Report of the Institute of Control and Computation Engineering Warsaw University of Technology

On Mean-Risk Models Consistent with Stochastic Dominance

Włodzimierz Ogryczak and Małgorzata Opolska-Rutkowska

May, 2004

Report nr: 04–08

Warsaw University of Technology
Institute of Control & Computation Engineering
ul. Nowowiejska 15/19, 00–665 Warsaw, Poland
phone (48-22)-660-7397, fax (48-22)-825-3719

On Mean-Risk Models
Consistent with Stochastic Dominance*

Włodzimierz Ogryczak — *W.Ogryczak@ia.pw.edu.pl*
Małgorzata Opolska-Rutkowska[†] — *M.Rutkowska@mini.pw.edu.pl*

May, 2004

Abstract

Comparing uncertain prospects is one of fundamental interests of the economic decision theory. Two methods are frequently used for modeling the choice among uncertain outcomes: stochastic dominance and mean-risk approaches. The former is based on the axiomatic model of risk-averse preferences but does not provide a convenient computational recipe. It is, in fact, a multiple criteria model with a continuum of criteria. The mean-risk approach quantifies the problem in a lucid form of only two criteria: the mean, representing the expected outcome, and the risk: a scalar measure of the variability of outcomes. The mean-risk model is appealing to decision makers and allows a simple trade-off analysis, analytical or geometrical. On the other hand, for typical dispersion statistics used as risk measures, the mean-risk approach may lead to inferior conclusions. Several risk measures, however, can be combined with the mean itself into the robust optimization criteria thus generating SSD consistent performances (safety) measures. In this paper we introduce general conditions for risk measures sufficient to provide the SSD consistency of the corresponding safety measures.

Key words. Decisions under risk, portfolio theory, stochastic dominance, mean-risk, coherent risk measures.

1 Introduction

Typical economical or managerial decisions result can be evaluated only from the perspective of uncertain future outcomes. Therefore, comparing uncertain outcomes is one of fundamental interests of decision theory. Our objective is to analyze relations between the existing approaches and to provide some tools to facilitate the analysis. We consider decisions with real-valued outcomes, such as return, net profit or number of lives saved. A leading example, originating from finance, is the problem of choice among investment opportunities or portfolios having uncertain returns. Although we discuss the consequences of our analysis in the portfolio selection context, we do not assume any specificity related to this or another application. We consider the general problem of comparing real-valued random variables (distributions), assuming that larger outcomes are preferred. It is a general framework: the random

*The research was supported by the grant PBZ-KBN-016/P03/99 from The State Committee for Scientific Research.
[†]Faculty of Mathematics and Information Science, Warsaw University of Technology.

variables considered may be discrete, continuous, or mixed although some specific results are limited to some type of distributions, like discrete or symmetric. Owing to that, our analysis covers a variety of problems of choosing among uncertain prospects that occur in economics and management.

Two methods are frequently used for modeling choice among uncertain prospects: stochastic dominance (Whitmore and Findlay, 1978; Levy, 1992), and mean-risk analysis (Markowitz, 1987). The former is based on an axiomatic model of risk-averse preferences: it leads to conclusions which are consistent with the axioms. Unfortunately, the stochastic dominance approach does not provide us with a simple computational recipe. It is, actually, a multiple criteria model with a continuum of criteria. The mean-risk approach quantifies the problem in a lucid form of only two criteria: the mean, representing the expected outcome, and the risk: a scalar measure of the variability of outcomes. The mean-risk model is appealing to decision makers and allows a simple trade-off analysis, analytical or geometrical. On the other hand, mean-risk approaches are not capable of modeling the entire gamut of risk-averse preferences. Moreover, for typical dispersion statistics used as risk measures, the mean-risk approach may lead to inferior conclusions.

The seminal Markowitz (1952) portfolio optimization model uses the variance as the risk measure in the mean-risk analysis. The Markowitz model provided the main theoretical background for the modern portfolio theory. Nevertheless, many authors have pointed out that the mean-variance model is, in general, not consistent with stochastic dominance rules. Several other risk measures have been later considered thus creating the entire family of mean–risk (Markowitz type) models. While the original Markowitz model forms a quadratic programming problem, following the initial works on its linear programming (LP) approximation (Sharpe, 1971) many attempts have been made to linearize the optimization procedure (Speranza, 1993; Mansini, Ogryczak and Speranza, 2003). The risk measures, although nonlinear, can be LP computable in the case of discrete random variables, i.e., in the case of returns defined by their realizations under the specified scenarios. This applies, in particular, to the mean absolute deviation from the mean. Konno and Yamazaki (1991) presented and analyzed the complete portfolio optimization model based on the mean absolute deviation used as a risk measure - the so-called MAD model. Yitzhaki (1982) introduced the mean-risk model using the Gini's mean (absolute) difference as the risk measure. Recently, Young (1998) analyzed the LP solvable portfolio optimization model based on risk defined by the worst case scenario (minimax approach) and several new models have appeared (Mansini, Ogryczak and Speranza, 2003). Opposite to the mean-variance approach, for general random variables some consistency with the stochastic dominance rules was shown for the Gini's mean difference (Yitzhaki, 1982), for the MAD model (Ogryczak and Ruszczyński, 1999) and for many other LP solvable models as well (Ogryczak, 2000).

It can be argued that the variability of rate of return above the mean should not be penalized since the investors concern of an underperformance rather than the overperformance of a portfolio. This led Markowitz (1959) to propose downside risk measures such as (downside) semivariance to replace variance as the risk measure. Consequently, one observes growing popularity of downside risk models for portfolio selection (Fishburn, 1970; Sortino and Forsey, 1996). Many authors pointed out that the MAD model opens up opportunities for more specific modeling of the downside risk (Speranza, 1993). In fact most of the LP solvable models may be viewed as based on some downside risk measures. Some consistency with the stochastic dominance rules have been recently also shown for downside standard semideviation and similar higher order semideviations (Ogryczak and Ruszczyński, 2001).

In this paper we introduce general conditions for risk measures sufficient to provide the SSD consistency of the corresponding models, Actually, we show that under simple and natural conditions on the risk measures they can be combined with the mean itself into the robust optimization criteria thus generating SSD consistent performances (safety) measures. The analysis is perform for general distributions but we also pay attention to special cases such as discrete or symmetric distributions. Recently, in Artzner et al. (1999), a class of coherent risk measures has been defined by means of several axioms. Again, the coherence has been shown for the MAD model (Ogryczak and Ruszczyński, 2002) and for some other LP computable measures (Acerbi and Tasche, 2002). We analyze also when our conditions guarantee also the coherency of the corresponding performance functions.

The paper is organized as follows. In the next section we recall the basics of the stochastic dominance and mean-risk approaches. We also demonstrate that the mean-risk approaches with typical dispersion type risk measures may always result in some dominated choices with respect to stochastic dominance although many of them can be combined with the mean itself into the corresponding SSD consistent safety measures. In Section 3 we analyze sufficient conditions which allow us to transform several risk measures into SSD consistent safety measures. We demonstarate that, while considering risk measures depending only on the distributions, the conditions similar to those for the coherency are, essentially, sufficient for SSD consistency.

2 Stochastic dominance and mean-risk models

2.1 The models

Stochastic dominance is based on an axiomatic model of risk-averse preferences (Fishburn, 1964). It originated in the majorization theory (Hardy, Littlewood and Polya, 1934) for the discrete case and was later extended to general distributions (Hanoch and Levy, 1969; Rothschild and Stiglitz, 1969). Since that time it has been widely used in economics and finance (see Bawa, 1982; Levy, 1992; Whitmore and Findlay, 1978; for numerous references). In the stochastic dominance approach random variables are compared by pointwise comparison of some performance functions constructed from their distribution functions.

Let X be a random variable representing some returns. The first performance function $F_1(X, r)$ is defined as the right-continuous cumulative distribution function itself: $F_1(X, r) = \mathbb{P}[X \leq r]$ for $r \in R$. We say that X dominates Y under the FSD rules ($X \succ_{_{FSD}} Y$), if $F_1(X, r) \leq F_1(Y, r)$ for all $r \in R$, where at least one strict inequality holds. Actually, the stochastic dominance is a stochastic order thus defined on distributions rather than on random variables themselves. Nevertheless, it is a common convention, that in the case of random variables X and Y having distributions P_X and P_Y, the stochastic order relation $P_X \succeq P_Y$ might be viewed as a relation on random variables $X \succeq Y$ (Mueller and Stoyan, 2002). It must be emphasized, however, that the dominance relation on random variables is no longer an order as it is antisymmetric.

Note that $F_1(X, r)$ expresses the probability of underachievement for a given target value r. Thus the first degree stochastic dominance is based on the multidimensional (continuum-dimensional) objective defined by the probabilities of underachievement for all target values (Roy, 1952). The FSD is the most general relation. If $X \succ_{_{FSD}} Y$, then X is preferred to Y within all models preferring larger outcomes, no matter how risk-averse or risk-seeking they are. In terms of expected utility theory, the FSD relation covers all increasing utility functions.

For decision making under risk most important is the second degree stochastic dominance relation, associated with the second performance function $F_2(X, r)$ given by areas below the cumulative distribution function itself, i.e.: $F_2(X, r) = \int_{-\infty}^{r} F_1(X, t)dt$ for $r \in R$. Similarly to FSD, we say that X dominates Y under the SSD rules ($X \succ_{_{SSD}} Y$), if $F_2(X, r) \leq F_2(Y, r)$ for all $r \in R$, with at least one inequality strict. Certainly, $X \succ_{_{FSD}} Y$ implies $X \succ_{_{SSD}} Y$. Function $F_2(X, r)$, used to define the SSD relation can also be presented as follows (Ogryczak and Ruszczyński, 1999):

$$F_2(X, r) = \mathbb{P}[X \leq r]\mathbb{E}[r - X | X \leq r] = \mathbb{E}[\max\{r - X, 0\}]$$

Hence, the SSD relation is the Pareto dominance for mean below-target deviations from infinite number (continuum) of targets.

If $X \succ_{SSD} Y$, then X is preferred to Y within all risk-averse preference models that prefer larger outcomes. In terms of the expected utility theory the SSD relation represent all the preferences modeled with increasing and concave utility functions. It is therefore a matter of primary importance that an approach to the comparison of random outcomes be consistent with the second degree stochastic dominance relation. Our paper focuses on the consistency of mean-risk approaches with SSD.

Alternatively, the stochastic dominance order can be expressed on the inverse cumulative functions (quantile functions). Namely, for random variable X, one may consider the performance function $F_{-1}(X, p)$ defined as is the left-continuous inverse of the cumulative distribution function $F_1(X, r)$, i.e., $F_{-1}(X, p) = \inf \{\eta : F_1(X, \eta) \geq p\}$. Obviously, X dominates Y under the FSD rules ($X \succ_{FSD} Y$), if $F_{-1}(X, p) \geq F_{-1}(Y, p)$ for all $p \in [0, 1]$, where at least one strict inequality holds. Further, the second quantile function (or the so-called *Absolute Lorenz Curve* ALC) is defined by integrating F_{-1}:

$$F_{-2}(X, p) = \int_0^p F_{-1}(X, \alpha) d\alpha \quad \text{for } 0 < p \leq 1 \quad \text{and} \quad F_{-2}(X, 0) = 0. \tag{1}$$

Actually, as shown by Ogryczak and Ruszczyński (2002),

$$F_{-2}(X, p) = \max_{r \in R} [pr - F_2(X, r)] = \max_{r \in R} [pr - \mathbb{E}[\max\{r - X, 0\}]]. \tag{2}$$

Hence, by the theory of convex conjugent (dual) functions, the pointwise comparison of ALCs provides an alternative characterization of the SSD relation in the sense that $X \succeq_{SSD} Y$ if and only if $F_{-2}(X, \beta) \geq F_{-2}(Y, \beta)$ for all $0 < \beta \leq 1$.

Mean-risk approaches are based on comparing two scalar characteristics (summary statistics), the first, denoted $\mu(X)$, represents the expected outcome (reward), and the second, denoted $\varrho(X)$, is some measure of risk. The weak relation of mean-risk dominance is defined as follows: We say that X dominates Y under the μ/ϱ rules ($X \succ_{\mu/\varrho} Y$), if $\mu(X) \geq \mu(Y)$ and $\varrho(X) \leq \varrho(Y)$, and at least one of these inequalities is strict. Note that quite different random variables X and Y may have $\mu(X) = \mu(Y)$ and $\varrho(X) = \varrho(Y)$ which make them indifferent under the μ/ϱ rules thus generating a tie in the mean-risk comparison model.

An important advantage of mean-risk approaches is the possibility of a pictorial trade-off analysis. Having assumed a trade-off coefficient λ between the risk and the mean, one may directly compare real values of $\mu(X) - \lambda\varrho(X)$ and $\mu(Y) - \lambda\varrho(Y)$. Indeed, $X \succ_{\mu/\varrho} Y$ implies $\mu(X) - \lambda\varrho(X) > \mu(Y) - \lambda\varrho(Y)$ for all $\lambda > 0$. In other words, the trade-off approach is consistent with the mean-risk dominance.

The original Markowitz (1952) portfolio optimization model uses the variance or the standard deviation. Several other risk measures have been later considered thus creating the entire family of mean-risk models. Risk measures in Markowitz-type mean-risk models, similar to the standard deviation, are translation invariant and risk relevant deviation type measures (dispersion parameters). Thus, they are not affected by any shift of the outcome scale $\varrho(X + a) = \varrho(X)$ for any real number a and they are equal to 0 in the case of a risk-free portfolio while taking positive values for any risky portfolio. Unfortunately, such risk measures are not consistent with the stochastic dominance order (Mueller and Stoyan, 2002) or other axiomatic models of risk-averse preferences (Rotschild and Stiglitz, 1970) and risk measurement

(Artzner et al., 1999). Indeed, in the Markowitz model its efficient set may contain SSD inferior portfolios characterized by a small risk but also very low return (Porter and Gaumnitz, 1972; Porter, 1974). Unfortunately, it is a common flaw of all Markowitz-type mean-risk models where risk is measured with some dispersion measures. Although, the SSD relation $X \succ_{SSD} Y$ implies $\mu(X) \geq \mu(Y)$ (Ogryczak and Ruszczyński, 1999), this is not enough to guarantee the μ/ϱ dominance, due to the lack of similar consistency relation for the risk measures. For dispersion type risk measures $\varrho(X)$, it may occur that $X \succ_{SSD} Y$ and simultaneously $\varrho(X) > \varrho(Y)$. This can be illustrated by two portfolios X and Y (with rates of return given in percents):

$$\mathbb{P}[X = r] = \begin{cases} 1/2, & r = 3.0 \\ 1/2, & r = 5.0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbb{P}[Y = r] = \begin{cases} 1, & r = 1.0 \\ 0, & \text{otherwise} \end{cases}$$

Note that the risk-free portfolio Y with the guaranteed result 1.0 is obviously worse than the risky portfolio X giving 3.0 or 5.0. In all preference models based on the risk aversion axioms (Levy, 1972; Artzner et al., 1999) outcome Y is dominated by X, in particular $X \succ_{SSD} Y$. On the other hand, when a dispersion type risk measure $\varrho(X)$ is used, then both the outcomes are efficient in the corresponding mean-risk model since for each such a measure $\varrho(X) > 0$ while $\varrho(Y) = 0$.

2.2 Safety measures and consistency results

As discussed earlier, the Markowitz model is frequently criticized as not consistent with axiomatic models of preferences for choice under risk. Namely, except for the case of returns meeting the multivariate normal distribution, the mean-variance model may lead to inferior conclusions with respect to the stochastic dominance order. As pointed out in the previous section it is a common flaw of all dispersion type risk measures typically used in the mean-risk models. In order to overcome this flaw of the Markowitz model, already Baumol (1964) suggested to consider a performance measure, he called the expected gain-confidence limit criterion, $\mu(X) - \lambda\sigma(X)$ to be maximized instead of the minimization of $\sigma(X)$ itself. Similarly, Yitzhaki (1982) considered maximization of the criterion $\mu(X) - \varrho(X)$ for the Gini's mean difference and he demonstrated its SSD consistency. Recently, similar consistency results have been introduced (Ogryczak and Ruszczyński, 1999, 2001) for measures corresponding to the standard semideviation and to the mean semideviation (half of the mean absolute deviation).

Hereafter, for any dispersion type risk measure $\varrho(X)$, the performance function $S(X) = \mu(X) - \varrho(X)$ will be referred to as the corresponding safety measure. Note that risk measures, we consider, are defined as translation invariant and risk relevant dispersion parameters. Hence, the corresponding safety measures are translation equivariant in the sense that any shift of the outcome scale results in an equivalent change of the safety measure value (with opposite sign as safety measures are maximized), or in other words, the safety measures distinguish (and order) various risk-free portfolios (outcomes) according to their values. The safety measures, we consider, are risk relevant but in the sense that the value of a safety measure for any risky portfolio is less than the value for the risk-free portfolio with the same expected returns. Moreover, when risk measure $\varrho(X)$ is a convex function of X, then the corresponding safety measure $S(X)$ is concave.

Comparison of random variables is usually related to the problem of choice among risky alternatives in a given feasible set Q. For instance, in the simplest problem of portfolio selection (Markowitz, 1987) the feasible set of random variables is defined as all convex combinations (weighted averages with nonnegative weights totaling 1) of a given number of investment opportunities (securities). A feasible random variable $X \in Q$ is called efficient under the relation SSD if there is no $Y \in Q$ such that

$Y \succ_{SSD} X$. If the safety measure $S(X) = \mu(X) - \varrho(X)$ is SSD consistent in the sense that $X \succ_{SSD} Y$ implies $\mu(X) - \varrho(X) \geq \mu(Y) - \varrho(Y)$, then except for alternatives (portfolios) with identical values of $\mu(X)$ and $\varrho(X)$, every efficient solution of the bicriteria problem

$$\max\{[\mu(X), \mu(X) - \varrho(X)] : \quad X \in Q\} \tag{3}$$

is SSD efficient.

This can be demonstrated as follows. Let $X^0 \in Q$ be an efficient solution of (3). Suppose that X^0 is not SSD efficient. This means, there exists X such that $X \succ_{SSD} X^0$. Then $\mu(X) \geq \mu(X^0)$ and simultaneously $\mu(X) - \varrho(X) \geq \mu(X^0) - \varrho(X^0)$, by virtue of the SSD consistency. Since X^0 is efficient to (3) no inequality can be strict, which implies $\mu(X) = \mu(X^0)$ and $\varrho(X) = \varrho(X^0)$. In the case of the strong SSD safety consistency, the supposition $X \succ_{SSD} X^0$ implies $\mu(X) \geq \mu(X^0)$ and $\mu(X) - \varrho(X) > \mu(X^0) - \varrho(X^0)$ which contradicts the efficiency of X^0 with respect to (3). Hence, X^0 is SSD efficient.

Relation of the SSD consistency of the safety measures directly involves criterion $\mu(X) - \varrho(X)$. However, the SSD dominance always implies the means inequality. Hence, in the case of $X \succ_{SSD} Y$ we have both $\mu(X) \geq \mu(Y)$ and $\mu(X) - \varrho(X) \geq \mu(Y) - \varrho(Y)$. Thus, by combining inequalities, one may easily notice that $X \succ_{SSD} Y$ implies $\mu(X) - \lambda\varrho(X) \geq \mu(Y) - \lambda\varrho(Y)$ for all $0 \leq \lambda \leq 1$. Consequently, if a risk measure $\varrho(X)$ generates a SSD consistent safety measure, then the mean-risk model is consistent with the SSD efficiency, provided that the trade-off coefficients are bounded from above by 1. On the other hand, one may just consider $\varrho_\beta(X) = \beta\varrho(X)$ as a basic risk measure, like the mean absolute semideviation equal to the half of the mean absolute deviation itself. In such a case one may gets another (possibly higher) upper bound for the trade-off coefficient guaranteeing the SSD consistency. Therefore, following Ogryczak and Ruszczyński (1999), in this paper we say that the (deviation) risk measure is SSD α-safety consistent if *there exists a positive constant* α *such that for all* X *and* Y

$$X \succeq_{SSD} Y \quad \Rightarrow \quad \mu(X) \geq \mu(Y) \quad \text{and} \quad \mu(X) - \alpha\,\varrho(X) \geq \mu(Y) - \alpha\,\varrho(Y). \tag{4}$$

For the sake of simplicity, the SSD 1-safety consistency of a risk measure we will usually call simply SSD safety consistency. Obviously, the SSD α-safety consistency guarantees that the random variable Y with worse trade-off performance value $\mu(X) - \lambda\varrho(X) > \mu(Y) - \lambda\mu(Y)$ for some $0 \leq \lambda \leq \alpha$ cannot dominate X according to the SSD rules. Note that the trade-off performance function $\mu(X) - \lambda\varrho(X)$ may interpreted as an aggregation of bicriteria optimization (3).

If the risk measure $\varrho(X)$ is SSD α-safety consistent, then except for random variables with identical $\mu(X)$ and $\varrho(X)$, every random variable that is maximal by $\mu(X) - \lambda\varrho(X)$ with $0 < \lambda < \alpha$ is efficient under the SSD rules. In other words, the optimal solution of the problem

$$\max\{\mu(X) - \lambda\varrho(X) \ : \ X \in Q\} \tag{5}$$

with $0 < \lambda < \alpha$, if it is unique, is efficient under the SSD rules. However, in the case of nonunique optimal solutions, we only know that the optimal set of (5) contains a solution which is efficient under the SSD rules. The optimal set may contain, however, also some SSD-dominated solutions. Exactly, an optimal solution $X \in Q$ can be SSD dominated only by another optimal solution $Y \in Q$ which is indifferent from X in the mean-risk model (i.e., $\mu(Y) = \mu(X)$ and $\varrho(Y) = \varrho(X)$). Recall that in many applications, especially in the portfolio selection problem, the mean-risk model is analyzed with the so-called critical line algorithm (Markowitz, 1987). This is a technique for identifying the μ/ϱ efficient frontier by parametric optimization (5) for varying $\lambda > 0$. According to our analysis, the SSD α-safety consistency of the risk measure guarantees that the part of the efficient frontier (in the μ/ϱ image space) corresponding to trade-off coefficients $0 < \lambda < \alpha$ is also efficient under the SSD rules. Therefore, it is very important to identify clear sufficient conditions for the SSD safety consistency of risk measures.

2.3 Specific risk measures

The original Markowitz (1952) portfolio optimization model uses the variance $\sigma^2(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$ or rather the standard deviation $\sigma(X) = \sqrt{\mathbb{E}[(X - \mathbb{E}[X])^2]}$ as a risk measure. The simplest alternative risk measures are generated by the use of absolute values replacing the squares in the variance formula. In particular, the mean absolute deviation from the mean (the MAD measure) is defined as $\delta(X) = \mathbb{E}[|X - \mathbb{E}[X]|]$. Several other risk measures have been later considered thus creating the entire family of mean-risk models. In particular, following obvious arguments that the variability of returns above the mean should not be penalized since the investors concern of an underperformance rather than the overperformance, led Markowitz (1959) to propose downside risk measures such as (downside) semivariance $\bar{\sigma}^2(X) = \mathbb{E}[(\max\{\mathbb{E}[X] - X, 0\})^2]$ or the standard semideviation $\bar{\sigma}(X) = \sqrt{\mathbb{E}[(\max\{\mathbb{E}[X] - X, 0\})^2]}$. Similarly, one may consider (downside) mean semideviation $\bar{\delta}(X) = \mathbb{E}[\max\{\mathbb{E}[X] - X, 0\}]$, although the mean absolute deviation is a symmetric measure which causes that $\delta(X) = 2\bar{\delta}(X)$ for any (possibly not symmetric) random variable X. For discrete random variables (or rather random variables with bounded support) there is well defined the (downside) maximum semideviation $\Delta(X) = \min\{r \in R : r \geq \mathbb{E}[X] - X\}$.

The stochastic dominance partial orders are defined on distributions. The risk measures are commonly considered as functions of random variables. One may focus on a linear space of random variables $\mathcal{L} = L^k(\Omega, \mathcal{F}, \mathbb{P})$ with some $k \geq 1$ (assuming $k \geq 2$ whenever variance or any related measure is considered). Although defined for random variables, typical risk measures depend only on the corresponding distributions themselves and we focus on such measures. In other words, we assume that $\varrho(X) = \varrho(\hat{X})$ whenever random variables X and $\hat{X}$ have the same distribution, i.e. $F_1(X, r) = F_1(\hat{X}, r)$ for all $r \in R$ or equivalently $F_{-1}(X, p) = F_{-1}(\hat{X}, p)$ for all $p \in [0, 1]$.

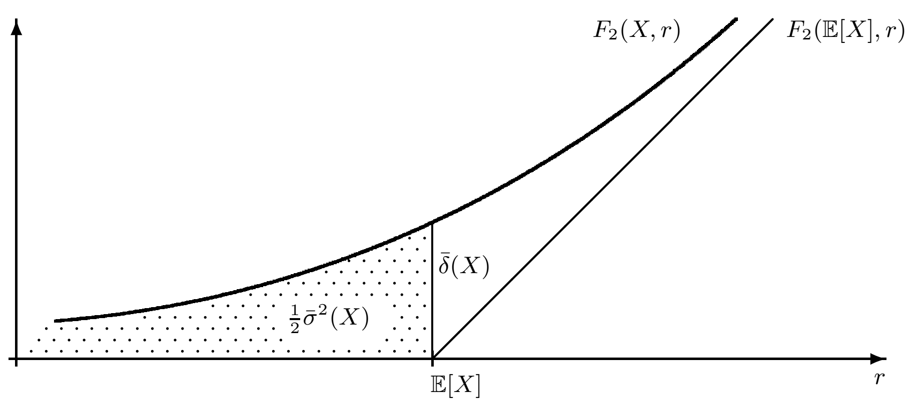

Figure 1: The O-R diagram and risk measures

Note that in terms of the SSD relation the riskiness of a random variable X can be depicted by the difference between SSD characteristic of X and its risk-free equivalent $\mathbb{E}[X]$, as in the so-called O-R diagrams. This difference may considered either in the standard (primal) SSD characteristics as

$F_2(X,r) - F_2(\mathbb{E}[X],r)$ (Ogryczak and Ruszczyński, 1999, 2001) or in the quantile (dual) SSD characteristics $F_{-2}(\mathbb{E}[X],p) - F_{-2}(X,p)$ (Ogryczak and Ruszczyński, 2002). Indeed, most of the risk measures considered in typical mean-risk models turn out (Mansini, Ogryczak and Speranza, 2003) to be defined in this way. One may notice (Ogryczak and Ruszczyński, 1999) that $F_2(\mathbb{E}[X],r) = \max\{r - \mathbb{E}[X], 0\}$ while variance and semivariance can be expressed, respectively, by the following formula:

$$\sigma^2(X) = 2\int_{-\infty}^{\infty} [F_2(X,t) - F_2(\mathbb{E}[X],t)]dt,$$

$$\bar{\sigma}^2(X) = 2\int_{-\infty}^{\mathbb{E}(X)} [F_2(X,t) - F_2(\mathbb{E}[X],t)]dt.$$

The mean absolute semideviation $\bar{\delta}(X) = F_2(X,\mathbb{E}[X]) = F_2(X,\mathbb{E}[X]) - F_2(\mathbb{E}[X],\mathbb{E}[X])$ (Fig, 1) but it can be also expressed as the maximum value of both primal and dual differences:

$$\bar{\delta}(X) = \max_{t\in R}[F_2(X,t) - F_2(\mathbb{E}[X],t)],$$

$$\bar{\delta}(X) = \max_{p\in[0,1]}[F_{-2}(\mathbb{E}[X],p) - F_{-2}(X,p)].$$

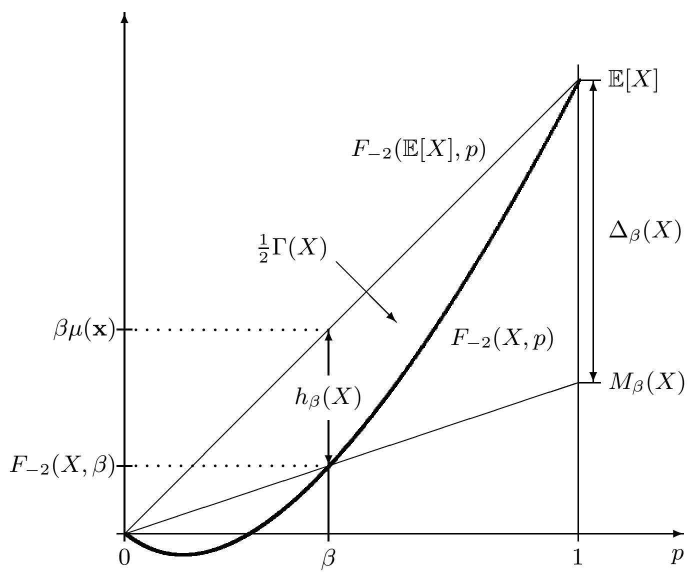

Figure 2: The absolute Lorenz curve and risk measures

Note that $F_{-2}(\mathbb{E}[X],p) = \mathbb{E}[X]p$ and the conditional β-semideviations, related to the so-called CVaR models (Rockafellar and Uryasev, 2000: Ogryczak and Rusczyński, 2002), can be expressed as dual differences (Fig. 2)

$$\Delta_\beta(X) = \frac{1}{\beta}[F_{-2}(\mathbb{E}[X],\beta) - F_{-2}(X,\beta)] \quad \text{for } \beta \in (0,1]$$

while the maximum semideviation represents it upper limit

$$\Delta(X) = \lim_{\beta\to 0_+} \frac{1}{\beta}[F_{-2}(\mathbb{E}[X],\beta) - F_{-2}(X,\beta)] = \sup_{\beta\in(0,1]} \frac{1}{\beta}[F_{-2}(\mathbb{E}[X],\beta) - F_{-2}(X,\beta)].$$

Finally, the Gini's mean-difference can be expressed as

$$\Gamma(X) = 2\int_0^1 [F_{-2}(\mathbb{E}[X],\beta) - F_{-2}(X,\beta)]d\beta.$$

As shown by Ogryczak and Ruszczyński (1999, 2001, 2002) within the class of arbitrary uncertain prospects allowing to consider stochastic dominance (the class of random variables with finite expectations $\mathbb{E}[|X|] < \infty$, or $\mathbb{E}[X^2] < \infty$ while for standard deviation) the following risk measures are SSD safety consistent:

(i) the standard semideviation $\bar{\sigma}(X)$,
(ii) the mean semideviation $\bar{\delta}(X)$,
(iii) the conditional β-semideviation $\Delta_\beta(X)$,
(iv) the maximum semideviation $\Delta(X)$,
(v) the Gini's mean difference $\Gamma(X)$.

The consistency results are summarized in Table 1, where the maximum value of $alpha$ is presented for general distributions and for the subclass of symmetric distributions as well. It turns out that when limiting the analysis to outcomes described with the symmetric distribution some consistency levels α increase and one gets additionally SSD 1-safety consistency of the standard deviation $\sigma(X) = \sqrt{2}\bar{\sigma}(X)$, the mean absolute deviation $\delta(X) = 2\bar{\delta}(X)$ and the doubled Gini's mean difference $2\Gamma(X)$.

Table 1: SSD consistency limits for typical measures

Risk Measure	Consistency limit α	
$\varrho(X)$	general case	symmetric case
Standard semideviation $\bar{\sigma}(X)$	1	$\sqrt{2}$
Mean absolute semideviation $\bar{\delta}(X)$	1	2
Conditional β-semideviation $\Delta_\beta(X)$	1	1
Maximum semideviation $\Delta(X)$	1	1
Gini's mean difference $\Gamma(X)$	1	2

We emphasize that, in general case, despite the standard semideviation is SSD safety consistent, the consistency is not valid for variance, semivariance and even for the standard deviation itself. Although, for symmetric distribution one gets the SSD safety consistency of the standard deviation $\sigma(X) = \sqrt{2}\bar{\sigma}(X)$, even then the SSD consistency results are not valid for variance itself.

3 SSD consistency conditions

3.1 SSD consistency and coherency

The risk measures we consider from the perspective of the stochastic dominance are defined as (real valued) functions of distributions rather than random variables themselves. Nevertheless, in many various applications it might be more convenient to analyze their properties as functions of random variables. Recently, Artzner et al. (1999) have defined a class of coherent risk measures by means of several axioms. The axioms depicts the most important issues in the risk comparison for economic decisions. therefore, they have been quite commonly recognized as the standard requirements for risk measures. Let us consider a linear space of random variables $\mathcal{L} = L^k(\Omega, \mathcal{F}, \mathbb{P})$ with some $k \geq 1$ (recall, we assume

$k \geq 2$ whenever variance or any related measure is considered). A real valued performance function $C : \mathcal{L} \to \mathbb{R}$ is called a coherent risk measure on $\mathcal{L}$ if for any $X, Y \in \mathcal{L}$ it is

(i) monotonous: $X \geq Y$ implies $C(X) \leq C(Y)$,
(ii) subadditive: $C(X + Y) \leq C(X) + C(Y)$,
(iii) positively homogeneous: $C(hX) = hC(X)$ for positive real number h,
(iv) translation equivariant: $C(X + a) = C(X) - a$, for any real number a,
(v) risk relevant: $X \leq 0$ and $X \neq 0$ implies $C(X) > 0$,

where or inequalities on random variables are understood in terms 'a.s.'. Note that if the performance function is positively homogeneous, then the axiom of subadditivity is equivalent to the standard convexity requirement:

(iia) $C(\alpha X + (1 - \alpha)Y) \leq \alpha C(X) + (1 - \alpha)C(Y)$ for any $0 \leq \alpha \leq 1$.

The requirement of subadditivity (or convexity) is crucial to guarantee that the outcomes diversification does not increase the risk. The last axiom (risk relevance) is, however, sometimes ignored or understood in a different way.

Note that typical dispersion type risk measures can define the corresponding coherent risk measures, provided that they are SSD safety consistent and positively homogeneous. Indeed, typical dispersion type risk measures are convex, positively homogeneous and translation invariant, which implies that the composite objective $-\mu(X) + \varrho(x)$ does satisfy the axioms of translation equivariance, positive homogeneity, subadditivity. Moreover, the SSD safety consistency implies then the monotonicity and justifies the coherence, Actually the following assertion is valid (Mansini, Ogryczak and Speranza, 2003).

Theorem 1 *Let $\varrho(X) \geq 0$ be a convex, positively homogeneous and translation invariant (dispersion type) risk measure. If the measure satisfies additionally the SSD safety consistency, i.e.,*

$$X \succeq_{SSD} Y \quad \Rightarrow \quad \mu(X) - \varrho(X) \geq \mu(Y) - \varrho(Y),$$

then the corresponding performance function $C(X) = -S(X) = \varrho(X) - \mu(X)$ fulfills the coherence axioms.

Coming back to the most common risk measures mentioned with shown SSD safety consistency, one may easily notice that all they are convex, positively homogeneous and translation invariant. Hence, the following (dispersion type) risk measures $\varrho(X)$ define the corresponding coherent risk measures $\varrho(X) - \mu(X)$:

(i) the standard semideviation $\bar{\sigma}(X)$,
(ii) the mean semideviation $\bar{\delta}(X)$,
(iii) the conditional β-semideviation $\Delta_\beta(X)$,
(iv) the maximum semideviation $\Delta(X)$,
(iv) the Gini's mean difference $\Gamma(X)$.

We emphasize that, despite the standard semideviation generates a coherent risk measure, it is not valid for variance itself which is even not positively homogeneous. Recall that, when limiting the analysis to outcomes described with the symmetric distribution, one get additionally SSD consistency results for the standard deviation $\sigma(X)$, the mean absolute deviation $\delta(X)$ and the doubled Gini's mean difference $2\Gamma(X)$. However, these results cannot be directly translated with Theorem 1 into the corresponding coherency results since the latter are defined for a linear space of random variables while symmetric distributions may not form any linear space.

Actually, the SSD consistency is not necessary for the coherence of $-S(X)$ if the the risk measure $\varrho(X)$ is upper bounded by the maximum downside deviation $\Delta(X)$. If $\varrho(X) \geq 0$ is a convex, positively homogeneous and translation invariant (dispersion type) risk measure, then the performance function $C(X) = \varrho(X) - \mu(X)$ does satisfy the axioms of translation equivariance, positive homogeneity, and subadditivity. Further, if $X \geq Y$, then $X = Y + (X - Y)$ and $X - Y \geq 0$. Hence, the subadditivity together with the assumed risk measure bound imply that the performance function $C(X)$ satisfies also the axioms of monotonicity and relevance. Exactly, the following assertion is valid (Mansini, Ogryczak and Speranza, 2003; Rockafellar, Uryasev and Zabarankin, 2003).

Theorem 2 *Let $\varrho(X) \geq 0$ be a convex, positively homogeneous and translation invariant (dispersion type) risk measure. If the measure is additionally expectation bounded*

$$X \geq 0 \Rightarrow \varrho(X) \leq \mu(X), \tag{6}$$

then the corresponding performance function $C(X) = \varrho(X) - \mu(X)$ fulfills the coherence axioms.

Theorem 2 provides a very attractive simple characteristic of dispersion type risk measures generating corresponding coherent measures. It leads us to question if one may introduce a similar simple characteristic of risk measures guaranteeing the SSD safety consistency. Generally, the stochastic dominance relation as based on the distributions are more subtle than the coherency requirement formulated on random variables. In particular, the a.s. monotonicity used as the first coherency axiom is much stronger than the FSD or SSD monotonicity. On the other hand, we consider risk measures defined on the distributions thus allowing for various random variables while allocating the same values to all random variables with the same distributions. This causes that, similarly to Theorem 2, the convexity, homogeneity properties and the expectation boundness turns out to be equally important for SSD consistency.

3.2 SSD separation conditions

In order to derive the SSD consistency conditions we will use the SSD separation results. Namely, the following result (Mueller and Stoyan, 2002, Theorem 1.5.14) allows us to split the SSD dominance into two simpler stochastic orders: the FSD dominance and the Rotschild-Stiglitz (RS) dominance (or concave stochastic order), where the latter is the SSD dominance restricted to the case of equal means.

Theorem 3 *Let X and Y be random variables with $X \succeq_{SSD} Y$. Then there is a random variable Z such that*

$$X \succeq_{FSD} Z \succeq_{RS} Y$$

The above theorem allows us to separate two important properties of the SSD dominance and the corresponding requirements for the risk measures.

Corollary 1 *Let $\varrho(X) \geq 0$ be a (dispersion type) risk measure. The measure is SSD 1-safety consistent if and only if it satisfies both the following conditions:*

$$X \succeq_{FSD} Y \quad \Rightarrow \quad \mu(X) - \varrho(X) \geq \mu(Y) - \varrho(Y), \tag{7}$$

$$X \succeq_{RS} Y \quad \Rightarrow \quad \varrho(X) \leq \varrho(Y). \tag{8}$$

Proof. If $X \succeq_{SSD} Y$, then according to separation theorem $X \succeq_{FSD} Z \succeq_{RS} Y$ where $\mathbb{E}[Z] = \mathbb{E}[Y]$. Hence, applying (7) and (8) one gets

$$X \succeq_{SSD} Y \quad \Rightarrow \quad \mu(X) - \varrho(X) \geq \mu(Z) - \varrho(Z) \geq \mu(Y) - \varrho(Y).$$

On the other hand, both the requirements are obviously necessary. □

Condition (7) represents the stochastic monotonicity and it may be replaced with more standard monotonicity requirement

$$X \geq Y \quad \Rightarrow \quad \mu(X) - \varrho(X) \geq \mu(Y) - \varrho(Y), \tag{9}$$

where the inequality $X \geq Y$ is to be viewed in the sense o holding almost surely (a.s.). Essentially, the FSD dominance is much stronger relation the a.s. inequality ($X \geq Y$ implies $X \succeq_{FSD} Y$, but not opposite). However, the relation $X \succeq_{FSD} Y$ is equivalent (Mueller and Stoyan, 2002, Theorem 1.2.4) to the existence of a probability space and random variables $\hat{X}$ and $\hat{Y}$ on it with the distribution functions the same as X and Y, respectively, such that $\hat{X} \geq \hat{Y}$. Hence, for risk measures depending only on distributions, we consider, one gets $\varrho(\hat{X}) = \varrho(X)$ and the monotonicity requirements (7) and (9) are equivalent. Note that for any $X \geq 0$ and $a \in R$ one gets $X + a \geq a$ while $\varrho(X + a) = \varrho(X)$ and, therefore, the monotonicity (9) implies $\varrho(X) \leq \mu(X)$. This justifies the expectation boundness (6) as a necessary condition for monotonicity (9) or (7).

Condition (8) represents the required convexity properties to model diversification advantages. Note that the second cumulative distribution functions $F_2(X, r)$ are convex with respect to random variables X (Ogryczak and Ruszczyński, 2001). Hence, taking two random variables Y' and Y'' both with the same distribution as X one gets $F_2(\alpha Y' + (1-\alpha)Y'', r) \leq F_2(X, r)$ for any $0 \leq \alpha \leq 1$ and any $r \in \mathbb{R}$. Thus, $\alpha Y' + (1-\alpha)Y'' \succeq_{RS} X$ and convexity of $\varrho(X)$ is necessary to meet the requirement (8).

The concept of separation risk measures properties following Theorem 3 is applicable while considering general (arbitrary) distributions. It may be, however, adjusted to some specific classes of distribution. In particular, we will show that it remains valid for a class of symmetric distributions. The proof of separation theorem give by Mueller and Stoyan (2002) does not allow us for such interpretation. Indeed, the separating random variable is defined there by the distribution function

$$F_1(Z, r) = \begin{cases} F_1(X, r), & F_2(X, r) < r - \mu(Y) \\ 1, & \text{otherwise} \end{cases}$$

which does not preserve any possible symmetry of X. However, this can be replaced with a more subtle construction

$$F_1(Z, r) = \begin{cases} F_1(X, r), & r < \mu(Y) \\ F_1(X, r + 2(\mu(X) - \mu(Y))), & r \geq \mu(Y) \end{cases} \tag{10}$$

which preserves symmetry leading us to the following assertion.

Theorem 4 *Let X and Y be symmetric random variables with $X \succeq_{SSD} Y$. Then there is a symmetric random variable Z such that*

$$X \succeq_{FSD} Z \succeq_{RS} Y$$

One may also notice that the 'a.s.' characteristic of the FSD relation (Mueller and Stoyan, 2002, Theorem 1.2.4) may be, respectively, enhanced for symmetric distributions. Namely, in the case of symmetric distributions, the relation $X \succeq_{FSD} Y$ is equivalent to the existence of a probability space and random variables $\hat{X}$ and $\hat{Y}$ on it with the distribution functions the same as X and Y, respectively, such that $\hat{X} \geq \hat{Y}$ and $\hat{X} - \hat{Y}$ has a symmetric distribution.

3.3 Specific conditions

Recall that we consider risk measures depending only on the distribution itself. Nevertheless, they are frequently defined as functions of random variables. In this section we try to identify properties of such functions of random variables sufficient to guarantee the SSD safety consistency. This can provide us with a simple tool for justification various risk measures. The simplest risk measures depending only on distributions can be expressed as expectations of random variables possibly transformed with some convex function:

$$\varrho(X) = f(\mathbb{E}[g(X)]), \quad g \text{ - convex}, \ f \text{ - increasing}. \tag{11}$$

Note that formula (11) covers both the moments of distribution (in particular mean absolute deviation and variance) and the partial moments (mean semideviation and semivariance) as well as the maximum semideviation. Moreover, the external function f allows for final rescaling thus covering also the standard deviation (semideviation).

Theorem 5 *Let $\varrho(X) \geq 0$ be a convex, positively homogeneous and translation invariant (dispersion type) risk measure of the form (11). The measure is then SSD 1-safety consistent if and only if it is additionally expectation bounded (6).*

Proof. By assumed form (11) of $\varrho(X)$, it satisfies the requirement (8). Further, if $X \geq Y$, then $X = Y + (X - Y)$ and $X - Y \geq 0$. Hence, the convexity together with the positive homogeneity and the expectation bound (6) imply that the safety measures $S(X) = \mu(X) - \varrho(X)$ satisfies the monotonicity requirement (9). Hence, Corollary 1 justifies the Theorem. □

Theorem 5 justifies the SSD 1-safety consistency of mean semideviation, standard semideviation and higher order below-mean semideviations when well-defined. Note that in the case of symmetric distributions, the requirement of expectation boundness (6) becomes valid not only for mean absolute semideviation $\bar{\delta}(X)$ itself but also for $2\bar{\delta}(X) = \delta(X)$ thus justifying the SSD 1-safety consistency of the entire mean absolute deviation. Similarly, in the case of symmetric distributions, the expectation bound (6) is valid for $\sqrt{2}\bar{\sigma}(X) = \sigma(X)$ thus justifying the consistency results for the standard deviation itself.

It follows from the majorization theory (Hardy, Littlewood and Polya, 1934; Marshall and Olkin, 1979) that

In the case of simple lotteries constructed as random variables corresponding n-dimensional real vectors (probability $1/n$ is assigned to each coordinate if they are different, while probability k/n is assigned to the value of k coinciding coordinates) the assumptions (11) can be dropped form Theorem 5. It can be derived from the majorization theory (Hardy, Littlewood and Polya, 1934; Marshall and Olkin, 1979). We will demonstrate this for more general space of lotteries. Hereafter, a *lottery* is a discrete random variable with a finite number of steps.

Lemma 1 *Lotteries X and Y satisfies $X \succeq_{RS} Y$ if and only if $F_{-2}(X,p) \geq F_{-2}(Y,p)$ for all p - cumulative probability of a step of $F_1(X,\alpha)$ or $F_1(Y,\alpha)$ and $F_{-2}(X,1) = F_{-2}(Y,1)$.*

Proof. ¿From quantile characterization of SSD we have inequality for all $p \in (0,1)$. On the other hand, we can see that p - cumulative probability of steps are sufficient.
Let $p_1, p_2, \ldots, p_m$ - cumulative probability of the steps of $F_1(X,\alpha)$ or $F_1(Y,\alpha)$, and let $c \in (p_i, p_{i+1})$. One may notice that

$$\begin{aligned} F_{-2}(X,c) &= F_{-2}(X,p_i) + (c - p_i)F_{-1}(X,p_{i+1}) \\ &= F_{-2}(X,p_{i+1}) - (p_{i+1} - c)F_{-1}(X,p_{i+1}) \end{aligned}$$

Hence, $F_{-2}(X, c) \geq F_{-2}(Y, c)$ whenever $F_{-2}(X, p_i) \geq F_{-2}(Y, p_i)$ and $F_{-2}(X, p_{i+1}) \geq F_{-2}(Y, p_{i+1})$. This follows from the first equation apllied to the case of $F_{-1}(X, p_{i+1}) \geq F_{-1}(Y, p_{i+1})$, or from the second equation applied to the case of $F_{-1}(X, p_{i+1}) < F_{-1}(Y, p_{i+1})$, respectively.
Furthermore, $F_{-2}(X, 1) = \mathbb{E}[X] = \mathbb{E}[Y] = F_{-2}(Y, 1)$ is necessary for RS-dominance. ∎

Let $X = Y_n \succeq_{RS} Y_{n-1} \succeq_{RS} \ldots \succeq_{RS} Y_1 = Y$ and for all k: $Y_k = \lambda_{k-1} Y'_{k-1} + (1 - \lambda_{k-1}) Y''_{k-1}$, where Y'_{k-1}, Y''_{k-1} are the same distributed as Y_{k-1} then it is obvious to say that Y is more risky that X for all ρ - convex risk measures. Rothschild and Stiglitz have formulated that RS-dominance between two random variables is equivalent to existing a sequence of mean preserving spreads (MPS) that transform one variable to the other. Two variables X and Y differ by MPS if there exists some interval that the distribution of X one gets from the distribution of Y by removing some of the mass from inside the interval and moving it to some place outside this interval. Gaining by MPS we'll show that for lotteries with rational probability X and Y if only X $\succeq_{RS}$ Y then Y is always more risky than X for all convex risk measures.

Theorem 6 *Let X, Y - lotteries with rational probability of steps. If X $\succeq_{RS}$ Y then there exists a sequence of lotteries $Y_1, Y_2, \ldots, Y_n$ satisfying the following conditions:*

1. $X = Y_n \succeq_{RS} Y_{n-1} \succeq_{RS} \ldots \succeq_{RS} Y_1 = Y$
2. $Y^{i+1} = (1 - \lambda^i) Y_1^i + \lambda^i Y_2^i$, *for* $i = 1, \ldots, n-1$, *where* $0 \leq \lambda^i \leq 1$ *and* Y_1^i, Y_2^i *are identically distributed as* Y^i.

Proof. We will construct a sequence of MPS – $Y_1, Y_2, \ldots, Y_n$ using quantile characterization of RS-dominance. From Rothschild and Stiglitz theorem (Leshno, Levy and Spector, 1997) one knows that the sequence $(Y_k)_{k=1,\ldots,n}$ exists. We will build it, however, as a convex combination of two identically distributed random variables.

Let $c_1, c_2, \ldots, c_m$ – cumulative probability of steps of $F_1(X, \alpha)$ or $F_1(Y, \alpha)$. It is sufficient to consider only the steps of distributions (Lemma 1).
Let $p_i = c_i - c_{i-1}$, for $i = 2, \ldots, m$ and $p_1 = c_1$, $\vec{p} = (p_1, \ldots, p_m)$,
x_i – c_i-quantile of X for $i = 1, \ldots, m$, $\vec{X} = (x_1, \ldots, x_m)$,
y_i – c_i-quantile of Y for $i = 1, \ldots, m$, $\vec{Y} = (y_1, \ldots, y_m)$.
There exists the first index i such that $x_i \neq y_i$ (and then $x_i > y_i$, due to the dominance) as well as there exists the last index i for which $x_i \neq y_i$ (and then $x_i < y_i$, due to the dominance) with the equality: $F_{-2}(X, 1) = \mathbb{E}[X] = \mathbb{E}[Y] = F_{-2}(Y, 1)$. With no loss of generality we can assume that the first index is 1 and the last one is m.
¿From F_{-2} definition, we get $F_{-2}(Y, c_i) = \sum_{j=1}^{i} p_j y_j$.
Define: $\Delta_j^i := x_j - y_j^i \ \forall_{i=1,\ldots,n,\, j=1,\ldots,m}$,

First step. $\Delta_1^1 > 0$ and $\Delta_k^1 < 0$, where $k = \min\{i : \Delta_i^1 < 0\}$, let $\Delta^1 = \min\{\Delta_1^1, -\Delta_k^1\}$ and $p^1 = \min\{p_1, p_k\}$

$\vec{p}^{\,1}$:	p_1,		p_2	$\ldots$	p_k,		$\ldots$	p_m
$\vec{Y}^{\,1}$:	y_1,		y_2	$\ldots$	y_k,		$\ldots$	y_m
$\vec{X}^{\,1}$:	x_1,		x_2	$\ldots$	x_k,		$\ldots$	x_m

$\vec{p}^{\,2}$:	p^1,	$p_1 - p^1$,	p_2	$\ldots$	p^1,	$p_k - p^1$	$\ldots$	p_m
$\vec{Y}^{\,2}$:	y_1,	y_1,	y_2	$\ldots$	y_k,	y_k	$\ldots$	y_m
$\vec{X}^{\,2}$:	x_1,	x_1,	x_2	$\ldots$	x_k,	x_k	$\ldots$	x_m

Note that $\vec{p}^{\,2}$ has at least one coordinate equal 0: $p_1 - p^1 = 0$ or $p_k - p^1 = 0$ while $\vec{Y}^{\,2}$ and $\vec{X}^{\,2}$ have at least one new the same coordinate: $x_1 = y_1^1 + \Delta^1$ or $x_k = y_k^1 - \Delta^1$.

With a finite number of steps we can transform y_1^1 to x_1 or y_k^1 to x_k. Thus, m coordinates of Y can be transformed with a finite number of steps to m coordinates of X.

The i-th step has the same idea: we choose first index where Δ_j^i is positive and it can be treated as Δ_1^1 in first step, because for all indexes before $\Delta_j^i = 0$. Then, we choose first index when Δ_j^i is negative as Δ_k^1 in first step. Δ^i, p^i are formed in the same way as Δ^1, p^1.

$\vec{Y}^{\,i}$ is built from $\vec{Y}^{\,i-1}$ by moving the same mass – Δ^i from one coordinate to another with the same probability – p^i. Let j, k be these coordinates, the rest of them are the same in the vectors.

$$\begin{array}{rllllll} \vec{p}^{\,i}: & \dots & p^i, & \dots & p^i, & \dots \\ \vec{Y}^{\,i}: & \dots & y_j^i, & \dots & y_k^i, & \dots \\ \vec{Y}'^{\,i}: & \dots & y_k^i, & \dots & y_j^i, & \dots \\ \vec{Y}^{\,i+1}: & \dots & y_j^i + \Delta^i, & \dots & y_k^i - \Delta^i, & \dots \end{array}$$

$\vec{Y}_1^{\,i}$ and $\vec{Y}^{\,i}$ are the same distributed and $(1-\lambda^i)\vec{Y}_1^{\,i} + \lambda^i \vec{Y}^{\,i} = \vec{Y}^{\,i+1}$ where $\lambda^i = \Delta^i/(y_k^i - y_j^i)$ □

$X, Y', Y'' \in (\Omega, \mathcal{F}, \mathbb{P})$; Y', Y'' – the same distributed lotteries with rational probability of steps. If $X = \lambda Y' + (1-\lambda)Y''$, then for all convex positive functions ϱ (where $\varrho(Y') < \infty$), one gets $\varrho(X) = \varrho(\lambda Y' + (1-\lambda)Y'') \leq \lambda\varrho(Y') + (1-\lambda)\varrho(Y'')$. Moreover, if ϱ depends only on distributions, then $\varrho(Y') = \varrho(Y'')$. Hence, $X \succeq_{RS} Y'$ implies then $\varrho(X) \leq \varrho(Y')$ which leads to the following assertion.

Theorem 7 *Let us consider a linear space $\mathcal{L} \subset L^k(\Omega, \mathcal{F}, \mathbb{P})$ of lotteries with rational probability of steps. If risk measure $\varrho(X) \geq 0$ depending only on distributions is convex, positively homogeneous, translation invariant and expectation bounded, then the measure is SSD 1-safety consistent on $\mathcal{L}$.*

Note that Theorem 7 applies to the important class of distributions where one may take advantages of the LP computable risk measures (Mansini, Ogryczak and Speranza, 2003). It justifies then the sufficient conditions for the coherency as simultaneously sufficient for SSD (safety) consistency.

4 Concluding remarks

Comparing uncertain prospects is one of fundamental interests of the economic decision theory. The mean-risk approach quantifies the problem in a lucid form of only two criteria: the mean, representing the expected outcome, and the risk: a scalar measure of the variability of outcomes. The mean-risk model is appealing to decision makers and allows a simple trade-off analysis, analytical or geometrical. On the other hand, mean-risk approaches are not capable of modeling the entire gamut of risk-averse preferences. Moreover, for typical dispersion statistics used as risk measures, the mean-risk approach may lead to inferior conclusions with respect to stochastic dominance or other axiomatic risk aversion models.

We have demonstrated that one may specify risk dependent performance functions which allow to transform several risk measures into SSD consistent safety measures. We have introduced sufficient conditions which allow us to justify various risk measures. In particular, such SSD consistency results can be applied to risk measures defined as the standard semideviation, the mean semideviation, or the maximum semideviation. Moreover, we have shown that these safety measures become the coherent risk measures, after simple change of the sign. On the other hand, while focusing on the space of finite lotteries where one may take advantages of the LP computable risk measures, it turns out that the sufficient conditions for the coherency are also sufficient for SSD consistency.

References

Acerbi, C., Tasche, D. (2002), On the coherence of expected shortfall. *Journal of Banking and Finance*, 26, 1491-1507.

Artzner, P., Delbaen, F., Eber, J.-M., Heath, D. (1999), Coherent measures of risk. *Mathematical Finance*, 9, 203-228.

Baumol, W.J. (1964), An expected gain-confidence limit criterion for portfolio selection. *Management Science*, 10, 174-182.

Bawa, V.S. (1982), Stochastic Dominance: A Research Bibliography, *Management Science*, 28, 698-712.

Dentcheva, D., Ruszczyński, A. (2003), Optimization with Stochastic Dominance Constraints, *SIAM J. Optimization*, 14, 548-566.

Fishburn, P.C. (1964), *Decision and Value Theory*, Wiley, New York.

Fishburn, P.C. (1977), Mean-Risk Analysis with Risk Associated with Below Target Returns, *The American Economic Review*, 67, 116-126.

Hanoch, G., Levy, H. (1969), The Efficiency Analysis of Choices Involving Risk, *Revue Economic Studies*, 36, 335-346.

Hardy, G.H., Littlewood, J.E., Polya, G. (1934), *Inequalities*, Cambridge University Press, Cambridge, MA.

Konno, H., Yamazaki, H. (1991), Mean-Absolute Deviation Portfolio Optimization Model and Its Application to Tokyo Stock Market, *Management Science*, 37, 519-531.

Leshno, M., Levy, H., Spector, Y. (1997), A Comment on Rothschild and Stiglitz's "Increasing Risk: I. A Definition", *Journal of EConomic Theory*, 77, 223-228.

Levy, H. (1992), Stochastic Dominance and Expected Utility: Survey and Analysis, *Management Science*, 38, 555-593.

Mansini, R., Ogryczak, W., Speranza, M.G. (2003), LP Solvable Models for Portfolio Optimization: A Classification and Computational Comparison, *IMA Journal of Management Mathematics*, 14, 187-220.

Markowitz, H.M. (1952), Portfolio Selection, *Journal of Finance*, 7, 77-91.

Markowitz, H.M. (1959), *Portfolio Selection*, Wiley, New York.

Markowitz, H.M. (1987), *Mean-Variance Analysis in Portfolio Choice and Capital Markets*, Blackwell, Oxford.

Marshall, A.W., Olkin, I. (1979), *Inequalities: Theory of Majorization and Its Applications*, Academic Press, NY, 1979.

Mueller, A., Stoyan, D. (2002), *Comparison Methods for Stochastic Models and Risks*, Wiley, New York.

Ogryczak, W. (2000), Multiple criteria linear programming model for portfolio selection, *Annals of Operations Research*, 97, 143-162.

Ogryczak, W., Ruszczyński, A. (1999), From stochastic dominance to mean-risk models: Semideviations as risk measures, *European Journal of Operational Research*, 116, 33-50.

Ogryczak, W., Ruszczyński, A. (2001), On stochastic dominance and mean-semideviation models. *Mathematical Programming*, 89, 217-232.

Ogryczak, W., Ruszczyński, A. (2002), Dual stochastic dominance and related mean-risk models, *SIAM J. Optimization*, 13, 60-78.

Porter, R.B. (1974), Semivariance and Stochastic Dominance: A Comparison, *The American Economic Review*, 64, 200-204.

Porter, R.B., Gaumnitz, J.E. (1972), Stochastic Dominance versus Mean-Variance Portfolio Analysis, *The American Economic Review*, 62, 438-446.

Rockafellar, R.T., Uryasev, S. (2000), Optimization of conditional value-at-risk. *Journal of Risk*, 2, 21-41.

Rockafellar, R.T., Uryasev, S., Zabarankin, M. (2003), Portfolio Analysis with General Deviation Measures, Univ. of Florida, ISE, RR 2003-8.

Rothschild, M., Stiglitz, J.E. (1970), Increasing Risk: I. A Definition, *Journal of Economic Theory*, 2, 225-243.

Roy, A.D. (1952), Safety-first and the holding of assets. *Econometrica*, 20, 431-449.

Sharpe, W.F. (1971), A linear programming approximation for the general portfolio analysis problem. *Journal of Financial and Quantitative Analysis*, 6, 1263-1275.

Shorrocks, A.F. (1983), Ranking Income Distributions, *Economica*, 50, 3-17.

Sortino, F.A., Forsey, H.J. (1996), On the use and misuse of downside risk, *Journal of Portfolio Management*, Winter 1996, 35-42.

Speranza, M.G. (1993), Linear programming models for portfolio optimization, *Finance*, 14, 107-123.

Whitmore, G.A, Findlay, M.C. (Eds.) (1978), *Stochastic Dominance: An Approach to Decision-Making Under Risk*, D.C.Heath, Lexington, MA.

Yitzhaki, S. (1982), Stochastic Dominance, Mean Variance, and Gini's Mean Difference, *The American Economic Review*, 72, 178-185.

Young, M.R. (1998), A minimax portfolio selection rule with linear programming solution, *Management Science*, 44, 673-683.