REFERENCE POINT METHOD WITH IMPORTANCE WEIGHTED ORDERED ACHIEVEMENTS

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The Reference Point Method (RPM) is a very convenient technique for interactive analysis of the multiple criteria optimization problems. The interactive analysis is navigated with the commonly accepted control parameters expressing reference levels for the individual objective functions. The final scalarizing achievement function is built as the augmented max-min aggregation of partial achievements with respect to the given reference levels. In order to avoid inconsistencies caused by the regularization, the max-min solution may be regularized by the Ordered Weighted Averages (OWA) with monotonic weights which combines all the partial achievements allocating the largest weight to the worst achievement, the second largest weight to the second worst achievement, and so on. Further following the concept of the Weighted OWA (WOWA) the importance weighting of several achievements may be incorporated into the RPM. Such a WOWA RPM approach uses importance weights to affect achievement importance by rescaling accordingly its measure within the distribution of achievements rather than by straightforward rescaling of achievement values. The recent progress in optimization methods for ordered averages allows one to implement the WOWA RPM quite effectively as extension of the original constraints and criteria with simple linear inequalities.

Keywords: Multicriteria Decision Making, Interactive Methods, Efficiency, Reference Point Method, OWA, WOWA.

1. Introduction

Consider a decision problem defined by m–criteria optimization:

$$\max \{ (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})) \ : \ \mathbf{x} \in Q \} \tag{1}$$

where $\mathbf{x}$ denotes a vector of decision variables to be selected within the feasible set $Q \subset R^n$, and $\mathbf{f}(x) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ is a vector function that maps the feasible set Q into the criterion space R^m. We refer to elements of the criterion space as outcome vectors. An outcome vector $\mathbf{y}$ is attainable if it expresses outcomes of a feasible solution, i.e. $\mathbf{y} = \mathbf{f}(\mathbf{x})$ for some $\mathbf{x} \in Q$.

In order to make the multiple criteria model operational for the decision support process, one needs assume some solution concept well adjusted to the DM preferences. This can be achieved with the so-called quasi-satisficing approach proposed and developed mainly by Wierzbicki[8] as the Reference Point Method (RPM) having led to efficient implementations with many successful applications.[1,9] The RPM is an interactive technique working as follows. The DM specifies requirements in terms of reference levels, i.e., by introducing reference values for several individual outcomes. Depending on the specified reference levels, a special scalarizing achievement function is built which may be directly interpreted as expressing utility to be maximized. Maximization of the scalarizing achievement function generates an efficient solution to the multiple criteria problem. The computed efficient solution is presented to the DM as the current solution in a form that allows comparison with the previous ones and modification of the reference levels if necessary.

The scalarizing achievement function can be viewed as two-stage transformation of the original outcomes. First, the strictly monotonic partial achievement functions are built to measure individual performance with respect to given reference levels. Having all the outcomes transformed into a uniform scale of individual achievements they are aggregated at the second stage to form a unique scalarization. The RPM is based on the so-called augmented (or regularized) max-min aggregation. Thus, the worst individual achievement is essentially maximized but the optimization process is additionally regularized with the average achievement. The generic scalarizing achievement function takes the following form:[8]

$$S(\mathbf{y}) = \min_{1 \le i \le m} \{s_i(y_i)\} + \frac{\varepsilon}{m} \sum_{i=1}^{m} s_i(y_i) \qquad (2)$$

where ε is an arbitrary small positive number and $s_i : R \to R$, for $i = 1, 2, \ldots, m$, are the partial achievement functions measuring actual achievement of the individual outcomes y_i with respect to the corresponding reference levels. Let a_i denote the partial achievement for the ith outcome ($a_i = s_i(y_i)$) and $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ represent the achievement vector. Various functions s_i provide a wide modeling environment for measuring partial achievements.[9] The basic RPM model is based on a single vector of the reference levels, the aspiration vector $\mathbf{r}^a$ and the piecewise linear functions s_i.

Real-life applications of the RPM methodology usually deal with more complex partial achievement functions defined with more than one reference point.[9] In particular, the models taking advantages of two reference vectors: vector of aspiration levels $\mathbf{r}^a$ and vector of reservation levels $\mathbf{r}^r$[1] are used, thus allowing the DM to specify requirements by introducing acceptable and required values for several outcomes. The partial achievement function s_i can be interpreted then as a measure of the DM's satisfaction with the current value of outcome the ith criterion. It is a strictly increasing function of outcome y_i with value $a_i = 1$ if $y_i = r_i^a$, and $a_i = 0$ for $y_i = r_i^r$. Various functions can be built meeting those requirements. We use the piece-wise linear partial achievement function:[4]

$$s_i(y_i) = \begin{cases} \gamma(y_i - r_i^r)/(r_i^a - r_i^r), & y_i \leq r_i^r \\ (y_i - r_i^r)/(r_i^a - r_i^r), & r_i^r < y_i < r_i^a \\ \alpha(y_i - r_i^a)/(r_i^a - r_i^r) + 1, & y_i \geq r_i^a \end{cases} \tag{3}$$

where α and γ are arbitrarily defined parameters satisfying $0 < \alpha < 1 < \gamma$.

2. WOWA extension of the RPM

The crucial properties of the RPM are related to the max-min aggregation of partial achievements while the regularization is only introduced to guarantee the aggregation monotonicity. Unfortunately, the distribution of achievements may make the max-min criterion partially passive when one specific achievement is relatively very small for all the solutions. Maximization of the worst achievement may then leave all other achievements unoptimized. Nevertheless, the selection is then made according to linear aggregation of the regularization term instead of the max-min aggregation, thus destroying the preference model of the RPM.[3]

In order to avoid inconsistencies caused by the regularization, the max-min solution may be regularized according to the ordered averaging rules.[10] This is mathematically formalized as follows. Within the space of achievement vectors we introduce map $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$ which orders the coordinates of achievements vectors in a nonincreasing order, i.e., $\Theta(a_1, a_2, \ldots, a_m) = (\theta_1(\mathbf{a}), \theta_2(\mathbf{a}), \ldots, \theta_m(\mathbf{a}))$ iff there exists a permutation τ such that $\theta_i(\mathbf{a}) = a_{\tau(i)}$ for all i and $\theta_1(\mathbf{a}) \geq \theta_2(\mathbf{a}) \geq \ldots \geq \theta_m(\mathbf{a})$. The standard max-min aggregation depends on maximization of $\theta_m(\mathbf{a})$ and it ignores values of $\theta_i(\mathbf{a})$ for $i \leq m-1$. In order to take into account all the achievement values, one needs to maximize the weighted combination of the ordered achievements thus representing the so-called Ordered Weighted Averaging (OWA) aggregation.[10] Note that the weights are then assigned to the specific positions within the ordered achievements rather than to the partial achievements themselves. With the OWA aggregation one gets the following RPM model:

$$\max\{\sum_{i=1}^{m} w_i\theta_i(\mathbf{a}) \ : \ a_i = s_i(f_i(\mathbf{x})) \ \forall\ i,\ \mathbf{x} \in Q\ \} \tag{4}$$

where $w_1 < w_2 < \ldots < w_m$ are positive and strictly increasing weights. Actually, they should be significantly increasing to represent regularization of the max-min order. Note that the standard RPM model with the scalarizing achievement function (2) can be expressed as the OWA model (4) with weights $w_1 = \ldots = w_{m-1} = \varepsilon/m$ and $w_m = 1 + \varepsilon/m$ thus strictly increasing in the case of $m = 2$. Unfortunately, for $m > 2$ it abandons the differences in weighting of the largest achievement, the second largest one etc ($w_1 = \ldots = w_{m-1} = \varepsilon/m$). The OWA RPM model (4) allows one to differentiate all the weights by introducing increasing series (e.g. geometric ones).

Typical RPM model allow weighting of several achievements only by straightforward rescaling of the achievement values.[6] The OWA RPM model enables one to introduce importance weights to affect achievement importance by rescaling accordingly its measure within the distribution of achievements as defined in the so-called Weighted OWA (WOWA) aggregation.[7] Let $\mathbf{w} = (w_1, \ldots, w_m)$ be a vector of preferential (OWA) weights and let $\mathbf{p} = (p_1, \ldots, p_m)$ denote the vector of importance weights ($p_i \geq 0$ for $i = 1, 2, \ldots, m$ as well as $\sum_{i=1}^{m} p_i = 1$). The corresponding Weighted OWA aggregation of achievements $\mathbf{a} = (a_1, \ldots, a_m)$ is defined as follows:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{i=1}^{m} \omega_i\theta_i(\mathbf{a}), \qquad \omega_i = w^*(\sum_{k\leq i} p_{\tau(k)}) - w^*(\sum_{k<i} p_{\tau(k)}) \tag{5}$$

where w^* is a monotone increasing function that interpolates points $(\frac{i}{m}, \sum_{k\leq i} w_k)$ together with the point (0.0) and τ representing the ordering permutation for $\mathbf{a}$ (i.e. $a_{\tau(i)} = \theta_i(\mathbf{a})$). The WOWA may be expressed with more direct formula where preferential (OWA) weights w_i are applied to averages of the corresponding portions of ordered achievements (quantile intervals) according to the distribution defined by importance weights p_i:[5]

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{i=1}^{m} w_i m \int_{\frac{i-1}{m}}^{\frac{i}{m}} F_{\mathbf{a}}^{(-1)}(\xi)\ d\xi \tag{6}$$

where $\overline{F}_{\mathbf{a}}^{(-1)}$ is the stepwise function $\overline{F}_{\mathbf{a}}^{(-1)}(\xi) = \theta_i(\mathbf{a})$ for $\beta_{i-1} < \xi \leq \beta_i$. It can also be mathematically formalized as follows. First, we introduce the right-continuous cumulative distribution function (cdf) $F_{\mathbf{a}}(d) = \sum_{i=1}^{m} p_i \delta_i(d)$ where $\delta_i(d) = 1$ if $a_i \leq d$ and 0 otherwise. Next, we introduce the quantile function $F_{\mathbf{a}}^{(-1)} = \inf \{\eta : F_{\mathbf{a}}(\eta) \geq \xi\}$ for $0 < \xi \leq 1$ as the left-continuous inverse of $F_{\mathbf{a}}$, ie., $F_{\mathbf{a}}^{(-1)}(\xi) = \inf \{\eta : F_{\mathbf{a}}(\eta) \geq \xi\}$for $0 < \xi \leq 1$, and finally $\overline{F}_{\mathbf{a}}^{(-1)}(\xi) = F_{\mathbf{a}}^{(-1)}(1 - \xi)$.

Formula (6) defines the WOWA value applying preferential weights w_i to importance weighted averages within quantile intervals. It may be reformulated to use the tail averages

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{k=1}^{m} w'_k m L(\mathbf{a}, \mathbf{p}, \frac{k}{m}), \qquad L(\mathbf{y}, \mathbf{p}, \xi) = \int_0^{\xi} F_{\mathbf{y}}^{(-1)}(\alpha) d\alpha \tag{7}$$

where weights $w'_k = w_{m-k+1} - w_{m-k}$ for $k = 1, \ldots, m-1$ and $w'_m = w_1$ and $L(\mathbf{y}, \mathbf{p}, \xi)$ is defined by left-tail integrating of $F_{\mathbf{y}}^{(-1)}$. Values $L(\mathbf{a}, \mathbf{p}, \xi)$ for any $0 \leq \xi \leq 1$ can be given by optimization:

$$L(\mathbf{a}, \mathbf{p}, \xi) = \min_{s_i} \{\sum_{i=1}^{m} a_i s_i : \sum_{i=1}^{m} s_i = \xi, \quad 0 \leq s_i \leq p_i \quad \forall\, i\ \} \tag{8}$$

Introducing dual variable t corresponding to the equation $\sum_{i=1}^{m} s_i = \xi$ and variables d_i corresponding to upper bounds on s_i one gets the following LP dual expression of $L(\mathbf{a}, \mathbf{p}, \xi)$

$$L(\mathbf{a}, \mathbf{p}, \xi) = \max_{t, d_i} \{\xi t - \sum_{i=1}^{m} p_i d_i : t - d_i \leq a_i,\ d_i \geq 0 \quad \forall\, i\} \tag{9}$$

Following (7) and (9) taking into account piecewise linear partial achievement functions (3) one gets finally the following model for the WOWA RPM with piecewise linear partial achievement functions (3):

$$\begin{array}{lll} \max \sum_{k=1}^{m} w'_k z_k & \text{s.t. } z_k = k t_k - m \sum_{i=1}^{m} p_i d_{ik} & \forall\, k \\ & \mathbf{x} \in Q,\ y_i = f_i(\mathbf{x}) & \forall\, i \\ & a_i \geq t_k - d_{ik},\ d_{ik} \geq 0 & \forall\, i, k \\ & a_i \leq \gamma(y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\ & a_i \leq (y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\ & a_i \leq \alpha(y_i - r_i^a)/(r_i^a - r_i^r) + 1 & \forall\, i \end{array} \tag{10}$$

thus allowing for implementation of the entire WOWA RPM model as an LP expansion of the original problem.

Conclusions

The OWA aggregation with monotonic weights combines all the partial achievements allocating the largest weight to the worst achievement, the second largest weight to the second worst achievement, and so on. Further following the concept of Weighted OWA[7] the importance weighting of several achievements may be incorporated into the RPM. Such a WOWA enhancement of the RPM uses importance weights to affect achievement importance by rescaling accordingly its measure within the distribution of achievements rather than straightforward rescaling of achievement values.[6] The ordered regularizations are more complicated in implementation due to the requirement of pointwise ordering of partial achievements. However, the recent progress in optimization methods for ordered averages allows one to implement the OWA RPM quite effectively by taking advantages of piecewise linear expression of the cumulated ordered achievements. Similar, model can be achieved for the WOWA RPM resulting in the original constraints and criteria with simple linear inequalities.

References

1. A.Lewandowski, A.P.Wierzbicki (1989). *Aspiration Based Decision Support Systems – Theory, Software and Applications*. Berlin, Springer.
2. K.Miettinen, M.M.Mäkelä (2002). On Scalarizing Functions in Multiobjective Optimization, *OR Spectrum*, vol. 24, 193–213.
3. W.Ogryczak (2006). Nucleolar reference point method. *18th MCDM 2006*, http://www.dpem.tuc.gr/fel/mcdm2006/Papers/Ogryczak.pdf
4. W.Ogryczak, K.Studziński, K.Zorychta (1992). DINAS: A Computer-Assisted Analysis System for Multiobjective Transshipment Pproblems with Facility Location. *Comp. Opns. Res.*, vol. 19, 637–647.
5. W.Ogryczak, T.Śliwiński (2007). On Optimization of the Importance Weighted OWA Aggregation of Multiple Criteria. In *ICCSA 2007*, *LNCS*, vol. 4705, 804-817.
6. F.Ruiz, M.Luque, J.M.Cabello (2007). The role of the weights in reference point based interactive methods. In *DSTIS 2007*, Warsaw, Poland.
7. V.Torra (1997). The weighted OWA operator. *Int. J. Intell. Syst.*, vol. 12, 153–166.
8. A.P.Wierzbicki (1982). A Mathematical Basis for Satisficing Decision Making. *Math. Modelling*, vol. 3, 391–405.
9. A.P.Wierzbicki, M.Makowski, J.Wessels (2000). *Model Based Decision Support Methodology with Environmental Applications*. Dordrecht: Kluwer.
10. R.R.Yager (1988). On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Tr. Sys. Man Cyber.*, vol. 18, 183–190.