Report of the Institute of Control and Computation Engineering Warsaw University of Technology

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May, 2005

Report nr: 05–03

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Abstract

While making location decisions, the distribution of distances (effects) among the service recipients (clients) is an important issue. In order to comply with the minimization of distances as well as with an equal consideration of the clients, the concept of equitable efficiency should be used for the location model. The concept is based on extension of the standard efficiency concept with the Pigou-Dalton equity theory and it is mathematically equivalent to the stochastic dominance. Although rich with equitably efficient preference models, these approaches do not offer simple solution generation tools. Therefore, rather simplified mean-equity approaches are considered which quantify the problem in a lucid form of only two criteria: the mean, representing the mean outcome, and the equity: a scalar measure of the inequality of outcomes The mean-equity model is appealing to decision makers and allows a simple trade-off analysis, analytical or geometrical. On the other hand, for typical dispersion parameters used as inequality measures, the mean-equity approach may lead to inferior conclusions. Several inequality measures, however, can be combined with the mean itself into the optimization criteria generalizing the concept of the worst outcome and generating equitably consistent underachievement measures. In this paper we introduce general conditions for inequality measures sufficient to provide the equitable consistency of the corresponding underachievement measures.

Key Words. Location, Multiple Criteria, Efficiency, Equity, Fairness, Inequality Measures.

1 Introduction

The spatial distribution of public goods and services is influenced by facility location decisions and the issue of equity (or fairness) is important in many location decisions. Equity is usually quantified with the so–called inequality measures to be minimized. Inequality measures were primarily studied in economics (Atkinson, 1970; Sen, 1973; Young, 1994). The simplest inequality measures are based on the absolute measurement of the spread of outcomes. Variance is the most commonly used inequality measure of this type and it was also widely analyzed within various location models (Maimon, 1986; Berman, 1990; Carrizosa, 1999). However, Marsh and Schilling (1994) describe twenty different measures proposed in the literature to gauge the level

*This work was partially supported by The State Committee for Scientific Research under the grant KBN–3T11C00527

of equity in facility location alternatives. In economics one usually considers relative inequality measures normalized by mean outcome. Among many inequality measures perhaps the most commonly accepted by economists is the Gini index (Lorenz measure), which has been also analyzed in the location context (Berman and Kaplan, 1990; Erkut 1993; Mandell, 1971). One can easily notice that a direct minimization of typical inequality measures (especially relative ones) contradicts the minimization of individual outcomes. As noticed by Erkut (1993), it is rather a common flaw of all the relative inequality measures that while moving away from the clients to be serviced one gets better values of the measure as the relative distances become closer to one–another. As an extreme, one may consider an unconstrained continuous (single–facility) location problem and find that the facility located at (or near) infinity will provide (almost) perfectly equal service (in fact, rather lack of service) to all the clients.

Although minimization of the inequality measures contradicts the minimization of individual outcomes, the inequality minimization itself can be consistently incorporated into locational models. The notion of equitable multiple criteria optimization (Kostreva and Ogryczak, 1999a) introduces the preference structure that complies with both the outcomes minimization and with the inequality minimization rules (Sen, 1973). The equitable efficient solutions represent a subset of all efficient (Pareto-optimal) solutions which takes into account the inequality minimization according to the Pigou-Dalton approach. The equitable optimization is well suited for the locational analysis (Kostreva and Ogryczak, 1999; Ogryczak, 2000). It turns out that equitably efficient solution concepts may be modeled with the standard multiple criteria optimization applied to the cumulative ordered outcomes. The center solution concept represent only first criterion and in order to guarantee the equitable efficiency of a selected location pattern one needs to take into account all the ordered outcomes like in the lexicographic center (Ogryczak, 1997) which is a lexicographic refinement of the center solution concept. The entire multiple criteria ordered model is rich with various equitably efficient solution concepts (Ogryczak and Zawadzki, 2002; Kostreva, Ogryczak and Wierzbicki, 2004). Although the the cumulated ordered outcomes can be expressed with linear programming models (Ogryczak and Tamir, 2003), these approaches requires the disaggregation of location problem with the client weights which usually dramatically increases the problem size.

For typical inequality measures a simplified bicriteria mean-equity model is computationally very attractive since both the criteria are well defined directly for the weighted location problem without necessity of its disaggregation but it may result in solutions which are inefficient. Therefore, we are interested in a proper use of the mean-equity models in a way to guarantee the equitable efficiency of selected solutions. It turns out that, under the assumption of bounded trade-offs, the bicriteria mean-equity approaches for selected absolute inequality measures (maximum upper deviation, mean semideviation or mean absolute difference) comply with the rules of equitable multiple criteria optimization (Ogryczak, 2000). In other words, several inequality measures can be combined with the mean itself into the optimization criteria generalizing the concept of the worst outcome and generating equitably consistent underachievement measures. We generalize those findings by introducing simple sufficient conditions for inequality measures to keep this consistency property. It allows us to identify more inequality measures which can be effectively used to incorporate equity factors into various location while preserving the consistency with distance minimization. Among others the standard upper semideviation turns out to be such a consistent inequality measure.

The paper is organized as follows. In the next section we introduce the problem and the basic inequality measures. In Section 3 the equitable optimization with the preference structure that complies with both the efficiency (Pareto-optimality) principle and with the Pigou-Dalton principle of transfers is discussed and the underachievement criteria are introduced. Further, in Section 4, the equitable consistency of the underachievement criteria is analyzed and sufficient conditions for the inequality measures to keep this consistency property are introduced. There is shown that properties of convexity and positive homogeneity together with some boundedness condition is sufficient for a typical inequality measure to guarantee the corresponding equitable consistency.

2 Efficiency and inequality measures

The generic location problem that we consider may be stated as follows. There is given a set $I = \{1, 2, \dots, m\}$ of m clients (service recipients). Each client is represented by a specific point in the geographical space. There is also given a set Q of location patterns (location decisions). For each client i ($i \in I$) a function $f_i(\mathbf{x})$ of the location pattern $\mathbf{x}$ has been defined. This function, called the individual objective function, measures the outcome (effect) $y_i = f_i(\mathbf{x})$ of the location pattern for client i (Marsh and Schilling, 1994). In the simplest problems an outcome usually expresses the distance. However, we emphasize to the reader that we do not restrict our considerations to the case of outcomes measured as distances. They can be measured (modeled) as travel time, travel costs as well as in a more subjective way as relative travel costs (e.g., travel costs by clients incomes) or ultimately as the levels of clients dissatisfaction (individual disutility) of location decisions. In typical formulations of location problems related to desirable facilities a smaller value of the outcome (distance) means a better effect (higher service quality or client satisfaction). This remains valid for location of obnoxious facilities if the distances are replaced with their complements to some large number or other (decreasing) disutility function of distances. Therefore, without loss of generality, we can assume that each individual outcome y_i is to be minimized. This allows us to consider the generic location problem as the multiple criteria minimization (Ogryczak, 1997, 1999):

$$\min \{\mathbf{f}(\mathbf{x}) \;:\; \mathbf{x} \in Q\} = \min \{\mathbf{y} \;:\; \mathbf{y} \in A\}, \tag{1}$$

where $\mathbf{f} = (f_1, \dots, f_m)$ is a vector-function that maps feasible decisions (locations) $\mathbf{x} \in Q$ into the attainable outcome vectors $\mathbf{y} \in A = \{\mathbf{y} \in R^m \;:\; \mathbf{y} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} \in Q\}$.

We do not assume any special form of the problem constraints allowing the feasible set to be a general, possibly discrete (nonconvex), set. Similarly, we do not assume any special form of the individual objective functions nor their special properties (like convexity) while analyzing properties of the solution concepts. We have only assumed a finite set of clients for the minimization of the individual outcomes. Therefore, the results of our analysis apply to various classes of location problems covering continuous as well as discrete and special network models (c.f., Love, Morris and Wesolowsky, 1988; Francis, McGinnis and White, 1992; Current, Min and Schilling, 1990; Mirchandani and Francis, 1990; Labbé, Peeters and Thisse, 1996).

Model (1) only says that we are interested in the minimization of all outcome functions f_i for $i \in I$. In order to make it operational, one needs to assume some solution concept. Typical solution concepts for locations problems are based on the minimization of some scalar achievement function $C(\mathbf{y})$ of outcome vectors $\mathbf{y}$. Most classical location studies focus on the minimization of the mean (or total) distance (the median concept) or the minimization of the maximum distance (the center concept) to the service facilities (Morrill and Symons, 1977).

Since for each outcome the smaller value is preferred, some outcome vectors are clearly dominated by others. We say that outcome vector $\mathbf{y}'$ *(Pareto) dominates* $\mathbf{y}''$ ($\mathbf{y}' \prec \mathbf{y}''$), iff $y'_i \leq y''_i$ for all $i \in I$ where at least one strict inequality holds. We say that a location pattern $\mathbf{x} \in Q$ is an *Pareto-efficient* solution of the multiple criteria problem (1), iff $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is nondominated. The latter refers to the commonly used definition of the efficient solutions as feasible solutions for which one cannot improve any criterion without worsening another (e.g., Steuer, 1986).

Frequently, one may be interested in putting into location model some additional client weights $v_i > 0$. Typically the model of distribution weights is introduced to represent the service demand thus defining distribution of outcomes $y_i = f_i(\mathbf{x})$ according to measures defined by the weights v_i for $i = 1, \ldots, m$. Note that the such distribution weights allows us for a clear interpretation of weights as the client repetitions at the same place. Splitting a client into two clients sharing the demand at the same geographical point does not cause any change of the final distribution of outcomes. For theoretical considerations one may assume that the problem is transformed (disaggregated) to the unweighted one (that means all the client weights are equal to 1). Note that such a disaggregation is possible for integer as well as rational client weights, but it usually dramatically increases the problem size. Therefore, we are interested in solution concepts which can be applied directly to the weighted problem.

Alternatively, scaling weights might be used as client importance factors thus defining outcomes $y_i = v_i f_i(\mathbf{x})$ uniformly distributed for $i = 1, \ldots, m$. Such an usage of weights represents actually redefinition of outcome values. Recall that we consider the outcome values $f_i(\mathbf{x})$ as distance dependent but allowing any specific form of this function thus any weighted scaling is already taken into account within the outcomes definition. Actually, the distance scaling model means the use of unweighted location problem with a very simple modification of distances. Therefore, our analysis is focused on the model of distribution weights.

As mentioned, for some theoretical considerations it might be convenient to disaggregate the weighted problem into the unweighted one. Therefore, to simplify the analysis we will assume integer weights v_i, although while discussing solution concepts we will use the normalized client weights

$$\bar{v}_i = v_i / \sum_{i=1}^{m} v_i \quad \text{for } i = 1, 2, \ldots, m$$

rather than the original quantities v_i. Note that, in the case of unweighted problem (all $v_i = 1$), all the normalized weights are given as $\bar{v}_i = 1/m$. Furthermore, to avoid possible misunderstandings between weighted and the corresponding unweighted form of outcomes we will use the following notation. Vector $\mathbf{y} = (y_i)_{i \in I} = (y_1, y_2, \ldots, y_m)$ denotes the unweighted outcomes (possibly disaggregated if necessary) while the equivalent weighted outcomes of the aggregated problem are denoted by vector $(y_{v_i})_{i \in I_v}$.

Note that the classical solution concepts of median and center are well defined for aggregated location models using (distribution) demand weights $v_i > 0$. Exactly, the *median* solution concept is defined by minimization the *mean* outcome

$$\mu(\mathbf{y}) = \frac{1}{m} \sum_{i=1}^{m} y_i = \sum_{i \in I_v} \bar{v}_i y_{v_i}, \tag{2}$$

i.e., by the optimization problem

$$\min \; \{\mu(\mathbf{f}(\mathbf{x})) \;\; : \quad \mathbf{x} \in Q\}. \tag{3}$$

In the above problem the objective function is defined as the mean (average) outcome but the problem (3) itself is also equivalent to minimization of the total outcome $\sum_{i=1}^{m} y_i = \sum_{i\in I_v} v_i y_{v_i}$. The *center* solution concept is defined by minimization of the maximum (worst) outcome

$$M(\mathbf{y}) = \max_{i\in I} y_i = \max_{i\in I_v} y_{v_i}, \tag{4}$$

thus resulting in the optimization problem

$$\min \{M(\mathbf{f}(\mathbf{x})) \;:\; \mathbf{x}\in Q\}. \tag{5}$$

Note the maximum outcome $M(\mathbf{y})$ is not affected by the distribution weights at all and the same applies to center solution itself. The weighted center solution concepts considered in some location models (Labbé, Peeters and Thisse, 1996) represent distance scaling weights rather than the distribution weights. In our analysis such scaling weights are considered as included within the outcome functions $f_i(\mathbf{x})$.

The individual outcomes in our multiple criteria location model express the same quantity (usually the distance) for various clients. Thus the outcomes are uniform in the sense of the scale used and their values are directly comparable. Moreover, especially when locating public facilities, we want to consider all the clients impartially and equally. Thus the distribution of distances (outcomes) among the clients is more important than the assignment of several distances (outcomes) to the specific clients. Both the center and the median solution concepts minimize only simple scalar characteristics of the distribution: the maximum (the worst) outcome and the mean outcome, respectively.

Equity is, essentially, an abstract socio–political concept that implies fairness and justice (Young, 1994). Nevertheless, equity is usually quantified with the so–called inequality measures to be minimized. Inequality measures were primarily studied in economics (Sen, 1973). However, Marsh and Schilling (1994) described twenty different measures proposed in the literature to gauge the level of equity in facility location alternatives. Typical inequality measures are some deviation type dispersion characteristics. They are *translation invariant*

$$\varrho(\mathbf{y} + a\mathbf{e}) = \varrho(\mathbf{y}) \quad \text{for any} \quad \text{outcome vector } \mathbf{y} \text{ and real number } a \tag{6}$$

where $\mathbf{e}$ vector of units $(1,\ldots,1)$, thus being not affected by any shift of the outcome scale. Moreover, the inequality measures are also *inequality relevant* which means that they are equal to 0 in the case of a perfectly equal outcomes while taking positive values for any unequal one.

The simplest inequality measures are based on the absolute measurement of the spread of outcomes, like the *mean (absolute) difference* (also called the Gini's mean difference)

$$D(\mathbf{y}) = \frac{1}{2m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}|y_i - y_j| = \frac{1}{2}\sum_{i\in I_v}\sum_{j\in I_v}|y_{v_i} - y_{v_j}|\bar{v}_i\bar{v}_j \tag{7}$$

or the *maximum (absolute) difference*

$$S(\mathbf{y}) = \max_{i,j=1,\ldots,m}|y_i - y_j| = \max_{i,j\in I_v}|y_{v_i} - y_{v_j}|. \tag{8}$$

In the location framework better intuitive appeal may have inequality measures related to deviations from the mean outcome (Mulligan, 1991) like the *mean (absolute) deviation*

$$\delta(\mathbf{y}) = \frac{1}{m}\sum_{i=1}^{m}|y_i - \mu(\mathbf{y})| = \sum_{i\in I_v}|y_{v_i} - \mu(\mathbf{y})|\bar{v}_i \tag{9}$$

or the *maximum (absolute) deviation* (López-de-los-Mozos and Mesa, 2001)

$$R(\mathbf{y}) = \max_{i=1,\ldots,m} |y_i - \mu(\mathbf{y})| = \max_{i \in I_v} |y_{v_i} - \mu(\mathbf{y})|. \tag{10}$$

Note that the *standard deviation* σ (or the *variance* σ^2) represents both the deviations and the spread measurement as

$$\begin{aligned} \sigma(\mathbf{y}) &= \sqrt{\frac{1}{m}\sum_{i=1}^{m}(y_i - \mu(\mathbf{y}))^2} = \sqrt{\frac{1}{2m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}(y_i - y_j)^2} \\ &= \sqrt{\sum_{i \in I_v}(y_{v_i} - \mu(\mathbf{y}))^2 \bar{v}_i} = \sqrt{\frac{1}{2}\sum_{i \in I_v}\sum_{j \in I_v}(y_{v_i} - y_{v_j})^2 \bar{v}_i \bar{v}_j}. \end{aligned} \tag{11}$$

Deviational measures may be focused on the upper semideviations as related to worsening of outcome while ignoring downside semideviations related to improvement of outcome. One may define the *maximum (upper) semideviation*

$$\Delta(\mathbf{y}) = \max_{i=1,\ldots,m} (y_i - \mu(\mathbf{y})) = \max_{i \in I_v}(y_{v_i} - \mu(\mathbf{y})), \tag{12}$$

the *mean absolute (upper) semideviation*

$$\bar{\delta}(\mathbf{y}) = \frac{1}{m}\sum_{y_i \geq \mu(\mathbf{y})} (y_i - \mu(\mathbf{y})) = \sum_{y_{v_i} \geq \mu(\mathbf{y})} (y_{v_i} - \mu(\mathbf{y}))\bar{v}_i, \tag{13}$$

and the *standard (upper) semideviation*

$$\bar{\sigma}(\mathbf{y}) = \sqrt{\frac{1}{m}\sum_{y_i \geq \mu(\mathbf{y})} (y_i - \mu(\mathbf{y}))^2} = \sqrt{\sum_{y_{v_i} \geq \mu(\mathbf{y})} (y_{v_i} - \mu(\mathbf{y}))^2 \bar{v}_i}. \tag{14}$$

In income economics, relative inequality measures (normalized by mean outcome) are commonly used with the Gini coefficient as a typical example. The latter is a relative measure of the mean absolute difference and has been also analyzed in the location context (Mandell, 1991; Mulligan, 1991; Erkut, 1993). One can easily notice that direct minimization of relative inequality measures contradicts the minimization of individual outcomes (Erkut, 1993). Unfortunately, the same applies to all dispersion type inequality measures, including the upper semideviations.

3 Equitable efficiency and underachievement criteria

As recalled in the previous section direct use of the inequality measure minimization may result in locations strictly worsening all the distances. In other words the inequality measures minimizations may contradict the outcomes minimization. It does not mean, however, that the inequality minimization itself cannot be consistently incorporated into the location models. There exist models of equitable optimization based on the majorization theory (Hardy, Littlewood and Pólya, 1934; Marshall and Olkin, 1979) which are consistent both with the Pareto-efficiency and theories of inequality measurement (in particular the Pigou–Dalton approach). Namely, the Pareto dominance relation is transitive and can be transitively extended with additional relations representing inequality minimization (Ogryczak, 1997a; Kostreva and Ogryczak, 1999a). The resulting notion of equitable multiple criteria optimization is based on the preference structure that complies with both the Pareto-efficiency and with the inequality measurement rules, and it is well suited for the locational analysis (Kostreva and Ogryczak, 1999, Ogryczak, 2000).

First of all, the equity requires impartiality of evaluation, thus focusing on the distribution of outcome values while ignoring their ordering (or individual assignment). That means, in the multiple criteria problem (1) we are interested in a set of outcome values without taking into account which outcome is taking a specific value. Hence, we assume that the dominance relation is impartial (anonymous, symmetric). In terms of the unweighted outcomes (disaggregated if necessary), it may be written as the following axiom

$$(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}) \cong (y_1, y_2, \ldots, y_m) \quad \text{for any } \tau \in \Pi(I) \tag{15}$$

where $\Pi(I)$ is the set of all permutations of the set I.

Further, one needs equitability of outcomes which causes that the preference model should satisfy the (Pigou–Dalton) principle of transfers. The principle of transfers states that a transfer of any small amount from an outcome to any other relatively worse–off outcome results in a more preferred outcome vector. Again, in terms of the problem with unweighted outcomes (disaggregated if necessary), the principle of transfers takes the form of the following property of the dominance relation

$$y_{i'} > y_{i''} \quad \Rightarrow \quad \mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''} \prec \mathbf{y} \quad \text{for } 0 < \varepsilon < y_{i'} - y_{i''};\ i', i'' \in I. \tag{16}$$

Note that the requirements of impartiality (15) and equitability (expressed with the principle of transfers (16)) themselves do not contradict to the Pareto dominance. Therefore, one may unify them by the transitivity rule getting a consistent concept of the equitable dominance. The relation of (weak) equitable dominance $\mathbf{y}' \preceq_e \mathbf{y}''$ denotes that there exists a finite sequence of vectors $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \ldots, \mathbf{y}^t$ such that $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'} + \varepsilon_k \mathbf{e}_{i''}$, $0 \leq \varepsilon_k \leq y^{k-1}_{i'} - y^{k-1}_{i''}$ for $k = 1, 2, \ldots, t$ and there exists a permutation τ such that $y'_{\tau(i)} \leq y^t_i$ for all $i \in I$. We say that outcome vector $\mathbf{y}'$ *equitably dominates* $\mathbf{y}''$ (the strict dominance relation $\mathbf{y}' \prec_e \mathbf{y}''$) if and only if, $\mathbf{y}' \preceq_e \mathbf{y}''$ but not $\mathbf{y}'' \preceq_e \mathbf{y}'$. Note that according to equitable dominance a solution generating all three outcomes equal to 2 is considered better than any solution generating individual outcomes: 4, 2 and 0 (due to principle of transfers), while it remains worse than a solution generating one outcome 0 and two other equal to 2 (due to the Pareto dominance). We say that a location pattern $\mathbf{x} \in Q$ is *equitably efficient*, if and only if there does not exist any $\mathbf{x}' \in Q$ such that $\mathbf{f}(\mathbf{x}')$ equitably dominates $\mathbf{f}(\mathbf{x})$.

The relation of equitable dominance $\preceq_e$ can be expressed as a vector inequality on the cumulative ordered outcomes. For the unweighted problem this can be mathematically formalized as follows. First, we introduce the ordering map $\Theta : R^m \rightarrow R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \geq \theta_2(\mathbf{y}) \geq \cdots \geq \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, 2, \ldots, m$. This allows us to focus on distributions of outcomes impartially. Next, we apply cumulation to the ordered outcome vectors to get quantities

$$\bar{\theta}_i(\mathbf{y}) = \sum_{j=1}^{i} \theta_j(\mathbf{y}) \quad \text{for } i = 1, 2, \ldots, m. \tag{17}$$

expressing, respectively, the largest outcome, the total of the two largest outcomes, the total of the three largest outcomes, etc. Pointwise comparison of the cumulated ordered outcomes $\bar{\Theta}(\mathbf{y})$ was extensively analyzed within the theory of majorization (Marshall and Olkin, 1979), where it is called the relation of weak submajorization. The theory of majorization includes the results which allows one to derive the following theorem (Kostreva and Ogryczak, 1999a).

Theorem 1 *Outcome vector $\mathbf{y}' \in Y$ equitably dominates $\mathbf{y}'' \in Y$, if and only if $\bar{\theta}_i(\mathbf{y}') \leq \bar{\theta}_i(\mathbf{y}'')$ for all $i \in I$ where at least one strict inequality holds.*

The equitable optimization for general weighted problems can be mathematically formalized as follows. First, we introduce the left-continuous right tail cumulative distribution function (cdf):

$$F_{\mathbf{y}}(d) = \sum_{i \in I_v} \bar{v}_i \delta_i(d) \quad \text{where} \quad \delta_i(d) = \begin{cases} 1 & \text{if } y_{v_i} \geq d \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

which for any real (outcome) value d provides the measure of outcomes greater or equal to d. Note that the requirement of impartiality means that two outcome vectors $\mathbf{y}'$ and $\mathbf{y}''$ resulting in identical cdf are indifferent. Next, we introduce the quantile function $F_{\mathbf{y}}^{(-1)}$ as the right-continuous inverse of the cumulative distribution function $F_{\mathbf{y}}$:

$$F_{\mathbf{y}}^{(-1)}(\beta) = \sup\ \{\eta : F_{\mathbf{y}}(\eta) \geq \beta\} \quad \text{for } 0 < \beta \leq 1.$$

By integrating $F_{\mathbf{y}}^{(-1)}$ one gets:

$$F_{\mathbf{y}}^{(-2)}(0) = 0 \quad \text{and} \quad F_{\mathbf{y}}^{(-2)}(\beta) = \int_0^\beta F_{\mathbf{y}}^{(-1)}(\alpha) d\alpha \quad \text{for } 0 < \beta \leq 1, \tag{19}$$

where $F_{\mathbf{y}}^{(-2)}(1) = \mu(\mathbf{y})$. Graphs of functions $F_{\mathbf{y}}^{(-2)}(v)$ (with respect to v) take the form of concave curves, the *(upper) absolute Lorenz curves.* The absolute Lorenz curves defines the relation (partial order) equivalent to the equitable dominance. Exactly, outcome vector $\mathbf{y}'$ equitably dominates $\mathbf{y}''$, if and only if $F_{\mathbf{y}'}^{(-2)}(\beta) \leq F_{\mathbf{y}''}^{(-2)}(\beta)$ for all $\beta \in (0;1]$ where at least one strict inequality holds. Note that for the expanded form to the unweighted outcomes, the absolute Lorenz curve is completely defined by the values of the (cumulated) ordered outcomes. Hence, $\bar{\theta}_i(\mathbf{y}) = mF_{\mathbf{y}}^{(-2)}(i/m)$ for $i = 1, 2, \ldots, m$, and pointwise comparison of cumulated ordered outcomes is enough to justify equitable dominance.

Alternatively, the equitable dominance can be expressed on the cumulative distribution functions. Having introduced the left-continuous right tail cumulative distribution function (18), one may further integrate it to get the second order cumulative distribution function $F_{\mathbf{y}}^{(2)}(\eta) = \int_\eta^\infty F_{\mathbf{y}}(\xi) d\xi$ for $\eta \in R$, representing average exceed over any real target τ. Graphs of functions $F_{\mathbf{y}}^{(2)}(\eta)$ (with respect to η) take the form of convex decreasing curves (Ogryczak, 1997a). By the theory of convex conjugent functions, the pointwise comparison of the second order cumulative distribution functions provides an alternative characterization of the equitable dominance relation (Ogryczak and Ruszczyński, 2002). Exactly, $\mathbf{y}'$ equitably dominates $\mathbf{y}''$, if and only if $F_{\mathbf{y}'}^{(2)}(\eta) \leq F_{\mathbf{y}''}^{(2)}(\eta)$ for all η where at least one strict inequality holds.

Furthermore, the classical results of Hardy, Littlewood and Pólya (1934) allow us to refer the equitable dominance to the mean utility. For any strictly convex, increasing utility function $u : R \rightarrow R$, if outcome vector $\mathbf{y}'$ equitably dominates $\mathbf{y}''$, then

$$\frac{1}{m}\sum_{i=1}^{m} u(y'_i) = \sum_{i\in I_v} \bar{v}_i u(y'_{v_i}) \leq \frac{1}{m}\sum_{i=1}^{m} u(y''_i) = \sum_{i\in I_v} \bar{v}_i u(y''_{v_i}).$$

Finally, there are three alternative analytical characterizations of the relation of equitable dominance as specified in the following theorem. Note that according to condition (iii), the equitable dominance is actually the so-called increasing convex order which is more commonly known as the second degree stochastic dominance (SSD) or stop loss order (Mueller and Stoyan, 2002).

Theorem 2 *For any outcome vectors $\mathbf{y}', \mathbf{y}'' \in A$ each of the three following conditions is equivalent to the (weak) equitable dominance $\mathbf{y}' \preceq_e \mathbf{y}''$:*
(i) $F^{(-2)}_{\mathbf{y}'}(\beta) \leq F^{(-2)}_{\mathbf{y}''}(\beta)$ for all $\beta \in (0;1]$;
(ii) $F^{(2)}_{\mathbf{y}'}(\eta) \leq F^{(2)}_{\mathbf{y}''}(\eta)$ for all real η;
(iii) $\sum_{i\in I_v} \bar{v}_i u(y'_i) \leq \sum_{i\in I_v} \bar{v}_i u(y''_i)$ for any convex, increasing function u.

We say that a solution concept (achievement function) $C(\mathbf{y})$ is *equitably consistent* if

$$\mathbf{y}' \preceq_e \mathbf{y}'' \quad \Rightarrow \quad C(\mathbf{y}') \leq C(\mathbf{y}''). \tag{20}$$

The relation of equitable consistency is called *strong* if, in addition, the following holds $\mathbf{y}' \prec_e \mathbf{y}'' \quad \Rightarrow \quad C(\mathbf{y}') < C(\mathbf{y}'')$.

According to condition (iii) of Theorem 2, for any strictly convex, increasing function $u : R \rightarrow R$, the solution concept defined by achievement function $C(\mathbf{y}) = \sum_{i=1}^{m} u(y_i)$ is equitably consistent. Various convex functions u can be used to define such equitable solution concepts. In the case of the outcomes restricted to positive values, any p-power y^p is a strictly positive and convex function for $p > 1$. This justifies the L_p norms as a source of equitable solution concepts, since the minimization of any such norm $\|\mathbf{y}\|_p$ is then equivalent to the minimization of $\|\mathbf{y}\|_p^p = \sum_{i=1}^{m} y_i^p$.

Condition (i) of Theorem 2 (or directly Theorem 1) permits one to seek equitably efficient location patterns as efficient solutions of the multiple criteria problem with objectives $\bar{\Theta}(\mathbf{f}(\mathbf{x}))$ (c.f. Kostreva and Ogryczak, 1999):

$$\min \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) \;:\; \mathbf{x} \in Q\}. \tag{21}$$

The worst outcome (4) and the mean outcome (2) correspond, respectively, to the first and to the last (m–th) criterion in problem (21). Thus both the center and the median concepts use only single objective from the multiple criteria problem (21). It means that both the concepts are equitably consistent in the sense of (20). They are not strongly consistent and the solutions can be equitably dominated by some alternative center or median solutions, respectively. In order to guarantee the equitable efficiency of a selected location pattern one need to take into account all the criteria of (21) like in the *lexicographic center* (Ogryczak, 1997). The lexicographic center is a refinement of the center solution concept which corresponds to the lexicographic approach to multicriteria optimization in (21) (Kostreva and Ogryczak, 1999). Although the cumulated ordered outcomes (17) can be expressed with linear programming models (Ogryczak and Tamir, 2003), the multicriteria ordered model (21) is, in general, rather hard to implement as it requires the disaggregation of a location problem with the client weights v_i which usually dramatically increases the problem size.

As a simplified approach one may consider a bicriteria mean-equity model (Mandell, 1991):

$$\min \{(\mu(\mathbf{f}(\mathbf{x})), \varrho(\mathbf{f}(\mathbf{x}))) \ : \quad \mathbf{x} \in Q\} \tag{22}$$

taking into account both the efficiency with minimization of the mean outcome $\mu(\mathbf{y})$ and the equity with minimization of an inequality measure $\varrho(\mathbf{y})$. For typical inequality measures bicriteria model (22) is computationally very attractive since both the criteria are well defined directly for the weighted location problem without necessity of its disaggregation. We have pointed out that direct minimization of any dispersion type inequality measures may contradict the efficiency in the sense of outcomes minimization. Unfortunately, the same applies to the bicriteria mean-equity models. This can be illustrated by a trivial example of two alternative locations generating outcome vectors (say in kilometers) $\mathbf{y}' = (0, 1)$ and $\mathbf{y}'' = (5, 5)$, respectively. Note that the perfectly equal outcome vector $\mathbf{y}''$ with both the distances 5 is obviously worse than the unequal vector $\mathbf{y}'$ giving the distances 0 and 1, respectively. Actually, $\mathbf{y}'$ Pareto dominates $\mathbf{y}''$. Nevertheless, $\varrho(\mathbf{y}'') = 0$ for any dispersion type inequality measure ϱ while $\varrho(\mathbf{y}') > 0$ for each such a measure. Hence, one must accept that $\mathbf{y}''$ is efficient in the corresponding bicriteria mean-equity model.

Note that the lack of consistency with the equitable dominance applies also to the maximum semideviation $\Delta(\mathbf{y})$ (12) whereas adding this measure to the mean $\mu(\mathbf{y})+\Delta(\mathbf{y}) = M(\mathbf{y}) = \bar{\theta}_1(\mathbf{y})$ results in the worst outcome and thereby the first criterion of the ordered multicriteria model (21). In other words, although a direct use of the maximum semideviation contradicts the efficiency, the measure can be used complementary to the mean leading to the worst outcome criterion which is equitably consistent. This construction can be generalized for various (dispersion type) inequality measures. For any inequality measure ϱ we introduce the corresponding underachievement function defined as the sum of the mean outcome and the inequality measure itself, i.e.

$$M_{\varrho}(\mathbf{y}) = \mu(\mathbf{y}) + \varrho(\mathbf{y}). \tag{23}$$

In the case of maximum semideviation the corresponding underachievement $M_{\Delta}(\mathbf{y})$ function represents the worst outcome $M(\mathbf{y})$. Similarly, in the case of mean semideviation one gets the underachievement function

$$M_{\bar{\delta}}(\mathbf{y}) = \mu(\mathbf{y}) + \bar{\delta}(\mathbf{y}) = \frac{1}{m}\sum_{i=1}^{m} \max\{y_i, \mu(\mathbf{y})\} = \sum_{i \in I_v} \bar{v}_i \max\{y_{v_i}, \mu(\mathbf{y})\}$$

representing the mean underachievement, and in the case of mean absolute difference the corresponding underachievement function

$$M_D(\mathbf{y}) = \mu(\mathbf{y}) + D(\mathbf{y}) = \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m} \max\{y_i, y_j\} = \sum_{i \in I_v}\sum_{j \in I_v} \bar{v}_i \bar{v}_j \max\{y_{v_i}, y_{v_j}\}$$

represents the mean pairwise worse outcome. Both the above underachievement measures are equitably consistent (Ogryczak, 2000). This leads to us a very important problem of identification some clear conditions for inequality measures ϱ sufficient to guarantee that the corresponding underachievement measures are equitably consistent.

4 Consistency results

Inequality measures in mean-equity models are translation invariant (6) and inequality relevant deviation type measures (dispersion parameters). Thus, they are not affected by any shift of the outcome scale and they are equal to 0 in the case of a perfectly equal outcomes while taking positive values for any unequal one. Moreover, they depend only on the distribution of outcomes thus in terms of the unweighted location model they are impartial, i.e., $\varrho(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}) = \varrho(y_1, y_2, \ldots, y_m)$ for any permutation τ. Unfortunately, as discussed earlier, such inequality measures are not consistent with the equitable optimization or axiomatic models of equitable preferences (Marshall and Olkin, 1979; Kostreva and Ogryczak, 1999). Indeed, in the bicriteria mean-equity model its efficient set may contain equitably inferior locations characterized by a small inequality but also very high distances.

This flaw can be overcome by replacing the original mean-equity bicriteria optimization (22) with the following bicriteria problem:

$$\min \{(\mu(\mathbf{f}(\mathbf{x})), M_\varrho(\mathbf{f}(\mathbf{x}))) \;:\quad \mathbf{x} \in Q\} \tag{24}$$

where the second objective represents the corresponding underachievement measure (23). Note that for any inequality measure $\varrho(\mathbf{y}) \geq 0$ one gets $M_\varrho(\mathbf{y}) \geq \mu(\mathbf{y})$ thus really expressing underachievements (comparing to mean) from the perspective of outcomes being minimized.

The equitable consistency of inequality measures may be formalized as follows. We say that inequality measure $\varrho(\mathbf{y})$ is *mean-complementary equitably consistent* if the corresponding underachievement measure $M_\varrho(\mathbf{y})$ is equitably consistent, i.e.,

$$\mathbf{y}' \preceq_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') + \varrho(\mathbf{y}') \leq \mu(\mathbf{y}'') + \varrho(\mathbf{y}''). \tag{25}$$

The relation of equitable (mean-complementary) consistency is called *strong* if, in addition to (25), the following holds

$$\mathbf{y}' \prec_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') + \varrho(\mathbf{y}') < \mu(\mathbf{y}'') + \varrho(\mathbf{y}''). \tag{26}$$

Theorem 3 *If the inequality measure $\varrho(\mathbf{y})$ is mean-complementary equitably consistent (25), then except for outcomes with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$, every efficient solution of the bicriteria problem (24) is an equitably efficient location. In the case of strong consistency (26), every location pattern $\mathbf{x} \in Q$ efficient to (24) is, unconditionally, equitably efficient.*

Proof. Let $\mathbf{x}^0 \in Q$ be an efficient solution of (24). Suppose that $\mathbf{x}^0$ is not equitably efficient. This means, there exists $\mathbf{x} \in Q$ such that $\mathbf{y} = \mathbf{f}(\mathbf{x}) \prec_e \mathbf{y}^0 = \mathbf{f}(\mathbf{x}^0)$. Then, it follows $\mu(\mathbf{y}) \leq \mu(\mathbf{y}^0)$, and simultaneously $\mu(\mathbf{y}) + \varrho(\mathbf{y}) \leq \mu(\mathbf{y}^0) + \varrho(\mathbf{y}^0)$, by virtue of the mean-complementary equitable consistency (25). Since $\mathbf{x}^0$ is efficient to (24) no inequality can be strict, which implies $\mu(\mathbf{y}) = \mu(\mathbf{y}^0)$ and $\varrho(\mathbf{y}) = \varrho(\mathbf{y}^0)$.

In the case of the strong mean-complementary equitable consistency (26), the supposition $\mathbf{y} = \mathbf{f}(\mathbf{x}) \prec_e \mathbf{y}^0 = \mathbf{f}(\mathbf{x}^0)$ implies $\mu(\mathbf{y}) \leq \mu(\mathbf{y}^0)$ and $\mu(\mathbf{y}) + \varrho(\mathbf{y}) < \mu(\mathbf{y}^0) + \varrho(\mathbf{y}^0)$ which contradicts the efficiency of $\mathbf{x}^0$ with respect to (24). Hence, $\mathbf{x}^0$ is equitably efficient. □

An important advantage of mean-equity approaches is the possibility of a pictorial trade-off analysis. Having assumed a trade-off coefficient λ between the inequality measure $\varrho(\mathbf{y})$ and the mean outcome, one may directly compare real values of $\mu(\mathbf{y}) + \lambda\varrho(\mathbf{y})$. Note that $(1 - \lambda)\mu(\mathbf{y}) + \lambda(\mu(\mathbf{y}) + \varrho(\mathbf{y})) = \mu(\mathbf{y}) + \lambda\varrho(\mathbf{y})$. Hence, the complete weighting parameterization of the mean-underachievement model (24) with $0 < \lambda < 1$ is equivalent to the bounded trade-off analysis of the bicriteria mean-equity model (22). This allows us to use Theorem 3 to derive the consistency results for the trade-off approach defined by solving the optimization problem

$$\min\{\mu(\mathbf{f}(\mathbf{x})) + \lambda\varrho(\mathbf{f}(\mathbf{x})) \;:\quad \mathbf{x} \in Q\}. \tag{27}$$

Corollary 1 *If the inequality measure $\varrho(\mathbf{y})$ is mean-complementary equitably consistent (25), then except for location patterns with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$, every optimal solution of problem (27) with $0 < \lambda < 1$ is an equitably efficient solution. In the case of strong consistency (26), every location pattern $\mathbf{x} \in Q$ optimal to (27) with $0 < \lambda < 1$ is, unconditionally, equitably efficient.*

Typical dispersion type risk measures are convex, i.e.

$$\varrho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \leq \lambda\varrho(\mathbf{y}') + (1-\lambda)\varrho(\mathbf{y}'') \quad \text{for any } \mathbf{y}', \mathbf{y}'' \text{ and } 0 \leq \lambda \leq 1.$$

Actually, convexity of an inequality measure on equally distributed outcomes is necessary for its mean-complementary equitable consistency. Note, that for any two vectors $\mathbf{y}'$ and $\mathbf{y}''$ representing the same distribution of outcomes as $\mathbf{y}$ (i.e., $\mathbf{y}' = (y_{\tau'(1)}, \ldots, y_{\tau'(m)})$ for some permutation τ' and $\mathbf{y}'' = (y_{\tau''(1)}, \ldots, y_{\tau''(m)})$ for some permutation τ'') one gets $\bar{\theta}_i(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \leq \bar{\theta}_i(\mathbf{y})$ for all $i \in I$ and any $0 \leq \lambda \leq 1$. Hence, $\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'' \preceq_e \mathbf{y}$ and $M_\varrho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \leq M_\varrho(\mathbf{y})$ is necessary for the equitable consistency. Thus, due to equal means $\mu(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') = \mu(\mathbf{y}') = \mu(\mathbf{y}'') = \mu(\mathbf{y})$, the inequality measure depending only on distribution $\varrho(\mathbf{y}') = \varrho(\mathbf{y}'') = \varrho(\mathbf{y})$ must satisfy $\varrho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \leq \varrho(\mathbf{y}) = \lambda\varrho(\mathbf{y}') + (1-\lambda)\varrho(\mathbf{y}'')$ which represents the convexity of $\varrho(\mathbf{y})$. Certainly, the underachievement function $M_\varrho(\mathbf{y})$ must be also monotonic for the equitable consistency which enforces more restrictions on the inequality measures. We will show further that convexity together with positive homogeneity and some boundedness of an inequality measure is sufficient to guarantee monotonicity of the corresponding underachievement measure and thereby to guarantee the mean-complementary equitable consistency of inequality measure itself.

We say that (dispersion type) inequality measure $\varrho(\mathbf{y}) \geq 0$ is Δ-*bounded* if it upper bounded by the maximum upper deviation, i.e.,

$$\varrho(\mathbf{y}) \leq \Delta(\mathbf{y}) \quad \forall \mathbf{y}. \tag{28}$$

Moreover, we say that $\varrho(\mathbf{y}) \geq 0$ is strictly Δ-bounded if inequality (28) a strict bound, except from the case of perfectly equal outcomes, i.e.,

$$\varrho(\mathbf{y}) < \Delta(\mathbf{y}) \quad \text{for any } \mathbf{y} \text{ such that } \Delta(\mathbf{y}) > 0. \tag{29}$$

Theorem 4 *Let $\varrho(\mathbf{y}) \geq 0$ be a convex, positively homogeneous and translation invariant (dispersion type) inequality measure. If the measure is additionally Δ-bounded (28), then the corresponding underachievement function $M_\varrho(\mathbf{y}) = \mu(\mathbf{y}) + \varrho(\mathbf{y})$ is:*

(i) monotonous: $\mathbf{y}' \leq \mathbf{y}''$ implies $M_\varrho(\mathbf{y}') \leq M_\varrho(\mathbf{y}'')$,

(ii) convex: $M_\varrho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \leq \lambda M_\varrho(\mathbf{y}') + (1-\lambda)M_\varrho(\mathbf{y}'')$ for any $0 \leq \lambda \leq 1$,

(iii) positively homogeneous: $M_\varrho(h\mathbf{y}) = hM_\varrho(\mathbf{y})$ for positive real number h,

(iv) translation equivariant: $M_\varrho(\mathbf{y} + a\mathbf{e}) = M_\varrho(\mathbf{y}) + a$, for any real number a.

If the inequality measure $\varrho(\mathbf{y})$ is strictly Δ-bounded (29), then the corresponding underachievement function $M_\varrho(\mathbf{y})$ is:

(i′) strictly monotonous: $\mathbf{y}' \leq \mathbf{y}''$ and $\mathbf{y}' \neq \mathbf{y}''$ implies $M_\varrho(\mathbf{y}') < M_\varrho(\mathbf{y}'')$.

Proof. If $\varrho(\mathbf{y}) \geq 0$ is a convex, positively homogeneous and translation invariant (dispersion type) inequality measure, then the underachievement function $M_\varrho(\mathbf{y}) = \mu(\mathbf{y}) + \varrho(\mathbf{y})$ does satisfy the requirements of translation equivariance, positive homogeneity, and convexity. Further, if $\mathbf{y}' \leq \mathbf{y}''$, then $\mathbf{y}' = \mathbf{y}'' + (\mathbf{y}' - \mathbf{y}'')$ and $\mathbf{y}' - \mathbf{y}'' \leq 0$. Hence, due to convexity and positive homogeneity, $M_\varrho(\mathbf{y}') \leq M_\varrho(\mathbf{y}'') + M_\varrho(\mathbf{y}' - \mathbf{y}'')$. Moreover, due to the bound (28), $M_\varrho(\mathbf{y}' - \mathbf{y}'') \leq \mu(\mathbf{y}' - \mathbf{y}'') + \Delta(\mathbf{y}' - \mathbf{y}'') \leq \mu(\mathbf{y}' - \mathbf{y}'') + 0 - \mu(\mathbf{y}' - \mathbf{y}'') = 0$. Thus, $M_\varrho(\mathbf{y})$ satisfies also the requirement of monotonicity.
Note that strict upper bound (29) causes that $M_\varrho(\mathbf{y}' - \mathbf{y}'') < 0$ for $\mathbf{y}' \neq \mathbf{y}''$, thus showing strict monotonicity of $M_\varrho(\mathbf{y})$. □

Monotonicity and convexity of the underachievement function turns out to be sufficient for its equitable consistency. Therefore, the following assertion is valid.

Theorem 5 *Let $\varrho(\mathbf{y}) \geq 0$ be a convex and Δ-bounded positively homogeneous inequality measure. Then $\varrho(\mathbf{y})$ is mean-complementary equitably consistent in the sense that of (25).*

Proof. The relation of equitable dominance $\mathbf{y}' \preceq_e \mathbf{y}''$ denotes that there exists a finite sequence of vectors $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \ldots, \mathbf{y}^t$ such that $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'} + \varepsilon_k \mathbf{e}_{i''}$, $0 \leq \varepsilon_k \leq y_{i'}^{k-1} - y_{i''}^{k-1}$ for $k = 1, 2, \ldots, t$ and there exists a permutation τ such that $y'_{\tau(i)} \leq y_i^t$ for all $i \in I$. Note that the underachievement function $M_\varrho(\mathbf{y})$, similar as $\varrho(\mathbf{y})$ depends only on the distribution of outcomes and, due to Theorem 4, is monotonous. Hence, $M_\varrho(\mathbf{y}') \leq M_\varrho(\mathbf{y}^t)$. Further, let us notice that $\mathbf{y}^k = \lambda \bar{\mathbf{y}}^{k-1} + (1-\lambda)\mathbf{y}^{k-1}$ where $\bar{\mathbf{y}}^{k-1} = \mathbf{y}^{k-1} - (y_{i'} - y_{i''})\mathbf{e}_{i'} + (y_{i'} - y_{i''})\mathbf{e}_{i''}$ and $\lambda = \varepsilon/(y_{i'} - y_{i''})$. Vector $\bar{\mathbf{y}}^{k-1}$ has the same distribution of coefficients as $\mathbf{y}^{k-1}$ (actually it represents results of swapping $y_{i'}$ and $y_{i''}$). Hence, due to convexity of $M_\varrho(\mathbf{y})$, one gets $M_\varrho(\mathbf{y}^k) \leq \lambda M_\varrho(\bar{\mathbf{y}}^{k-1}) + (1-\lambda) M_\varrho(\mathbf{y}^{k-1}) = M_\varrho(\mathbf{y}^{k-1})$. Thus, $M_\varrho(\mathbf{y}') \leq M_\varrho(\mathbf{y}'')$ which justifies the mean-complementary equitable consistency of the inequality measure $\varrho(\mathbf{y})$. □

For strict equitable consistency some strict monotonicity and convexity properties of the achievement function are needed. Obviously, there does not exist any inequality measure which is positively homogeneous and simultaneously strictly convex. However, one may notice from the proof of Theorem 5 that only convexity properties on equally distributed outcome vectors are important for monotonous achievement functions. We say that function $C(\mathbf{y})$ is strictly convex on equally distributed outcome vectors, if

$$C(\lambda \mathbf{y}' + (1-\lambda)\mathbf{y}'') < \lambda C(\mathbf{y}') + (1-\lambda) C(\mathbf{y}'') \quad \text{for } 0 < \lambda < 1$$

for any two vectors $\mathbf{y}' \neq \mathbf{y}''$ but representing the same outcomes distribution as some $\mathbf{y}$, i.e., $\mathbf{y}' = (y_{\tau'(1)}, \ldots, y_{\tau'(m)})$ for some permutation τ' and $\mathbf{y}'' = (y_{\tau''(1)}, \ldots, y_{\tau''(m)})$ for some permutation τ''.

Theorem 6 *Let $\varrho(\mathbf{y}) \geq 0$ be a convex and strictly Δ-bounded positively homogeneous inequality measure. If $\varrho(\mathbf{y})$ is also strictly convex on equally distributed outcomes, then it is mean-complementary equitably strongly consistent in the sense that of (26).*

Proof. The relation of weak equitable dominance $\mathbf{y}' \preceq_e \mathbf{y}''$ denotes that there exists a finite sequence of vectors $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \ldots, \mathbf{y}^t$ such that $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'} + \varepsilon_k \mathbf{e}_{i''}$, $0 \leq \varepsilon_k \leq y_{i'}^{k-1} - y_{i''}^{k-1}$ for $k = 1, 2, \ldots, t$ and there exists a permutation τ such that $y'_{\tau(i)} \leq y_i^t$ for all $i \in I$. The strict equitable dominance $\mathbf{y}' \prec_e \mathbf{y}''$ means that $y'_{\tau(i)} < y_i^t$ for some $i \in I$ or at least one ε_k is strictly positive. Note that the underachievement function $M_\varrho(\mathbf{y})$ is strictly monotonous and strictly convex on equally distributed outcome vectors. Hence, $M_\varrho(\mathbf{y}') < M_\varrho(\mathbf{y}'')$ which justifies the mean-complementary equitable strong consistency of the inequality measure $\varrho(\mathbf{y})$. □

Corollary 2 *Let $\varrho(\mathbf{y}) \geq 0$ be a convex, positively homogeneous and Δ-bounded (dispersion type) inequality measure, Then except for location patterns with identical mean $\mu(\mathbf{y})$ and inequality measure $\varrho(\mathbf{y})$, every efficient solution to the bicriteria problem (24) is an equitably efficient solution of the location problem (1). If the measure also strictly Δ-bounded and strictly convex on equally distributed outcome vectors, then every location $\mathbf{x} \in Q$ efficient to (24) is, unconditionally, equitably efficient.*

As mentioned, typical inequality measures are convex and many of them are positively homogeneous. Moreover, the measures such as the mean absolute (upper) semideviation $\bar{\delta}(\mathbf{y})$ (13), the standard upper semideviation $\bar{\sigma}(\mathbf{y})$ (14), and the mean absolute difference $D(\mathbf{y})$ (7) are Δ-bounded. Obviously, Δ-bounded is also the maximum absolute upper deviation $\Delta(\mathbf{y})$ itself. The same applies to the quantile generalizations of the maximum upper deviations, i.e., to the worst conditional k-semideviations defined by the formula (Ogryczak and Zawadzki, 2002):

$$\Delta_k(\mathbf{y}) = \frac{1}{m}\sum_{i=1}^{k}(\theta_i(\mathbf{y}) - \mu(\mathbf{y})). \tag{30}$$

Thus, the following assertion is valid.

Corollary 3 *The following inequality measures $\varrho(\mathbf{y}) \geq 0$ are mean-complementary equitably consistent in the sense of (25):*

1. *the maximum upper deviation $\Delta(\mathbf{y})$ (12),*
2. *the mean absolute (upper) semideviation $\bar{\delta}(\mathbf{y})$ (13),*
3. *the standard upper semideviation $\bar{\sigma}(\mathbf{y})$ (14),*
4. *the mean absolute difference $D(\mathbf{y})$ (7),*
5. *the worst conditional k-semideviation $\Delta_k(\mathbf{y})$ (30).*

Corollary 3 enumerates only the simplest inequality measures studied in the locational context which satisfy the assumptions of Theorem 5 and thereby they are mean-complementary equitably consistent. Theorem 5 allows one to show this property for many other measures. In particular, one may easily find out that any convex combination of mean-complementary equitably efficient inequality measures remains also consistent. On the other hand, among typical inequality measures the mean absolute difference seems to be the only one meeting the stronger assumptions of Theorem 6 and thereby maintaining the strong consistency.

Corollary 4 *The mean absolute difference $D(\mathbf{y})$ (7) is mean-complementary equitably strongly consistent in the sense of (26).*

The mean absolute deviation being a mean-complementary equitably strongly consistent may be used to regularize other consistent but not strongly consistent inequality measures, Namely, if $\varrho(\mathbf{y})$ is a mean-complementary equitably consistent inequality measure, then for any $0 < \varepsilon < 1$ the convex combination $(1-\varepsilon)\varrho(\mathbf{y}) + \varepsilon D(\mathbf{y})$ satisfies strict forms of both Δ-boundedness and convexity requirements and therefore it is mean-complementary equitably strongly consistent. By using arbitrary small positive ε, this approach allows one to build strongly consistent forms (actually regularizations) of maximum semideviation $(1-\varepsilon)\Delta(\mathbf{y})+\varepsilon D(\mathbf{y})$, of the mean absolute semideviation $(1-\varepsilon)\bar{\delta}(\mathbf{y}) + \varepsilon D(\mathbf{y})$, or other equitably consistent inequality measures.

We emphasize that, despite the standard semideviation is mean-complementary equitably consistent inequality measure, the consistency is not valid for variance, semivariance and even for the standard deviation. These measures, in general, do not satisfy all assumptions of Theorem 5. In particular, the standard deviation although convex and positively homogeneous is not Δ-bounded which may result in the lack of consistency. This can be illustrated with a simple example of two outcome vectors: $\mathbf{y}'$ consisted of one outcome 0 (say $y'_1 = 0$) and 9 outcomes 10 (say $y'_i = 10$ for $i = 2, \ldots, 10$); $\mathbf{y}''$ consisted of all 10 outcomes 10 ($y''_i = 10$ for $i = 1, \ldots, 10$). Note that $\mathbf{y}' \le \mathbf{y}''$ and therefore $\mathbf{y}' \preceq_e \mathbf{y}''$ (actually the relation of dominance is strict). Nevertheless, $\mu(\mathbf{y}') + \sigma(\mathbf{y}') = 9 + 3 > \mu(\mathbf{y}'') + \sigma(\mathbf{y}'') = 10$, which contradicts the consistency (25).

One may notice that the mean absolute semideviations are symmetric in the sense that the upper semideviation is always equal to the downside one. In other words, $\bar{\delta}(\mathbf{y}) = \frac{1}{2}\delta(\mathbf{y})$ and thereby Theorem 3 justifies also equitable robust consistency of the half mean absolute deviation. In general, one may just consider $\varrho_\alpha(X) = \alpha\varrho(X)$ as a basic risk measure, like the mean absolute semideviation equal to the half of the mean absolute deviation itself. In order to avoid creation of new inequality measures by simple scaling we rather parameterize the equitable consistency concept. We will say that an inequality measure ϱ is equitably α-consistent if

$$\mathbf{y}' \preceq_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') + \alpha\varrho(\mathbf{y}') \le \mu(\mathbf{y}'') + \alpha\varrho(\mathbf{y}'') \tag{31}$$

The relation of equitable α-consistency will be called *strong* if, in addition to (31), the following holds

$$\mathbf{y}' \prec_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') + \alpha\varrho(\mathbf{y}') < \mu(\mathbf{y}'') + \alpha\varrho(\mathbf{y}''). \tag{32}$$

Note that the equitable 1-consistency represent our basic relation of the mean-complementary equitable consistency. On the other hand, the equitable α-consistency of measure $\varrho(\mathbf{y})$ is equivalent to the mean-complementary equitable consistency of measure $\alpha\varrho(\mathbf{y})$. Hence, the equitable α-consistency of measure $\varrho(\mathbf{y})$ guarantees that then except for outcomes with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$, every efficient solution of the bicriteria problem $\min \{(\mu(\mathbf{f}(\mathbf{x})), \mu(\mathbf{f}(\mathbf{x})) + \alpha\varrho(\mathbf{f}(\mathbf{x}))) \;:\; \mathbf{x} \in Q\}$ is an equitably efficient location and in the case of strong α-consistency every efficient location pattern is, unconditionally, equitably efficient. In terms of the trade-off approach it leads to the following statement.

Corollary 5 *If the inequality measure $\varrho(\mathbf{y})$ is equitably α-consistent (31), then except for location patterns with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$, every optimal solution of problem (27) with $0 < \lambda < \alpha$ is an equitably efficient solution. In the case of strong α-consistency (32), every location pattern $\mathbf{x} \in Q$ optimal to (27) with $0 < \lambda < \alpha$ is, unconditionally, equitably efficient.*

Theorem 7 *Let $\varrho(\mathbf{y}) \ge 0$ be a convex, positively homogeneous and translation invariant (dispersion type) inequality measure. If $\alpha\varrho(\mathbf{y})$ is Δ-bounded, then $\varrho(\mathbf{y})$ is equitably α-consistent in the sense that of (31).*

Theorem 8 *Let $\varrho(\mathbf{y}) \geq 0$ be a convex and positively homogeneous inequality measure. If $\varrho(\mathbf{y})$ is also strictly convex on equally distributed outcomes and $\alpha\varrho(\mathbf{y})$ is strictly Δ-bounded, then $\varrho(\mathbf{y})$ is equitably strongly α-consistent in the sense that of (32).*

As mentioned, the mean absolute semideviation is twice the mean absolute upper semideviation which means that $\alpha\delta(\mathbf{y})$ is Δ-bounded for any $0 < \alpha \leq 0.5$. One may also find out that (for m-dimensional outcome vectors), the maximum absolute deviation satisfies the inequality $1/(m-1)R(\mathbf{y}) \leq \Delta(\mathbf{y})$, while the maximum absolute difference fulfills the inequality $1/mS(\mathbf{y}) \leq \Delta(\mathbf{y})$. For the standard deviation, $\alpha\sigma(\mathbf{y})$ is strictly Δ-bounded for any $0 < \alpha \leq 1/\sqrt{m}$. These leads us to the following corollary.

Corollary 6 *The following inequality measures $\varrho(\mathbf{y}) \geq 0$ are equitably α-consistent within the specified intervals of α:*

1. *the mean absolute semideviation with $0 < \alpha \leq 0.5$,*
2. *the maximum absolute deviation with $0 < \alpha \leq 1/(m-1)$,*
3. *the maximum absolute difference with $0 < \alpha \leq 1/m$,*
4. *the standard deviation with $0 < \alpha \leq 1/\sqrt{m}$.*

Moreover, the α-consistency of the standard deviation is strong.

Following Corollary 6, the standard deviation maintains strong mean-complementary equitable consistency provided that it is used with a very small coefficient. Note that this allows us to consider the standard deviation as a regularization term transforming any mean-complementary equitably consistent inequality measure $\varrho(\mathbf{y})$ into a strongly consistent measure $(1-\varepsilon)\varrho(\mathbf{y}) + \varepsilon\sigma(\mathbf{y})$. Although usage of the mean absolute difference for this purpose seems to be simpler, due to its linear programming computability.

5 Concluding remarks

While making location decisions, the distribution of distances among the service recipients (clients) is an important issue. In order to comply with the minimization of distances as well as with an equal consideration of the clients, the concept of equitable efficiency must be used for the multiple criteria model. Equitably efficient solution concepts may be modeled with the standard multiple criteria optimization applied to the cumulative ordered outcomes. Although rich with equitably efficient solutions, these approaches, in general, are hard to implement since the ordering of outcomes requires the disaggregation of location problem with the client weights which usually dramatically increases the problem size. Therefore, rather simplified mean-equity approaches are applied. Unfortunately, for typical inequality measures, the mean-equity approach may lead to inferior conclusions.

It turns out, however, that several inequality measures can be combined with the mean itself into the optimization criteria generalizing the concept of the worst outcome and generating equitably consistent underachievement measures. In this paper we have introduced general conditions for inequality measures sufficient to provide the equitable consistency of the corresponding underachievement measures. We have shown that properties of convexity and positive homogeneity together with boundedness by the maximum upper semideviation are sufficient for a typical inequality measure to guarantee the corresponding equitable consistency. It allows us to identify various inequality measures which can be effectively used to incorporate equity factors into various location while preserving the consistency with outcomes minimization. Among others the standard upper semideviation turns out to be such a consistent inequality measure while the mean absolute difference is strongly consistent.

Our analysis is related to the properties of location models. It has been shown how equity factors can be consistently included into the location models. We do not analyze algorithmic issues of the models. Many of the inequality measures, we analyzed, can be implemented with auxiliary linear programming constraints. Nevertheless, further research on efficient computational algorithms for solving the corresponding equitable location models is necessary.

This paper is focused on location problems. However, the location decisions are analyzed from the perspective of their effects for individual clients. Therefore, the general concept of the proposed approaches can be used for optimization of various systems which serve many users.

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