming (LP) This

method determines the OWA operator weights by minimizing the maximum difference between two adjacent weights under a given level of orness. The min-max disparity approach was further extended [1,13] and related to the minimum variance approaches [6]. The maximum entropy approach was generalized for various Minkowski metrics [19,20] and in some cases expressed with LP models [15]. In this paper we analyze a possibility to use another LP solvable models. In particular, we develop the LP model to determine the OWA operator weights by minimizing the Mean Absolute Deviation (MAD) inequality measure. In addition to the LP model an analytical formula is also derived.

2 Orness and Inequality Measures

Let $\mathbf{w} = (w_1, \dots, w_m)$ be a weighting vector of dimension m such that $w_i \geq 0$ for $i = 1, \dots, m$ and $\sum_{i=1}^m w_i = 1$. The corresponding OWA aggregation of outcomes $\mathbf{y} = (y_1, \dots, y_m)$ can be mathematically formalized as follows [17]. First, we introduce the ordering map $\Theta : R^m \rightarrow R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \dots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \geq \theta_2(\mathbf{y}) \geq \dots \geq \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, \dots, m$. Further, we apply the weighted sum aggregation to ordered achievement vectors $\Theta(\mathbf{y})$, i.e. the OWA aggregation has the following form:

$$A_{\mathbf{w}}(\mathbf{y}) = \sum_{i=1}^m w_i \theta_i(\mathbf{y}) \quad (1)$$

The OWA aggregation may model various preferences from the optimistic (max) to the pessimistic (min). Yager [17] introduced a well appealing concept of the orness measure to characterize the OWA operators. The degree of orness associated with the OWA operator $A_{\mathbf{w}}(\mathbf{y})$ is defined as

$$\text{orness}(\mathbf{w}) = \sum_{i=1}^m \frac{m-i}{m-1} w_i \quad (2)$$

For the max aggregation representing the fuzzy ‘or’ operator with weights $\mathbf{w} = (1, 0, \dots, 0)$ one gets $\text{orness}(\mathbf{w}) = 1$ while for the min aggregation representing the fuzzy ‘and’ operator with weights $\mathbf{w} = (0, \dots, 0, 1)$ one has $\text{orness}(\mathbf{w}) = 0$. For the average (arithmetic mean) one gets $\text{orness}((1/m, 1/m, \dots, 1/m)) = 1/2$. Actually, one may consider a complementary measure of andness defined as $\text{andness}(\mathbf{w}) = 1 - \text{orness}(\mathbf{w})$. OWA aggregations with orness greater or equal $1/2$ are considered or-like whereas the aggregations with orness smaller or equal $1/2$ are treated as and-like. The former corresponds to rather optimistic preferences while the latter represents rather pessimistic preferences.

The OWA aggregations with monotonic weights are either or-like or and-like. Exactly, decreasing weights $w_1 \geq w_2 \geq \dots \geq w_m$ define an or-like OWA operator, while increasing weights $w_1 \leq w_2 \leq \dots \leq w_m$ define an and-like OWA operator. Actually, the orness and the andness properties of the OWA operators

with monotonic weights are total in the sense that they remain valid for any subaggregations defined by subsequences of their weights.

Yager [18] proposed to define the OWA weighting vectors via the regular increasing monotone (RIM) quantifiers, which provide a dimension independent description of the aggregation. A fuzzy subset Q of the real line is called a RIM quantifier if Q is (weakly) increasing with $Q(0) = 0$ and $Q(1) = 1$. The OWA weights can be defined with a RIM quantifier Q as $w_i = Q(i/m) - Q((i - 1)/m)$, and the orness measure can be extended to a RIM quantifier (according to $m \rightarrow \infty$) as follows [18]

$$\text{orness}(Q) = \int_0^1 Q(\alpha) d\alpha \tag{3}$$

Thus, the orness of a RIM quantifier is equal to the area under it.

Monotonic weights can be uniquely defined by their distribution. First, we introduce the right-continuous cumulative distribution function (cdf):

$$F_{\mathbf{w}}(d) = \sum_{i=1}^m \frac{1}{m} \delta_i(d) \quad \text{where} \quad \delta_i(d) = \begin{cases} 1 & \text{if } w_i \leq d \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

which for any real value d provides the measure of weights smaller or equal to d . Alternatively one may use the left-continuous right tail cumulative distribution function $\overline{F}_{\mathbf{w}}(d) = 1 - F_{\mathbf{w}}(d)$ which for any real value d provides the measure of weights greater or equal to d .

Next, we introduce the quantile function $F_{\mathbf{w}}^{(-1)} = \inf \{ \eta : F_{\mathbf{w}}(\eta) \geq \xi \}$ for $0 < \xi \leq 1$ as the left-continuous inverse of the cumulative distribution function $F_{\mathbf{w}}$, i.e., $F_{\mathbf{w}}^{(-1)}(\xi) = \inf \{ \eta : F_{\mathbf{w}}(\eta) \geq \xi \}$ for $0 < \xi \leq 1$. Similarly, we introduce the right tail quantile function $\overline{F}_{\mathbf{w}}^{(-1)}$ as the right-continuous inverse of the cumulative distribution function $\overline{F}_{\mathbf{w}}$, i.e., $\overline{F}_{\mathbf{w}}^{(-1)}(\xi) = \sup \{ \eta : \overline{F}_{\mathbf{w}}(\eta) \geq \xi \}$ for $0 < \xi \leq 1$. Actually, $\overline{F}_{\mathbf{w}}^{(-1)}(\xi) = F_{\mathbf{w}}^{(-1)}(1 - \xi)$. It is the stepwise function $\overline{F}_{\mathbf{w}}^{(-1)}(\xi) = \theta_i(\mathbf{w})$ for $\frac{i-1}{m} < \xi \leq \frac{i}{m}$.

Dispersion of the weights distribution can be described with the Lorenz curves and related inequality measures. Classical Lorenz curve used in income economics as a cumulative population versus income curve to compare equity of income distributions. Although, the Lorenz curve for any distribution may be viewed [5] as a normalized integrated quantile function. In particular, for distribution of weights \mathbf{w} one gets

$$L_{\mathbf{w}}(\xi) = \frac{1}{\mu(\mathbf{w})} \int_0^\xi F_{\mathbf{w}}^{(-1)}(\alpha) d\alpha = m \int_0^\xi F_{\mathbf{w}}^{(-1)}(\alpha) \tag{5}$$

where while dealing with normalized weights w_i we have always $\mu(\mathbf{w}) = 1/m$. Graphs of functions $L_{\mathbf{w}}(\xi)$ take the form of piecewise linear convex curves. They are also nondecreasing, due to nonnegative weights w_i . Any perfectly equal distribution of income has the diagonal line as the Lorenz curve (the same independently from the income value).

Alternatively, the upper Lorenz curve may be used which integrates the right tail quantile function. For distribution of weights \mathbf{w} one gets

$$\bar{L}_{\mathbf{w}}(\xi) = \frac{1}{\mu(\mathbf{w})} \int_0^\xi \bar{F}_{\mathbf{w}}^{(-1)}(\alpha) d\alpha = m \int_0^\xi \bar{F}_{\mathbf{w}}^{(-1)}(\alpha) \quad (6)$$

Graphs of functions $\bar{L}_{\mathbf{w}}(\xi)$ take the form of piecewise linear concave curves. They are also nondecreasing, due to nonnegative weights w_i . Similar to $L_{\mathbf{w}}$, the vector of perfectly equal weights has the diagonal line as the upper Lorenz curve. Actually, both the classical (lower) and the upper Lorenz curves are symmetric with respect to the diagonal line in the sense that the differences

$$\bar{d}_{\mathbf{w}}(\xi) = \bar{L}_{\mathbf{w}}(\xi) - \xi \quad \text{and} \quad d_{\mathbf{w}}(\xi) = \xi - L_{\mathbf{w}}(\xi) \quad (7)$$

are equal for symmetric arguments $\bar{d}_{\mathbf{w}}(\xi) = d_{\mathbf{w}}(1 - \xi)$. Hence,

$$\bar{L}_{\mathbf{w}}(\xi) + L_{\mathbf{w}}(1 - \xi) = 1 \quad \text{for any } 0 \leq \xi \leq 1 \quad (8)$$

Note that in the case of nondecreasing OWA weights $0 \leq w_1 \leq \dots \leq w_m \leq 1$ the corresponding Lorenz curve $L_{\mathbf{w}}(\xi)$ is (weakly) increasing with $L_{\mathbf{w}}(0) = 0$ and $L_{\mathbf{w}}(1) = 1$ as well as the OWA weights can be defined with L as $w_i = L_{\mathbf{w}}(i/m) - L_{\mathbf{w}}((i-1)/m)$. Hence, $L_{\mathbf{w}}$ may be considered then as a RIM quantifier generating weights \mathbf{w} [10]. Following (3), the orness measure of RIM quantifier is given as $\text{orness}(L) = \int_0^1 L(\alpha) d\alpha$, thus equal to the area under $L_{\mathbf{w}}$. Certainly, for any finite m the RIM orness $\text{orness}(L_{\mathbf{w}})$ differs from the orness, but the difference depends only on the value of m , exactly,

$$\text{orness}(L_{\mathbf{w}}) = \sum_{i=1}^m \frac{m-i}{m} w_i + \sum_{i=1}^m \frac{1}{2m} w_i = \frac{m-1}{m} \text{orness}(\mathbf{w}) + \frac{1}{2m} \quad (9)$$

In the case of nonincreasing OWA weights $1 \geq w_1 \geq \dots \geq w_m \geq 0$ the corresponding upper Lorenz curve $\bar{L}_{\mathbf{w}}(\xi)$ is (weakly) increasing with $\bar{L}_{\mathbf{w}}(0) = 0$ and $\bar{L}_{\mathbf{w}}(1) = 1$ as well as the OWA weights can be defined with \bar{L} as $w_i = \bar{L}_{\mathbf{w}}(i/m) - \bar{L}_{\mathbf{w}}((i-1)/m)$. Hence, $\bar{L}_{\mathbf{w}}$ may be considered then as a RIM quantifier generating weights \mathbf{w} . Similar to (9), the difference between the RIM orness $\text{orness}(L_{\mathbf{w}})$ and $\text{orness}(\mathbf{w})$ depends only on the value of m .

Typical inequality measures are some deviation type dispersion characteristics. They are *inequality relevant* which means that they are equal to 0 in the case of perfectly equal outcomes while taking positive values for unequal ones.

The simplest inequality measures are based on the absolute measurement of the spread of outcomes, like the (*Gini's*) *mean absolute difference*

$$\Gamma(\mathbf{w}) = \frac{1}{2m^2} \sum_{i=1}^m \sum_{j=1}^m |w_i - w_j| \quad (10)$$

In most application frameworks a better intuitive appeal may have inequality measures related to deviations from the mean value like the Mean Absolute Deviation (MAD) from the mean

$$\delta(\mathbf{w}) = \frac{1}{m} \sum_{i=1}^m |w_i - \mu(\mathbf{w})| \tag{11}$$

In economics one usually considers relative inequality measures normalized by the mean. Among many inequality measures perhaps the most commonly accepted by economists is the *Gini index*, which is the relative mean difference.

$$G(\mathbf{w}) = \Gamma(\mathbf{w})/\mu(\mathbf{w}) = m\Gamma(\mathbf{w}) \tag{12}$$

Similar, one may consider the relative mean deviation which is known as the *Schutz index*

$$S(\mathbf{w}) = \delta(\mathbf{w})/\mu(\mathbf{w}) = m\delta(\mathbf{w}) \tag{13}$$

Note that due to $\mu(\mathbf{w}) = 1/m$, the relative inequality measures are proportional to their absolute counterparts and any comparison of the relative measures is equivalent to comparison of the corresponding absolute measures.

The above inequality measures are closely related to the Lorenz curve [8] and its differences from the diagonal (equity) line (7). First of all

$$G(\mathbf{w}) = 2 \int_0^1 \bar{d}_{\mathbf{w}}(\alpha) d\alpha = 2 \int_0^1 d_{\mathbf{w}}(\alpha) d\alpha \tag{14}$$

thus

$$G(\mathbf{w}) = 2 \int_0^1 \bar{L}_{\mathbf{w}}(\alpha) d\alpha - 1 = 1 - 2 \int_0^1 L_{\mathbf{w}}(\alpha) d\alpha \tag{15}$$

Recall that in the case of nondecreasing OWA weights $0 \leq w_1 \leq \dots \leq w_m \leq 1$ the corresponding Lorenz curve $L_{\mathbf{w}}(\xi)$ may be considered as a RIM quantifier generating weights \mathbf{w} . Following (9), one gets

$$G(\mathbf{w}) = 1 - 2\text{orness}(L_{\mathbf{w}}) = \frac{m-1}{m}(1 - 2\text{orness}(\mathbf{w})) \tag{16}$$

enabling easy recalculation of the orness measure into the Gini index and vice versa. Similarly, in the case of nonincreasing OWA weights $1 \geq w_1 \geq \dots \geq w_m \geq 0$, one gets

$$G(\mathbf{w}) = 2\text{orness}(\bar{L}_{\mathbf{w}}) - 1 = \frac{m-1}{m}(2\text{orness}(\mathbf{w}) - 1) \tag{17}$$

3 Mean Absolute Deviation Minimization

We focus on the case of monotonic weights. Following (16) and (17), the Gini index is then uniquely defined by a given orness value. Nevertheless, one may still select various weights by minimization the MAD measure. Although related to the Lorenz curve it is not uniquely defined by the Gini index and the orness measure. Actually, the MAD minimization approach may be viewed as the generalized entropy maximization based on the first Minkowski metric [15].

Let us define differences

$$\bar{d}_i(\mathbf{w}) = \bar{L}_{\mathbf{w}}\left(\frac{i}{m}\right) - \frac{i}{m} \quad \text{and} \quad d_i(\mathbf{w}) = \frac{i}{m} - L_{\mathbf{w}}\left(\frac{i}{m}\right) \quad \text{for } i = 1, \dots, m \quad (18)$$

where due to nonnegativity of weights, for all $i = 1, \dots, m - 1$

$$\bar{d}_i(\mathbf{w}) \leq \frac{1}{m} + \bar{d}_{i+1}(\mathbf{w}) \quad \text{and} \quad d_i(\mathbf{w}) \leq \frac{1}{m} + d_{i-1}(\mathbf{w}) \quad (19)$$

with $d_0(\mathbf{w}) = \bar{d}_0(\mathbf{w}) = 0$ and $d_m(\mathbf{w}) = \bar{d}_m(\mathbf{w}) = 0$. Thus

$$\bar{d}_{m-i}(\mathbf{w}) \leq \frac{i}{m} \quad \text{and} \quad d_i(\mathbf{w}) \leq \frac{i}{m} \quad \text{for } i = 1, \dots, m - 1 \quad (20)$$

The Gini index represents the area defined by $\bar{d}_i(\mathbf{w})$ or $d_i(\mathbf{w})$, respectively,

$$G(\mathbf{w}) = \frac{2}{m} \sum_{i=1}^{m-1} \bar{d}_i(\mathbf{w}) = \frac{2}{m} \sum_{i=1}^{m-1} d_i(\mathbf{w}) \quad (21)$$

while the relative MAD (Schutz index) may be represented [8] as

$$S(\mathbf{w}) = m\delta(\mathbf{w}) = \max_{i=1, \dots, m-1} \bar{d}_i(\mathbf{w}) = \max_{i=1, \dots, m-1} d_i(\mathbf{w}) \quad (22)$$

Assume there is given some orness value $0.5 \leq \alpha \leq 1$ and we are looking for monotonic weights $1 \geq w_1 \geq \dots \geq w_m \geq 0$ such that $\text{orness}(\mathbf{w}) = \alpha$ and $S(\mathbf{w})$ is minimal. Following (17), (21) and (22) it leads us to the problem

$$\begin{aligned} & \min \max_{i=1, \dots, m-1} m\bar{d}_i(\mathbf{w}) \\ & \text{s.t. } \frac{2}{m} \sum_{i=1}^{m-1} \bar{d}_i(\mathbf{w}) = \frac{m-1}{m}(2\alpha - 1) \end{aligned} \quad (23)$$

with additional constraints (19). This allows us to form the following LP model

$$\min \quad md \quad (24)$$

$$\text{s.t. } \bar{d}_i \leq d \quad \text{for } i = 1, \dots, m - 1 \quad (25)$$

$$\bar{d}_1 + \dots + \bar{d}_{m-1} = (m - 1)(\alpha - 0.5) \quad (26)$$

$$0 \leq \bar{d}_i \leq \frac{1}{m} + \bar{d}_{i+1} \quad \text{for } i = 1, \dots, m - 1 \quad (27)$$

with variables \bar{d}_i for $i = 1, \dots, m - 1$, auxiliary variable d and constant $\bar{d}_m = 0$. Having solved the above LP problem, the corresponding weights can be simply calculated according to the following formula (with $\bar{d}_0 = \bar{d}_m = 0$):

$$w_i = \bar{d}_i - \bar{d}_{i-1} + \frac{1}{m} \quad \text{for } i = 1, \dots, m \quad (28)$$

Symmetrically, having given an orness value $0 \leq \alpha \leq 0.5$ and looking for monotonic weights $0 \leq w_1 \leq \dots \leq w_m \leq 1$ such that $\text{orness}(\mathbf{w}) = \alpha$ and $S(\mathbf{w})$ is minimal, following (16), (21) and (22), one gets the problem

$$\begin{aligned} & \min \max_{i=1, \dots, m-1} md_i(\mathbf{w}) \\ & \text{s.t. } \frac{2}{m} \sum_{i=1}^{m-1} d_i(\mathbf{w}) = \frac{m-1}{m}(1-2\alpha) \end{aligned} \tag{29}$$

with additional constraints (19). Thus leading to the LP problem

$$\begin{aligned} & \min md \\ & \text{s.t. } d_i \leq d \quad \text{for } i = 1, \dots, m-1 \\ & \quad d_1 + \dots + d_{m-1} = (m-1)(0.5 - \alpha) \\ & \quad 0 \leq d_i \leq \frac{1}{m} + d_{i-1} \quad \text{for } i = 1, \dots, m-1 \end{aligned} \tag{30}$$

with variables d_i for $i = 1, \dots, m-1$, auxiliary variable d and constant $d_0 = 0$. The corresponding weights can be found according to the formula

$$w_i = d_{i-1} - d_i + \frac{1}{m} \quad \text{for } i = 1, \dots, m \tag{31}$$

where $d_0 = d_m = 0$.

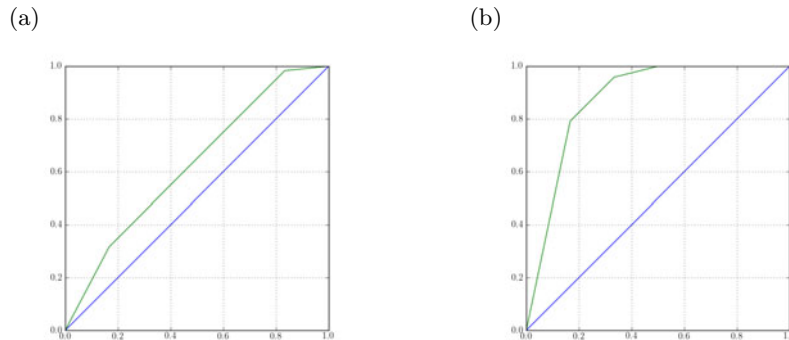


Fig. 1. Lorenz curve for MAD minimization: (a) $0.5 \leq \alpha \leq 0.5 + 1/m$, (b) $\alpha > 0.5 + 1/m$

LP models (24)–(27) and (30) allow for application standard optimization techniques to solve them. However, their structure is so simple that the problem of MAD minimization can also be solved analytically. We will show this in details for the case of $0.5 \leq \text{orness}(\mathbf{w}) \leq 1$ and the corresponding model (24)–(27).

One may take advantage of the fact that an optimal solution to the minimax problem $\min\{\max_{i \in I} y_i : \sum_{i \in I} y_i = b\}$ are perfectly equal values $y_i = b/|I|$ for all $i \in I$. Hence, when the required orness level is small enough (still not below 0.5), then the optimal solution is defined by

$$\bar{d}_1 = \dots = \bar{d}_{m-1} = \Delta$$

where $\Delta = \alpha - 0.5$ is defined by the orness equation (26) while leaving inequalities (27) inactive. Exactly, this is a case when $0.5 \leq \alpha \leq 0.5 + 1/m$ as illustrated in Fig. 1(a). Note that such a solution is generated by weights:

$$w_1 = \frac{1}{m} + \Delta, w_2 = \dots = w_{m-1} = \frac{1}{m}, w_m = \frac{1}{m} - \Delta$$

When the required orness level is larger then some constraints (27) become active thus setting some κ tail differences on their upper limits: $\bar{d}_{m-k} = \frac{k}{m}$ for $k = 1, \dots, \kappa$, as illustrated in Fig. 1(b). This leads us to the solution

$$\bar{d}_1 = \dots = \bar{d}_{m-\kappa-1} = \Delta, \quad \bar{d}_{m-k} = \frac{k}{m} \quad \text{for } k = 1, \dots, \kappa \tag{32}$$

where, following the orness equation (26),

$$\Delta = \frac{m-1}{m-\kappa-1}(\alpha - 0.5) - \frac{\kappa(\kappa+1)}{2m(m-\kappa-1)} \tag{33}$$

Exactly, formulas (32)-(33) are valid when

$$\frac{(2m-1-\kappa)\kappa}{2m(m-1)} \leq \alpha - 0.5 \leq \frac{(2m-1-(\kappa+1))(\kappa+1)}{2m(m-1)}$$

which means that κ can be simply computed as a function of the orness level α

$$\kappa = \kappa(\alpha) = \lfloor m-1 - \sqrt{2m(m-1)(1-\alpha) + 0.25} \rfloor$$

Following (32)-(33) and (28), the OWA weights are then given by the formula

$$w_1 = \frac{1}{m} + \Delta, w_2 = \dots = w_{m-\kappa-1} = \frac{1}{m}, w_{m-\kappa} = \frac{\kappa+1}{m} - \Delta$$

$$w_{m-k} = 0 \quad \text{for } k = 1, \dots, \kappa - 1$$

4 Conclusion

The determination of ordered weighted averaging (OWA) operator weights is a crucial issue of applying the OWA operator for decision making. We have considered determining monotonic weights of the OWA operator by minimization of the mean absolute deviation inequality measure. This leads us to a linear programming model which can also be solved analytically. The analytic approach results in simple direct formulas. The LP models allow us to find weights by the use of efficient LP optimization techniques and they enable easy enhancement of the preference model with additional requirements on the weights properties. The latter is the main advantage over the standard method of entropy maximization. Both the standard method and the proposed one do have their analytical solutions. However, if we try to elaborate them further by adding some auxiliary (linear) constraints on the OWA weights, then the entropy minimization model forms a difficult nonlinear optimization task while the MAD minimization is still easily LP-solvable.

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