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Abstract

Resource allocation problems are concerned with the allocation of limited resources among competing activities so as to achieve the best performances. However, in systems which serve many users there is a need to respect some fairness rules while looking for the overall efficiency. The so-called Max-Min Fairness is widely used to meet these goals. However, allocating the resource to optimize the worst performance may cause a dramatic worsening of the overall system efficiency. Therefore, several other fair allocation schemes are searched and analyzed. In this paper we show how the concepts of multiple criteria equitable optimization can effectively be used to generate various fair and efficient allocation schemes. First, we demonstrate how the scalar inequality measures can be consistently used in bicriteria models to search for a compromise fair and efficient allocations. Further, two alternative multiple criteria models equivalent to equitable optimization are introduced thus allowing to generate a larger variety of fair and efficient resource allocation schemes.

Key Words. Multiple Criteria, Efficiency, Fairness, Equity, Inequality Measures.

1 Introduction

Resource allocation problems are concerned with the allocation of limited resources among competing activities (Ibaraki and Katoh, 1988). In this paper, we focus on approaches that, while allocating resources to maximize the system efficiency, they also attempt to provide a fair treatment of all the competing activities (Luss, 1999). The problems of efficient and fair resource allocation arise in various systems which serve many users, like in telecommunication systems among others. This applies among others to networking where a central issue is how to allocate bandwidth to flows efficiently and fairly (Denda, Banchs and Effelsberg, 2000; Pióro and Medhi, 2004). The issue of equity is widely recognized in location analysis of public services, where the clients of a system are entitled to fair treatment according to community regulations. In such problems, the decisions often concern the placement of a service center or other facility in a position so that the users are treated in an equitable way, relative to certain criteria (Ogryczak, 2000). Moreover, uniform individual outcomes may be associated with some events rather than

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physical users, like in many dynamic optimization problems where uniform individual criteria represent a similar event in various periods and all they are equally important. Recently, several research publications relating the fairness and equity concepts to the multiple criteria optimization methodology have appeared (Luss, 1999; Ogryczak and Śliwiński, 2002; Kaliszewski, 2004; Kostreva, Ogryczak and Wierzbicki, 2004).

The generic resource allocation problem may be stated as follows. Each activity is measured by an individual performance function that depends on the corresponding resource level assigned to that activity. A larger function value is considered better, like the performance measured in terms of quality level, capacity, service amount available, etc. Models with an (aggregated) objective function that maximizes the mean (or simply the sum) of individual performances are widely used to formulate resource allocation problems, thus defining the so-called mean solution concept. This solution concept is primarily concerned with the overall system efficiency. As based on averaging, it often provides solution where some smaller services are discriminated in terms of allocated resources. An alternative approach depends on the so-called Max-Min solution concept, where the worst performance is maximized. The max-min approach is consistent with Rawlsian (Rawls, 1971) theory of justice, especially when additionally regularized with the lexicographic order. On the other hand, allocating the resources to optimize the worst performances may cause a large worsening of the overall (mean) performances.

Fairness is, essentially, an abstract socio-political concept that implies impartiality, justice and equity (Rawls, 1958; Young, 1994). Nevertheless, fairness was usually quantified with the so-called inequality measures to be minimized (Atkinson, 1970; Rothschild and Stiglitz, 1973; Sen, 1973). Unfortunately, direct minimization of typical inequality measures contradicts the maximization of individual outcomes and it may lead to inferior decisions. The concept of fairness has been studied in various areas beginning from political economics problems of fair allocation of consumption bundles (Dalton, 1920; Pigou, 1912; Rawls, 1958) to abstract mathematical formulation (Steinhaus, 1949). In order to ensure fairness in a system, all system entities have to be equally well provided with the system's services. This leads to concepts of fairness expressed by the equitable efficiency (Ogryczak, 1997; Kostreva and Ogryczak, 1999: Luss, 1999). The concept of equitably efficient solution is a specific refinement of the Pareto-optimality taking into account the inequality minimization according to the Pigou-Dalton approach.

The paper is organized as follows. In the next section the equitable optimization with the preference structure that complies with both the efficiency (Pareto-optimality) and with the Pigou-Dalton principle of transfers is used to formalize the fair solution concepts. In Section 3 the use of scalar inequality measures in bicriteria models to search for a compromise fair and efficient allocations is analyzed. There is shown that properties of convexity and positive homogeneity together with some boundedness condition are sufficient for a typical inequality measure to guarantee the corresponding fair consistency. Further, two alternative multiple criteria models equivalent to equitable optimization are introduced thus allowing to generate a larger variety of fair and efficient resource allocation schemes. In Section 4 the multiple criteria model of the cumulative ordered outcomes is analyzed. This model covers the MMF and the Ordered Weighted Averaging (Yager, 1988) as special cases but it allows also for the reference point approaches. In Section 5 the alternative model of multiple targets is introduced. There is shown that the fair dominance can be expressed by pointwise comparison of mean shortfalls to several real target values. The model requires a finite set of outcome values or its approximation with some grid but it is very attractive computationally.

2 Equity and fairness

The generic resource allocation problem may be stated as follows. There is a system dealing with a set I of m services. There is given a measure of services realization within a system. In applications we consider, the measure usually expresses the service quality. However, outcomes can be measured (modeled) as service time, service costs, service delays as well as in a more subjective way. There is also given a set Q of allocation patterns (allocation decisions). For each service $i \in I$ a function $f_i(\mathbf{x})$ of the decision $\mathbf{x}$ has been defined. This function, called the individual objective function, measures the outcome (effect) $y_i = f_i(\mathbf{x})$ of the allocation pattern for service i. In typical formulations a larger value of the outcome means a better effect (higher service quality or client satisfaction). Otherwise, the outcomes can be replaced with their complements to some large number. Therefore, without loss of generality, we can assume that each individual outcome y_i is to be maximized which allows us to view the generic resource allocation problem as a vector maximization model:

$$\max \{\mathbf{f}(\mathbf{x}) \; : \; \mathbf{x} \in Q\} \tag{1}$$

where

$\mathbf{f}(\mathbf{x})$	is a vector–function that maps the decision space $X = R^n$ into the criterion space $Y = R^m$,
$Q \subset X$	denotes the feasible set,
$\mathbf{x} \in X$	denotes the vector of decision variables.

Model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. In order to make it operational, one needs to assume some solution concept specifying what it means to maximize multiple objective functions. The solution concepts may be defined by properties of the corresponding preference model. The preference model is completely characterized by the relation of weak preference (Vincke, 1992), denoted hereafter with $\succeq$. Namely, the corresponding relations of strict preference $\succ$ and indifference $\cong$ are defined by the following formulas:

$$\begin{aligned} \mathbf{y}' \succ \mathbf{y}'' \quad &\Leftrightarrow \quad (\mathbf{y}' \succeq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \not\succeq \mathbf{y}'), \\ \mathbf{y}' \cong \mathbf{y}'' \quad &\Leftrightarrow \quad (\mathbf{y}' \succeq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \succeq \mathbf{y}'). \end{aligned}$$

The standard preference model related to the Pareto-optimal (efficient) solution concept assumes that the preference relation $\succeq$ is *reflexive*:

$$\mathbf{y} \succeq \mathbf{y}, \tag{2}$$

transitive:

$$(\mathbf{y}' \succeq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \succeq \mathbf{y}''') \quad \Rightarrow \quad \mathbf{y}' \succeq \mathbf{y}''', \tag{3}$$

and *strictly monotonic*:

$$\mathbf{y} + \varepsilon \mathbf{e}_i \succ \mathbf{y} \quad \text{for } \varepsilon > 0; \; i = 1, \ldots, m, \tag{4}$$

where $\mathbf{e}_i$ denotes the i–th unit vector in the criterion space. The last assumption expresses that for each individual objective function more is better (maximization). The preference relations satisfying axioms (2)–(4) are called hereafter *rational preference relations*. The rational preference relations allow us to formalize the Pareto-optimality (efficiency) concept with the following definitions. We say that outcome vector $\mathbf{y}'$ rationally dominates $\mathbf{y}''$ ($\mathbf{y}' \succ_r \mathbf{y}''$), iff $\mathbf{y}' \succ \mathbf{y}''$ for all rational preference relations $\succeq$. We say that feasible solution $\mathbf{x} \in Q$ is a *Pareto-optimal (efficient)* solution of the multiple criteria problem (1), iff $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is rationally nondominated.

Simple solution concepts for multiple criteria problems are defined by aggregation (or utility) functions $g : Y \rightarrow R$ to be maximized. Thus the multiple criteria problem (1) is replaced with the maximization problem

$$\max \{g(\mathbf{f}(\mathbf{x})) \ : \ \mathbf{x} \in Q\} \tag{5}$$

In order to guarantee the consistency of the aggregated problem (5) with the maximization of all individual objective functions in the original multiple criteria problem (or Pareto-optimality of the solution), the aggregation function must be strictly increasing with respect to every coordinate.

The simplest aggregation functions commonly used for the multiple criteria problem (1) are defined as the mean (average) outcome

$$\mu(\mathbf{y}) = \frac{1}{m} \sum_{i=1}^{m} y_i \tag{6}$$

or the worst outcome

$$M(\mathbf{y}) = \min_{i=1,\ldots,m} y_i. \tag{7}$$

The mean (6) is a strictly increasing function while the minimum (7) is only nondecreasing. Therefore, the aggregation (5) using the sum of outcomes always generates a Pareto-optimal solution while the maximization of the worst outcome may need some additional refinement. The mean outcome maximization is primarily concerned with the overall system efficiency. As based on averaging, it often provides a solution where some services are discriminated in terms of performances. On the other hand, the worst outcome maximization, ie, the so-called Max-Min solution concept

$$\max\{ \min_{i=1,\ldots,m} f_i(\mathbf{x}) \ : \ \mathbf{x} \in Q \}. \tag{8}$$

is regarded as maintaining equity. Indeed, in the case of a simplified resource allocation problem with knapsack constraints, the Max-Min solution

$$\max\{ \min_{i=1,\ldots,m} y_i \ : \ \sum_{i=1}^{m} a_i y_i \leq b \} \tag{9}$$

takes the form $\bar{y}_i = mb / \sum_{i=1}^{m} a_i$ for all $i \in I$ thus meeting the perfect equity requirement $\bar{y}_1 = \bar{y}_2 = \ldots = \bar{y}_m$. In the general case, with possibly more complex feasible set structure, this property is not fulfilled (Ogryczak, 2001). Nevertheless, the following assertion is valid.

Theorem 1 *If there exists a nondominated outcome vector $\bar{\mathbf{y}} \in Y$ satisfying the perfect equity requirement $\bar{y}_1 = \bar{y}_2 = \ldots = \bar{y}_m$, then $\bar{\mathbf{y}}$ is the unique optimal solution of the Max-Min problem*

$$\max\{ \min_{i=1,\ldots,m} y_i \ : \ \mathbf{y} \in Y \}. \tag{10}$$

Proof. Let $\bar{\mathbf{y}} \in Y$ be a nondominated outcome vector satisfying the perfect equity requirement. This means, there exists a number α such that $\bar{y}_i = \alpha$ for $i = 1, 2, \ldots, m$. Let $\mathbf{y} \in Y$ be an optimal solution of the Max-Min problem (10). Suppose, there exists some index i_0 such that $y_{i_0} \neq \bar{y}_{i_0}$. Due to the optimality of $\mathbf{y}$, we have:

$$y_i \geq \min_{1\leq j\leq m} y_j \geq \min_{1\leq j\leq m} \bar{y}_j = \alpha = \bar{y}_i \quad \forall\ i = 1,\ldots,m$$

which together with $y_{i_0} \neq \bar{y}_{i_0}$ contradicts the assumption that $\bar{\mathbf{y}}$ is nondominated. $\square$

According to Theorem 1, the perfectly equilibrated outcome vector is a unique optimal solution of the Max-Min problem if one cannot improve any of its individual outcome without worsening some others. Unfortunately, it is not a common case and, in general, the optimal set to the Max-Min aggregation (8) may contain numerous alternative solutions including dominated ones. While using standard algorithmic tools to identify the Max-Min solution, one of many solutions is then selected randomly.

Actually, the distribution of outcomes may make the Max-Min criterion partially passive when one specific outcome is relatively very small for all the solutions. For instance, while allocating clients to service facilities, such a situation may be caused by existence of an isolated client located at a considerable distance from all the facilities. Maximization of the worst service performances is then reduced to maximization of the service performances for that single isolated client leaving other allocation decisions unoptimized. This is a clear case of inefficient solution where one may still improve other outcomes while maintaining fairness by leaving at its best possible value the worst outcome. The Max-Min solution may be then regularized according to the Rawlsian principle of justice. Rawls (1971) considers the problem of ranking different "social states"which are different ways in which a society might be organized taking into account the welfare of each individual in each society, measured on a single numerical scale. Applying the Rawlsian approach, any two states should be ranked according to the accessibility levels of the least well–off individuals in those states; if the comparison yields a tie, the accessibility levels of the next–least well–off individuals should be considered, and so on. Formalization of this concept leads us to the lexicographic Max-Min concepts or the so-called Max-Min Fairness (Marchi and Oviedo, 1992; Klein, Luss and Rothblum, 1993; Luss, 1999).

The concept of fairness has been studied in various areas beginning from political economics problems of fair allocation of consumption bundles (Dalton, 1920; Pigou, 1912; Rawls, 1958) to abstract mathematical formulation (Steinhaus, 1949). In order to ensure fairness in a system, all system entities have to be equally well provided with the system's services. This leads to concepts of fairness expressed by the equitable rational preferences (Ogryczak, 1997; Kostreva and Ogryczak, 1999). First of all, the fairness requires impartiality of evaluation, thus focusing on the distribution of outcome values while ignoring their ordering. That means, in the multiple criteria problem (1) we are interested in a set of outcome values without taking into account which outcome is taking a specific value. Hence, we assume that the preference model is impartial (anonymous, symmetric). In terms of the preference relation it may be written as the following axiom

$$(y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(m)}) \cong (y_1, y_2, \ldots, y_m) \quad \text{for any permutation } \pi \text{ of } I \tag{11}$$

which means that any permuted outcome vector is indifferent in terms of the preference relation. Further, fairness requires equitability of outcomes which causes that the preference model should satisfy the (Pigou–Dalton) principle of transfers. The principle of transfers states that a transfer of any small amount from an outcome to any other relatively worse–off outcome results in a more preferred outcome vector. As a property of the preference relation, the principle of transfers takes the form of the following axiom

$$y_{i'} > y_{i''} \quad \Rightarrow \quad \mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''} \succ \mathbf{y} \quad \text{for } 0 < \varepsilon < y_{i'} - y_{i''} \tag{12}$$

The rational preference relations satisfying additionally axioms (11) and (12) are called hereafter *fair (equitable) rational preference relations*. We say that outcome vector $\mathbf{y}'$ *fairly (equitably) dominates* $\mathbf{y}''$ ($\mathbf{y}' \succ_e \mathbf{y}''$), iff $\mathbf{y}' \succ \mathbf{y}''$ for all fair rational preference relations $\succeq$. In other words, $\mathbf{y}'$ fairly dominates $\mathbf{y}''$, if there exists a finite sequence of vectors $\mathbf{y}^j$ $(j = 1, 2, \ldots, s)$ such that $\mathbf{y}^1 = \mathbf{y}''$, $\mathbf{y}^s = \mathbf{y}'$ and $\mathbf{y}^j$ is constructed from $\mathbf{y}^{j-1}$ by application of either permutation of coordinates, equitable transfer, or increase of a coordinate. An allocation pattern $\mathbf{x} \in Q$ is called *fairly (equitably) efficient* or simply *fair* if $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is fairly nondominated. Note that each fairly efficient solution is also Pareto-optimal, but not vice verse.

In order to guarantee fairness of the solution concept (5), additional requirements on the class of aggregation (utility) functions must be introduced. In particular, the aggregation function must be additionally symmetric (impartial), i.e. for any permutation π of I,

$$g(y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(m)}) = g(y_1, y_2, \ldots, y_m) \tag{13}$$

as well as be equitable (to satisfy the principle of transfers)

$$g(y_1, \ldots, y_{i'} - \varepsilon, \ldots, y_{i''} + \varepsilon, \ldots, y_m) > g(y_1, y_2, \ldots, y_m) \tag{14}$$

for any $0 < \varepsilon < y_{i'} - y_{i''}$. In the case of a strictly increasing function satisfying both the requirements (13) and (14), we call the corresponding problem (5) a *fair (equitable) aggregation* of problem (1). Every optimal solution to the fair aggregation (5) of a multiple criteria problem (1) defines some fair (equitable) solution.

Note that both the simplest aggregation functions, the sum (6) and the minimum (7), are symmetric although they do not satisfy the equitability requirement (14). To guarantee the fairness of solutions, some enforcement of concave properties is required. For any strictly concave, increasing utility function $u : R \to R$, the function $g(\mathbf{y}) = \sum_{i=1}^{m} u(y_i)$ is a strictly monotonic and equitable thus defining a family of the fair aggregations (Ogryczak, 1997)

$$\max \{\sum_{i=1}^{m} u(f_i(\mathbf{x})) \; : \; \mathbf{x} \in Q\} \tag{15}$$

Various concave functions utility s can be used to define fair aggregations (15) and the resulting fair solution concepts. In the case of the outcomes restricted to positive values, one may use logarithmic function thus resulting in the *Proportional Fairness* (PF) solution concept (Kelly, Mauloo and Tan, 1997; Pióro, Malicskó and Fodor, 2002). Actually, it corresponds to the so-called Nash criterion (Nash, 1950) which maximizes the product of additional utilities compared to the status quo. Again, in the case of a simplified resource allocation problem with knapsack constraints, the PF solution

$$\max\{\sum_{i=1}^{m} \log(y_i) \; : \; \sum_{i=1}^{m} a_i y_i \le b \; \} \tag{16}$$

takes the form $\bar{y}_i = b/a_i$ for all $i \in I$ thus allocating the resource inversely proportional to the consumption of particular services.

For a common case of upper bounded outcomes $y_i \leq u^*$ one may maximize power functions $-\sum_{i=1}^{m} (u^* - x_i)^p$ for $p > 1$ which corresponds to the minimization of the corresponding p-norm distances from the common upper bound u^* (Kostreva, Ogryczak and Wierzbicki, 2004). Various other concave functions s can be used to define fair aggregations and the resulting resource allocation schemes. In particular a parametric class of utility functions:

$$u(y_i, \alpha) = \begin{cases} y_i^{1-\alpha}/(1-\alpha) & \text{if } \alpha \neq 1 \\ \log(y_i) & \text{if } \alpha = 1 \end{cases}$$

may be used on positive outcomes generating various fair solution concepts for $\alpha > 0$ (Mo and Warland, 2000).

For $\alpha = 1$, it represents the PF approach while with α tending to the infinity it converges to the MMF. However, every such approach requires to build (or to guess) a utility function prior to the analysis and later it gives only one possible compromise solution. It is very difficult to identify and formalize the preferences at the beginning of the decision process. Moreover, apart from the trivial case of the total output maximization all the utility functions that really take into account any fairness preferences are nonlinear. Many decision models considered with fair outcomes are originally Linear Programming (LP) or Mixed Integer LP (MILP) models. Nonlinear objective functions applied to such models may result in computationally hard optimization problems. In the following, we shall describe an approach that allows to search for such compromise solutions with multiple linear criteria rather than the use nonlinear objective functions.

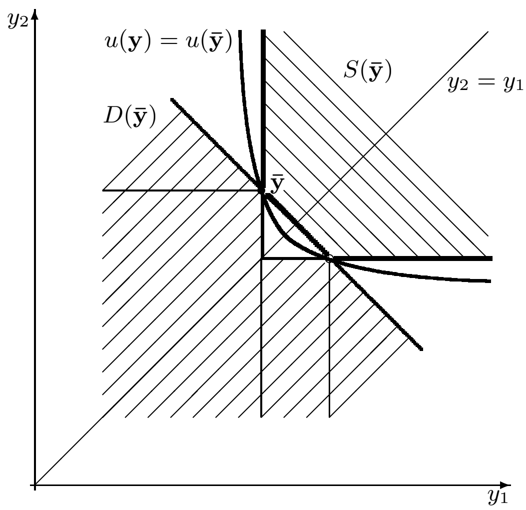

Figure 1: Structure of the fair dominance: $D(\bar{\mathbf{y}})$ – the set fairly dominated by $\bar{\mathbf{y}}$, $S(\bar{\mathbf{y}})$ – the set of outcomes fairly dominating $\bar{\mathbf{y}}$.

Fig. 1 presents the structure of fair dominance for two-dimensional outcome vectors. For any outcome vector $\bar{\mathbf{y}}$, the fair dominance relation distinguishes set $D(\bar{\mathbf{y}})$ of dominated outcomes (obviously worse for all fair rational preferences) and set $S(\bar{\mathbf{y}})$ of dominating outcomes (obviously better for all fair rational preferences). However, some outcome vectors are left (in white areas) and they can be differently classified by various specific fair rational preferences. The MMF fairness assigns the entire interior of the inner white triangle to the set of preferred outcomes while classifying the interior of the external open triangles as worse outcomes. Isolines of various utility functions split the white areas in different ways. For instance, there is no fair dominance between vectors $(0.01, 1)$ and $(0.02, 0.02)$ and the MMF considers the latter as better while the proportional fairness points out the former. On the other hand, vector $(0.02, 0.99)$ fairly dominates $(0.01, 1)$ and all fairness models (including MMF and PF) prefers the former. One may notice that the set $D(\bar{\mathbf{y}})$ of directions leading to outcome vectors being dominated by a given $\bar{\mathbf{y}}$ is, in general, not a cone and it is not convex. Although, when we consider the set $S(\bar{\mathbf{y}})$ of directions leading to outcome vectors dominating given $\bar{\mathbf{y}}$ we get a convex set. While the MMF nondominated optimal solution may be characterized by the lack of a possibility to increase of any outcome without decreasing of some smaller outcome, the general fairly nondominated vectors are characterized by the lack of a possibility to increase of any outcome without decreasing of some smaller outcome or greater decrease of some larger outcome.

3 Inequality measures and bicriteria models

Equity is, essentially, an abstract socio–political concept, but it is usually quantified with the so–called inequality measures to be minimized. Inequality measures were primarily studied in economics (Sen. 1973) while recently they become very popular tools in Operations Research. For instance, Marsh and Schilling (1994) describe twenty different measures proposed in the literature to gauge the level of equity in facility location alternatives. Typical inequality measures are some deviation type dispersion characteristics. They are *translation invariant* in the sense that $\varrho(\mathbf{y} + a\mathbf{e}) = \varrho(\mathbf{y})$ for any outcome vector $\mathbf{y}$ and real number a (where $\mathbf{e}$ vector of units $(1, \ldots, 1)$), thus being not affected by any shift of the outcome scale. Moreover, the inequality measures are also *inequality relevant* which means that they are equal to 0 in the case of a perfectly equal outcomes while taking positive values for any unequal one.

The simplest inequality measures are based on the absolute measurement of the spread of outcomes, like the *mean (absolute) difference*

$$\Gamma(\mathbf{y}) = \frac{1}{2m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} |y_i - y_j| \tag{17}$$

or the *maximum (absolute) difference*

$$d(\mathbf{y}) = \max_{i,j=1,\ldots,m} |y_i - y_j|. \tag{18}$$

In most application frameworks better intuitive appeal may have inequality measures related to deviations from the mean outcome like the mean (absolute) deviation

$$\delta(\mathbf{y}) = \frac{1}{m} \sum_{i=1}^{m} |y_i - \mu(\mathbf{y})|. \tag{19}$$

or the *maximum (absolute) deviation*

$$R(\mathbf{y}) = \max_{i \in I} |y_i - \mu(\mathbf{y})|. \tag{20}$$

Note that the *standard deviation* σ (or the *variance* σ^2) represents both the deviations and the spread measurement as

$$\sigma(\mathbf{y}) = \sqrt{\frac{1}{m}\sum_{i\in I}(y_i - \mu(\mathbf{y}))^2} = \sqrt{\frac{1}{2m^2}\sum_{i\in I}\sum_{j\in I}(y_i - y_j)^2}. \tag{21}$$

Deviational measures may be focused on the downside semideviations as related to worsening of outcome while ignoring upper semideviations related to improvement of outcome. One may define the *maximum (downside) semideviation*

$$\Delta(\mathbf{y}) = \max_{i\in I}(\mu(\mathbf{y}) - y_i) \tag{22}$$

and the *mean (downside) semideviation*

$$\bar{\delta}(\mathbf{y}) = \frac{1}{m}\sum_{i\in I}(\mu(\mathbf{y}) - y_i)_+ \tag{23}$$

where $(.)_+$ denotes the nonnegative part of a number. Similarly, the *standard (downside) semideviation* is given as

$$\bar{\sigma}(\mathbf{y}) = \sqrt{\frac{1}{m}\sum_{i\in I}(\mu(\mathbf{y}) - y_i)_+^2}. \tag{24}$$

In economics one usually considers relative inequality measures normalized by mean outcome. Among many inequality measures perhaps the most commonly accepted by economists is the Gini coefficient, which is the relative mean difference. One can easily notice that direct minimization of typical inequality measures (especially the relative ones) may contradict the optimization of individual outcomes (Erkut, 1993). Unfortunately, this flaw cannot be overcome with the standard bicriteria mean-equity model (Ogryczak, 2000):

$$\max\ \{(\mu(\mathbf{f}(\mathbf{x})), -\varrho(\mathbf{f}(\mathbf{x})))\ :\ \mathbf{x} \in Q\} \tag{25}$$

which takes into account both the efficiency with optimization of the mean outcome $\mu(\mathbf{y})$ and the equity with minimization of an inequality measure $\varrho(\mathbf{y})$. The bicriteria mean-equity model still does not completely eliminate contradiction to the optimization of individual outcomes. When considering a simple discrete problem with two allocation patterns P1 and P2 generating outcome vectors $\mathbf{y}' = (0, 0)$ and $\mathbf{y}'' = (2, 8)$, respectively, for any dispersion type inequality measure one gets $\varrho(\mathbf{y}'') > 0 = \varrho(\mathbf{y}')$ while $\mu(\mathbf{y}'') = 5 > 0 = \mu(\mathbf{y}')$. Hence, $\mathbf{y}''$ is not bicriteria dominated by $\mathbf{y}'$ and vice versa. Nevertheless, one must accept that for any dispersion type inequality measure ϱ, allocation P1 with obviously worse outcome vector than that for allocation P2 is a Pareto-optimal solution in the corresponding bicriteria mean-equity model.

Note that the lack of consistency with the dominance applies also to the maximum semideviation $\Delta(\mathbf{y})$ (22) whereas subtracting this measure from the mean $\mu(\mathbf{y}) - \Delta(\mathbf{y}) = M(\mathbf{y})$ results in the worst outcome and thereby the first criterion of the MMF model. In other words, although a direct use of the maximum semideviation contradicts the efficiency, the measure can be used complementary to the mean leading to the worst outcome criterion which is fairly consistent. This construction can be generalized for various (dispersion type) inequality measures. For any inequality measure ϱ we introduce the corresponding underachievement function defined as the difference of the mean outcome and the inequality measure itself, i.e.

$$M_{\varrho}(\mathbf{y}) = \mu(\mathbf{y}) - \varrho(\mathbf{y}). \tag{26}$$

This allows us to replace the original mean-equity bicriteria optimization (25) with the following bicriteria problem:

$$\max \{(\mu(\mathbf{f}(\mathbf{x})), \mu(\mathbf{f}(\mathbf{x})) - \varrho(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\} \tag{27}$$

where the second objective represents the corresponding underachievement measure $M_{\varrho}(\mathbf{y})$ (26). Note that for any inequality measure $\varrho(\mathbf{y}) \geq 0$ one gets $M_{\varrho}(\mathbf{y}) \leq \mu(\mathbf{y})$ thus really expressing underachievements (comparing to mean) from the perspective of outcomes being maximized.

The fair consistency of inequality measures may be formalized as follows. We say that inequality measure $\varrho(\mathbf{y})$ is *mean-complementary fairly consistent* if the corresponding underachievement measure $M_{\varrho}(\mathbf{y})$ is fairly consistent, i.e.,

$$\mathbf{y}' \succeq_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') - \varrho(\mathbf{y}') \geq \mu(\mathbf{y}'') - \varrho(\mathbf{y}''). \tag{28}$$

The relation of fair (mean-complementary) consistency is called *strong* if, in addition to (28), the following holds

$$\mathbf{y}' \succ_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') - \varrho(\mathbf{y}') > \mu(\mathbf{y}'') - \varrho(\mathbf{y}''). \tag{29}$$

Theorem 2 *If the inequality measure $\varrho(\mathbf{y})$ is mean-complementary fairly consistent (28), then except for outcomes with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$, every efficient solution of the bicriteria problem (27) is a fairly efficient allocation pattern. In the case of strong consistency (29), every solution $\mathbf{x} \in Q$ efficient to (27) is, unconditionally, fairly efficient.*

Proof. Let $\mathbf{x}^0 \in Q$ be an efficient solution of (27). Suppose that $\mathbf{x}^0$ is not fairly efficient. This means, there exists $\mathbf{x} \in Q$ such that $\mathbf{y} = \mathbf{f}(\mathbf{x}) \succ_e \mathbf{y}^0 = \mathbf{f}(\mathbf{x}^0)$. Then, it follows $\mu(\mathbf{y}) \geq \mu(\mathbf{y}^0)$, and simultaneously $\mu(\mathbf{y}) - \varrho(\mathbf{y}) \geq \mu(\mathbf{y}^0) - \varrho(\mathbf{y}^0)$, by virtue of the mean-complementary fair consistency (28). Since $\mathbf{x}^0$ is efficient to (27) no inequality can be strict, which implies $\mu(\mathbf{y}) = \mu(\mathbf{y}^0)$ and and $\varrho(\mathbf{y}) = \varrho(\mathbf{y}^0)$.

In the case of the strong mean-complementary fair consistency (29), the supposition $\mathbf{y} = \mathbf{f}(\mathbf{x}) \succ_e \mathbf{y}^0 = \mathbf{f}(\mathbf{x}^0)$ implies $\mu(\mathbf{y}) \geq \mu(\mathbf{y}^0)$ and $\mu(\mathbf{y}) - \varrho(\mathbf{y}) > \mu(\mathbf{y}^0) - \varrho(\mathbf{y}^0)$ which contradicts the efficiency of $\mathbf{x}^0$ with respect to (27). Hence, $\mathbf{x}^0$ is fairly efficient. □

Typical dispersion type inequality measures are convex, i.e. $\varrho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \leq \lambda\varrho(\mathbf{y}') + (1-\lambda)\varrho(\mathbf{y}'')$ for any $\mathbf{y}', \mathbf{y}''$ and $0 \leq \lambda \leq 1$. Actually, convexity of an inequality measure on equally distributed outcomes is necessary for its mean-complementary fair consistency. Note, that for any two vectors $\mathbf{y}'$ and $\mathbf{y}''$ representing the same distribution of outcomes as $\mathbf{y}$ (i.e., $\mathbf{y}' = (y_{\pi'(1)}, \ldots, y_{\pi'(m)})$ for some permutation π' and $\mathbf{y}'' = (y_{\pi''(1)}, \ldots, y_{\pi''(m)})$ for some permutation π'') due to concavity of $\bar{\theta}_i(\mathbf{y})$, one gets $\bar{\theta}_i(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \geq \lambda\bar{\theta}_i(\mathbf{y}') + (1-\lambda)\bar{\theta}_i(\mathbf{y}'') = \bar{\theta}_i(\mathbf{y})$ for all $i \in I$ and any $0 \leq \lambda \leq 1$. Hence, $\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'' \succeq_e \mathbf{y}$ and $M_{\varrho}(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \geq M_{\varrho}(\mathbf{y})$ is necessary for the fair consistency. Thus, due to equal means $\mu(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') = \mu(\mathbf{y}') = \mu(\mathbf{y}'') = \mu(\mathbf{y})$, the inequality measure depending only on distribution $\varrho(\mathbf{y}') = \varrho(\mathbf{y}'') = \varrho(\mathbf{y})$ must satisfy $\varrho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \leq \varrho(\mathbf{y}) = \lambda\varrho(\mathbf{y}') + (1-\lambda)\varrho(\mathbf{y}'')$ which represents the convexity of $\varrho(\mathbf{y})$. Certainly, the underachievement function $M_{\varrho}(\mathbf{y})$ must be also monotonic for the fair consistency which enforces more restrictions on the inequality measures. We will show further that convexity together with positive homogeneity and some boundedness of an inequality measure is sufficient to guarantee monotonicity of the corresponding underachievement measure and thereby to guarantee the mean-complementary fair consistency of inequality measure itself.

We say that (dispersion type) inequality measure $\varrho(\mathbf{y}) \geq 0$ is Δ-*bounded* if it upper bounded by the maximum downside deviation, i.e.,

$$\varrho(\mathbf{y}) \leq \Delta(\mathbf{y}) \quad \forall \mathbf{y}. \tag{30}$$

Moreover, we say that $\varrho(\mathbf{y}) \geq 0$ is strictly Δ-bounded if inequality (30) is a strict bound, except from the case of perfectly equal outcomes, i.e., $\varrho(\mathbf{y}) < \Delta(\mathbf{y})$ for any $\mathbf{y}$ such that $\Delta(\mathbf{y}) > 0$.

Theorem 3 *Let $\varrho(\mathbf{y}) \geq 0$ be a convex and Δ-bounded positively homogeneous inequality measure. Then $\varrho(\mathbf{y})$ is mean-complementary fairly consistent in the sense of (28), ie.*

$$\mathbf{y}' \succeq_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') - \varrho(\mathbf{y}') \geq \mu(\mathbf{y}'') - \varrho(\mathbf{y}'').$$

Proof. The relation of fair dominance $\mathbf{y}' \succeq_e \mathbf{y}''$ denotes that there exists a finite sequence of vectors $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \ldots, \mathbf{y}^t$ such that $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'} + \varepsilon_k \mathbf{e}_{i''}$, $0 \leq \varepsilon_k \leq y_{i'}^{k-1} - y_{i''}^{k-1}$ for $k = 1, 2, \ldots, t$ and there exists a permutation π such that $y'_{\pi(i)} \geq y_i^t$ for all $i \in I$. Note that the underachievement function $M_\varrho(\mathbf{y})$, similar as $\varrho(\mathbf{y})$ depends only on the distribution of outcomes. Further, if $\mathbf{y}' \geq \mathbf{y}''$, then $\mathbf{y}' = \mathbf{y}'' + (\mathbf{y}' - \mathbf{y}'')$ and $\mathbf{y}' - \mathbf{y}'' \geq 0$. Hence, due to concavity and positive homogeneity, $M_\varrho(\mathbf{y}') \geq M_\varrho(\mathbf{y}'') + M_\varrho(\mathbf{y}' - \mathbf{y}'')$. Moreover, due to the bound (30), $M_\varrho(\mathbf{y}' - \mathbf{y}'') \geq \mu(\mathbf{y}' - \mathbf{y}'') - \Delta(\mathbf{y}' - \mathbf{y}'') \geq \mu(\mathbf{y}' - \mathbf{y}'') - \mu(\mathbf{y}' - \mathbf{y}'') = 0$. Thus, $M_\varrho(\mathbf{y})$ satisfies also the requirement of monotonicity. Hence, $M_\varrho(\mathbf{y}') \geq M_\varrho(\mathbf{y}^t)$. Further, let us notice that $\mathbf{y}^k = \lambda \bar{\mathbf{y}}^{k-1} + (1-\lambda)\mathbf{y}^{k-1}$ where $\bar{\mathbf{y}}^{k-1} = \mathbf{y}^{k-1} - (y_{i'} - y_{i''})\mathbf{e}_{i'} + (y_{i'} - y_{i''})\mathbf{e}_{i''}$ and $\lambda = \varepsilon/(y_{i'} - y_{i''})$. Vector $\bar{\mathbf{y}}^{k-1}$ has the same distribution of coefficients as $\mathbf{y}^{k-1}$ (actually it represents results of swapping $y_{i'}$ and $y_{i''}$). Hence, due to concavity of $M_\varrho(\mathbf{y})$, one gets $M_\varrho(\mathbf{y}^k) \geq \lambda M_\varrho(\bar{\mathbf{y}}^{k-1}) + (1-\lambda) M_\varrho(\mathbf{y}^{k-1}) = M_\varrho(\mathbf{y}^{k-1})$. Thus, $M_\varrho(\mathbf{y}') \geq M_\varrho(\mathbf{y}'')$ which justifies the mean-complementary fair consistency of the inequality measure $\varrho(\mathbf{y})$. $\square$

For strict fair consistency some strict monotonicity and concavity properties of the achievement function are needed. Obviously, there does not exist any inequality measure which is positively homogeneous and simultaneously strictly convex. However, one may notice from the proof of Theorem 3 that only convexity properties on equally distributed outcome vectors are important for monotonous achievement functions. We say that function $C(\mathbf{y})$ is strictly convex on equally distributed outcome vectors, if $C(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') < \lambda C(\mathbf{y}') + (1-\lambda)C(\mathbf{y}'')$ for $0 < \lambda < 1$ and any two vectors $\mathbf{y}' \neq \mathbf{y}''$ but representing the same outcomes distribution as some $\mathbf{y}$, i.e., $\mathbf{y}' = (y_{\pi'(1)}, \ldots, y_{\pi'(m)})$ π' and $\mathbf{y}'' = (y_{\pi''(1)}, \ldots, y_{\pi''(m)})$ for some permutations π' and π'', respectively.

Theorem 4 *Let $\varrho(\mathbf{y}) \geq 0$ be a convex and strictly Δ-bounded positively homogeneous inequality measure. If $\varrho(\mathbf{y})$ is also strictly convex on equally distributed outcomes, then it is mean-complementary fairly strongly consistent in the sense of (29), ie.*

$$\mathbf{y}' \succ_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') - \varrho(\mathbf{y}') > \mu(\mathbf{y}'') - \varrho(\mathbf{y}'').$$

Proof. The relation of weak fair dominance $\mathbf{y}' \succeq_e \mathbf{y}''$ denotes that there exists a finite sequence of vectors $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \ldots, \mathbf{y}^t$ such that $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'} + \varepsilon_k \mathbf{e}_{i''}$, $0 \leq \varepsilon_k \leq y_{i'}^{k-1} - y_{i''}^{k-1}$ for $k = 1, 2, \ldots, t$ and there exists a permutation π such that $y'_{\pi(i)} \geq y_i^t$ for all $i \in I$. The strict fair dominance $\mathbf{y}' \succ_e \mathbf{y}''$ means that $y'_{\pi(i)} > y^t_i$ for some $i \in I$ or at least one ε_k is strictly positive. Note that the underachievement function $M_\varrho(\mathbf{y})$ is strictly monotonous and strictly convex on equally distributed outcome vectors. Hence, $M_\varrho(\mathbf{y}') > M_\varrho(\mathbf{y}'')$ which justifies the mean-complementary fair strong consistency of the inequality measure $\varrho(\mathbf{y})$. □

Let $\varrho(\mathbf{y}) \geq 0$ be a convex, positively homogeneous and Δ-bounded (dispersion type) inequality measure. It follows from Theorems 2–4 that except for allocation patterns with identical mean $\mu(\mathbf{y})$ and inequality measure $\varrho(\mathbf{y})$, every efficient solution to the bicriteria problem (27) is then a fairly efficient solution of the location problem (1). Moreover, if the measure is also strictly Δ-bounded and strictly convex on equally distributed outcome vectors, then every location $\mathbf{x} \in Q$ efficient to (27) is, unconditionally, fairly efficient.

As mentioned, typical inequality measures are convex and many of them are positively homogeneous. Moreover, the measures such as the mean absolute (downside) semideviation $\bar{\delta}(\mathbf{y})$ (23), the standard downside semideviation $\bar{\sigma}(\mathbf{y})$ (24), and the mean absolute difference $\Gamma(\mathbf{y})$ (17) are Δ-bounded. Indeed, one may easily notice that $y_i - \mu(\mathbf{y}) \leq \Delta(\mathbf{y})$ and therefore $\bar{\delta}(\mathbf{y}) \leq \frac{1}{m}\sum_{i\in I}\Delta(\mathbf{y}) = \Delta(\mathbf{y})$, $\bar{\sigma}(\mathbf{y}) \leq \sqrt{\Delta(\mathbf{y})^2} = \Delta(\mathbf{y})$ and $\Gamma(\mathbf{y}) = \frac{1}{m^2}\sum_{i\in I}\sum_{j\in I}(\max\{y_i, y_j\} - \mu(\mathbf{y})) \leq \Delta(\mathbf{y})$. Actually, all these inequality measures are strictly Δ-bounded since for any unequal outcome vector at least one outcome must be below the mean thus leading to strict inequalities in the above bounds. Obviously, Δ-bounded (but not strictly) is also the maximum absolute downside deviation $\Delta(\mathbf{y})$ itself. This us to justify the maximum downside deviation $\Delta(\mathbf{y})$ (22), the mean absolute (downside) semideviation $\bar{\delta}(\mathbf{y})$ (23), the standard downside semideviation $\bar{\sigma}(\mathbf{y})$ (24) and the mean absolute difference $\Gamma(\mathbf{y})$ (17) as mean-complementary fairly consistent in the sense of (28).

We emphasize that, despite the standard semideviation is mean-complementary fairly consistent inequality measure, the consistency is not valid for variance, semivariance and even for the standard deviation. These measures, in general, do not satisfy all assumptions of Theorem 3. Certainly, we have enumerated only the simplest inequality measures studied in the resource allocation context which satisfy the assumptions of Theorem 3 and thereby they are mean-complementary fairly consistent. Theorem 3 allows one to show this property for many other measures. In particular, one may easily find out that any convex combination of mean-complementary fairly efficient inequality measures remains also consistent. On the other hand, among typical inequality measures the mean absolute difference seems to be the only one meeting the stronger assumptions of Theorem 4 and thereby maintaining the strong consistency.

Note that the mean absolute semideviations are symmetric in the sense that the upper semideviation is always equal to the downside one. In other words, $\bar{\delta}(\mathbf{y}) = \frac{1}{2}\delta(\mathbf{y})$ and thereby the fair consistency of the mean absolute semideviation justifies also fair consistency of the half mean absolute deviation. In general, one may just consider $\varrho_\alpha(X) = \alpha\varrho(X)$ as a basic risk measure, like the mean absolute semideviation equal to the half of the mean absolute deviation itself. In order to avoid creation of new inequality measures by simple scaling we rather parameterize the fair consistency concept. We will say that an inequality measure ϱ is fairly α-consistent if

$$\mathbf{y}' \succeq_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') - \alpha\varrho(\mathbf{y}') \geq \mu(\mathbf{y}'') - \alpha\varrho(\mathbf{y}'') \tag{31}$$

The relation of fair α-consistency will be called *strong* if, in addition to (31), the following holds

$$\mathbf{y}' \succ_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') - \alpha\varrho(\mathbf{y}') > \mu(\mathbf{y}'') - \alpha\varrho(\mathbf{y}''). \tag{32}$$

Note that the fair 1-consistency represent our basic relation of the mean-complementary fair consistency. On the other hand, the fair α-consistency of measure $\varrho(\mathbf{y})$ is equivalent to the mean-complementary fair consistency of measure $\alpha\varrho(\mathbf{y})$. Thus the following assertion is valid.

Corollary 1 *If the inequality measure $\varrho(\mathbf{y})$ is fairly α-consistent (31), then except for outcomes with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$, every efficient solution of the bicriteria problem*

$$\max\{(\mu(\mathbf{f}(\mathbf{x})), \mu(\mathbf{f}(\mathbf{x})) - \alpha\varrho(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\} \tag{33}$$

is a fairly efficient allocation pattern. In the case of strong consistency (29), every allocation pattern $\mathbf{x} \in Q$ efficient to (33) is, unconditionally, fairly efficient.

Directly from Theorems 3 and 4 applied to the measures $\alpha\varrho(\mathbf{y})$ as in definition of α-consistency one gets the following sufficient conditions.

Corollary 2 *Let $\varrho(\mathbf{y}) \geq 0$ be a convex, positively homogeneous and translation invariant (dispersion type) inequality measure. If $\alpha\varrho(\mathbf{y})$ is Δ-bounded, then $\varrho(\mathbf{y})$ is fairly α-consistent in the sense of (31).*
If $\varrho(\mathbf{y})$ is also strictly convex on equally distributed outcomes and $\alpha\varrho(\mathbf{y})$ is strictly Δ-bounded, then $\varrho(\mathbf{y})$ is fairly strongly α-consistent in the sense of (32).

Note that the fair $\bar{\alpha}$-consistency of measure $\varrho(\mathbf{y})$ actually guarantees the mean-complementary fair consistency of measure $\alpha\varrho(\mathbf{y})$ for all $0 < \alpha \leq \bar{\alpha}$, and the same remain valid for the strong consistency properties. It follows from a possible expression of $\mu(\mathbf{y}) - \alpha\varrho(\mathbf{y})$ as the convex combination of $\mu(\mathbf{y}) - \bar{\alpha}\varrho(\mathbf{y})$ and $\mu(\mathbf{y})$. Hence, for any $\mathbf{y}' \succeq_e \mathbf{y}''$, due to $\mu(\mathbf{y}') \geq \mu(\mathbf{y}'')$ one gets $\mu(\mathbf{y}') - \alpha\varrho(\mathbf{y}') \geq \mu(\mathbf{y}'') - \alpha\varrho(\mathbf{y}'')$ in the case of the fair $\bar{\alpha}$-consistency of measure $\varrho(\mathbf{y})$ (or respective strict inequality in the case of strong consistency). Therefore, while analyzing specific inequality measures we seek a report the largest values α guaranteeing the corresponding fair efficiency.

As mentioned, the mean absolute semideviation is twice the mean absolute upper semideviation which means that $\alpha\delta(\mathbf{y})$ is Δ-bounded for any $0 < \alpha \leq 0.5$. The symmetry of mean absolute semideviations $\bar{\delta}(\mathbf{y}) = \sum_{i \in I}(y_i - \mu(\mathbf{y}))_+ = \sum_{i \in I}(\mu(\mathbf{y}) - y_i)_+$ can be also used to derive some Δ-boundedness relations for other inequality measures. In particular, one may find out that for m-dimensional outcome vectors of unweighted location problem, any downside semideviation from the mean cannot be larger than $m-1$ upper semideviations. Hence, the maximum absolute deviation satisfies the inequality $\frac{1}{m-1}R(\mathbf{y}) \leq \Delta(\mathbf{y})$, while the maximum absolute difference fulfills $\frac{1}{m}d(\mathbf{y}) \leq \Delta(\mathbf{y})$. Similarly, for the standard deviation one gets $\frac{1}{\sqrt{m-1}}\delta(\mathbf{y}) \leq \Delta(\mathbf{y})$. Actually, $\alpha\sigma(\mathbf{y})$ is strictly Δ-bounded for any $0 < \alpha \leq 1/\sqrt{m-1}$ since for any unequal outcome vector at least one outcome must be below the mean thus leading to strict inequalities in the above bounds. These allow us to justify the mean absolute semideviation with $0 < \alpha \leq 0.5$, the maximum absolute deviation with $0 < \alpha \leq \frac{1}{m-1}$, the maximum absolute difference with $0 < \alpha \leq \frac{1}{m}$ and the standard deviation with $0 < \alpha \leq \frac{1}{\sqrt{m-1}}$ as fairly α-consistent within the specified intervals of α. Moreover, the α-consistency of the standard deviation is strong.

The fair consistency results for basic dispersion type inequality measures considered in resource allocation problems are summarized in Table 1 where α values for unweighted as well as weighted problems are given and the strong consistency is indicated. Table 1 points out how the inequality measures can be used in resource allocation models to guarantee their harmony

Table 1: Fair consistency results for the basic dispersion type inequality measures

Measure			α–consistency	
Standard upper semideviation	$\bar{\sigma}(\mathbf{y})$	(24)	1	
Standard deviation	$\sigma(\mathbf{y})$	(21)	$\frac{1}{\sqrt{m-1}}$	strong
Mean absolute semideviation	$\bar{\delta}(\mathbf{y})$	(23)	1	
Mean absolute deviation	$\delta(\mathbf{y})$	(19)	$\frac{1}{2}$	
Maximum upper semideviation	$\Delta(\mathbf{y})$	(22)	1	
Maximum absolute deviation	$R(\mathbf{y})$	(20)	$\frac{1}{m-1}$	
Mean absolute difference	$\Gamma(\mathbf{y})$	(17)	1	strong
Maximum absolute difference	$d(\mathbf{y})$	(18)	$\frac{1}{m}$	

both with outcome maximization (Pareto-optimality) and with inequalities minimization (Pigou-Dalton equity theory). Exactly, for each inequality measure applied with the corresponding value α from Table 1 (or smaller positive value), every efficient solution of the bicriteria problem (33), ie. $\max\{(\mu(\mathbf{f}(\mathbf{x})), \mu(\mathbf{f}(\mathbf{x})) - \alpha\varrho(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}$, is a fairly efficient allocation pattern, except for outcomes with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$. In the case of strong consistency (as for mean absolute difference or standard deviation), every solution $\mathbf{x} \in Q$ efficient to (33) is, unconditionally, fairly efficient.

4 Ordered outcomes

Multiple criteria optimization defines the dominance relation $\mathbf{y}' \succeq_r \mathbf{y}''$ which may be expressed in terms of the vector inequality $\mathbf{y}' \geq \mathbf{y}''$. Hence, we can state that a feasible solution $\mathbf{x}^0 \in Q$ is a Pareto-optimal solution of the multiple criteria problem (1), if and only if, there does not exist $\mathbf{x} \in Q$ such that $\mathbf{f}(\mathbf{x}) \geq \mathbf{f}(\mathbf{x}^0)$. The latter refers to the commonly used definition of the Pareto-optimal (efficient) solutions as feasible solutions for which one cannot improve any criterion without worsening another (Vincke, 1992).

The theory of majorization (Marshall and Olkin, 1979) includes the results which allow us to express the relation of fair (equitable) dominance as a vector inequality on the cumulative ordered outcomes (Kostreva and Ogryczak, 1999). This can be mathematically formalized as follows. First, introduce the ordering map $\Theta : R^m \to R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \leq \theta_2(\mathbf{y}) \leq \cdots \leq \theta_m(\mathbf{y})$ and there exists a permutation π of set I such that $\theta_i(\mathbf{y}) = y_{\pi(i)}$ for $i = 1, \ldots, m$. Next, apply to ordered outcomes $\Theta(\mathbf{y})$, a linear cumulative map thus resulting in the cumulative ordering map $\bar{\Theta}(\mathbf{y}) = (\bar{\theta}_1(\mathbf{y}), \bar{\theta}_2(\mathbf{y}), \ldots, \bar{\theta}_m(\mathbf{y}))$ defined as

$$\bar{\theta}_i(\mathbf{y}) = \sum_{j=1}^{i} \theta_j(\mathbf{y}) \quad \text{for } i = 1, \ldots, m \tag{34}$$

The coefficients of vector $\bar{\Theta}(\mathbf{y})$ express, respectively: the smallest outcome, the total of the two smallest outcomes, the total of the three smallest outcomes, etc.

Note that fair solutions to problem (1) can be expressed as Pareto-optimal solutions for the multiple criteria problem with objectives $\bar{\Theta}(\mathbf{f}(\mathbf{x}))$

$$\max \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\} \tag{35}$$

Theorem 5 *A feasible allocation pattern* $\mathbf{x} \in Q$ *is a fair solution of the problem (1), iff it is a Pareto-optimal solution of the multiple criteria problem (35).*

Theorem 5 provides the relationship between fair solutions and the standard Pareto-optimality. Hence, the multiple criteria problem (35) may serve as a source of fair solution concepts. Although the definitions of quantities $\bar{\theta}_k(\mathbf{y})$, used as criteria in (35), are very complicated, the quantities themselves can be modeled with simple auxiliary variables and constraints. It is commonly known that the smallest outcome may be defined by the following optimization: $\bar{\theta}_1(\mathbf{y}) = \max \{t \ : \ t \leq y_i \quad \text{for } i = 1, \dots, m\}$, where t is an unrestricted variable. It turns out that this can be generalized to provide an effective modeling technique for quantities $\bar{\theta}_k(\mathbf{y})$ with arbitrary k (Ogryczak and Tamir, 2003). Let us notice that for any given vector $\mathbf{y}$, the quantity $\bar{\theta}_k(\mathbf{y})$ is defined by the following LP:

$$\begin{aligned} \bar{\theta}_k(\mathbf{y}) \quad &= \quad \min \sum_{i=1}^{m} y_i z_{ki} \\ &\text{s.t.} \quad \sum_{i=1}^{m} z_{ki} = k, \ 0 \leq z_{ki} \leq 1 \quad \text{for } i = 1, \dots, m. \end{aligned} \tag{36}$$

Exactly, the above problem is an LP for a given outcome vector $\mathbf{y}$ while it begins nonlinear for a variable $\mathbf{y}$. This difficulty can be overcome by taking advantages of the LP dual to (36):

$$\begin{aligned} \bar{\theta}_k(\mathbf{y}) \quad &= \quad \max \ kt_k - \sum_{i=1}^{m} d_{ik} \\ &\text{s.t.} \quad t_k - y_i \leq d_{ik}, \ d_{ik} \geq 0 \quad \text{for } i = 1, \dots, m \end{aligned} \tag{37}$$

where t_k is an unrestricted variable while nonnegative variables d_{ik} represent, for several outcome values y_i, their downside deviations from the value of t (Ogryczak and Tamir, 2003).

Theorem 5 allows one to generate fairly efficient solutions of (1) as Pareto-optimal solutions of multicriteria problem:

$$\begin{aligned} &\max \ (\eta_1, \eta_2, \dots, \eta_m) \\ &\text{subject to} \quad \mathbf{x} \in Q \\ &\eta_k = kt_k - \sum_{i=1}^{m} d_{ik} \quad \text{for } k = 1, \dots, m \\ &t_k - d_{ik} \leq f_i(\mathbf{x}), \quad d_{ik} \geq 0 \quad \text{for } i, k = 1, \dots, m \end{aligned} \tag{38}$$

The aggregation maximizing the sum of outcomes, corresponds to maximization of the last (m–th) objective (η_m) in problem (38). Similar, the Max-Min scalarization corresponds to maximization of the first objective (η_1). For modeling various fair preferences one may use some combinations the criteria. In particular, for the weighted sum $\sum_{i=1}^{m} w_i \eta_i$ on gets equivalent combination of the cumulative ordered outcomes $\bar{\theta}_i(\mathbf{y})$:

$$\sum_{i=1}^{m} w_i \bar{\theta}_i(\mathbf{y}). \tag{39}$$

Note that, due to the definition of map $\bar{\Theta}$ with (34), the above function can be expressed in the form with weights $v_i = \sum_{j=i}^{m} w_j$ $(i = 1, \dots, m)$ allocated to coordinates of the ordered

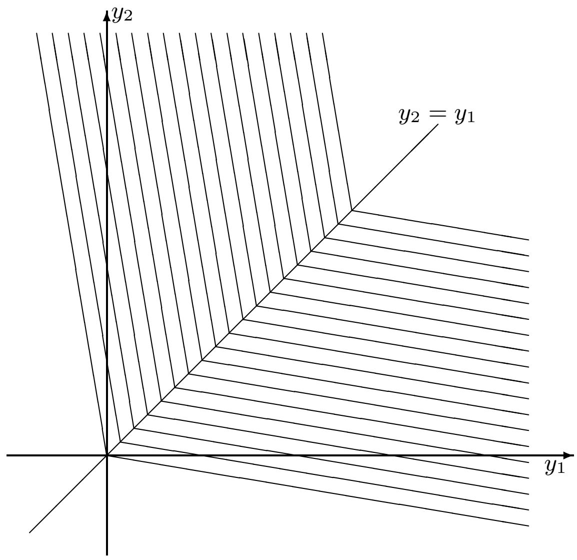

Figure 2: Isoline contours for a fair OWA aggregation.

outcome vector. Such an approach to aggregation of outcomes was introduced by Yager (1988) as the so–called Ordered Weighted Averaging (OWA). When applying OWA to problem (1) we get

$$\max \{\sum_{i=1}^{m} v_i \theta_i(\mathbf{f}(\mathbf{x})) \;:\; \mathbf{x} \in Q\} \tag{40}$$

The OWA aggregation is obviously a piece wise linear function since it remains linear within every area of the fixed order of arguments. If weights v_i are strictly decreasing and positive, i.e. $v_1 > v_2 > \cdots > v_{m-1} > v_m > 0$, then each optimal solution of the OWA problem (40) is a fair solution of (1).

While equal weights define the linear aggregation, several decreasing sequences of weights lead to various fair aggregation functions. Thus, the monotonic OWA aggregations provide a family of piece wise linear aggregations filling out the space between the piece wise linear aggregation functions (6) and (7) as shown in Fig. 3. Actually, formulas (39) and (37) allow us to formulate any monotonic (not necessarily strictly) OWA problem (40) as the following LP extension of the original multiple criteria problem:

$$\begin{aligned} &\max \sum_{k=1}^{m} w_k \eta_k \\ &\text{subject to} \quad \mathbf{x} \in Q \\ &\eta_k = k t_k - \sum_{i=1}^{m} d_{ik} \quad \text{for } k = 1, \ldots, m \\ &t_k - d_{ik} \le f_i(\mathbf{x}), \quad d_{ik} \ge 0 \quad \text{for } i, k = 1, \ldots, m \end{aligned} \tag{41}$$

where $w_m = v_m$ and $w_k = v_k - v_{k+1}$ for $k = 1, \ldots, m-1$.

When differences among weights tend to infinity, the OWA aggregation approximates the lexicographic ranking of the ordered outcome vectors (Ogryczak and Śliwiński, 2003). That

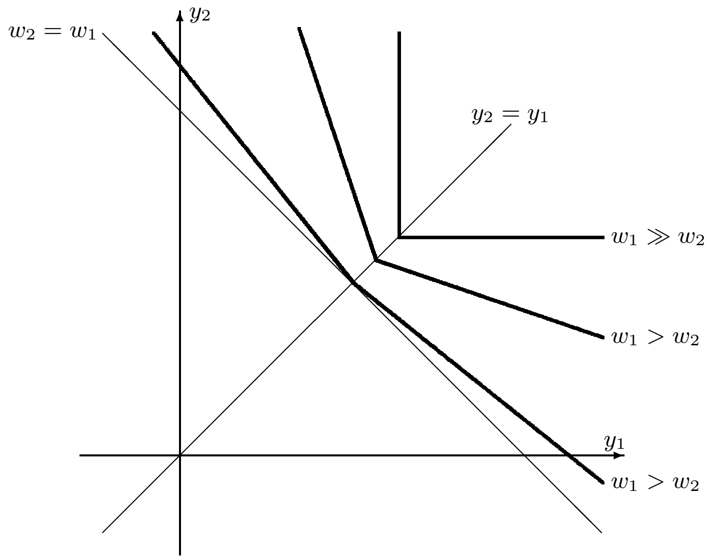

Figure 3: Isoline contours for various fair OWA aggregations.

means, as the limiting case of the OWA problem (40), we get the lexicographic problem:

$$\text{lexmax } \{(\theta_1(\mathbf{f}(\mathbf{x})), \theta_2(\mathbf{f}(\mathbf{x})), \ldots, \theta_m(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\} \tag{42}$$

which represents the MMF approach to the original problem (1). Problem (42) is a regularization of the standard Max-Min optimization (7), but in the former, in addition to the worst outcome, we maximize also the second worst outcome (provided that the smallest one remains as large as possible), maximize the third worst (provided that the two smallest remain as large as possible), and so on. Due to (34), the MMF problem (42) is equivalent to the problem:

$$\text{lexmax } \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\}$$

which leads us to a standard lexicographic optimization with predefined linear criteria defined according to (37)

$$\begin{array}{l} \text{lexmax } (\eta_1, \eta_2, \ldots, \eta_m) \\ \text{subject to} \quad \mathbf{x} \in Q \\ \eta_k = kt_k - \sum_{i=1}^{m} d_{ik} \quad \text{for } k = 1, \ldots, m \\ t_k - d_{ik} \le f_i(\mathbf{x}), \quad d_{ik} \ge 0 \quad \text{for } i, k = 1, \ldots, m \end{array} \tag{43}$$

Moreover, in the case of LP models, every fair solution can be identified as an optimal solution to some OWA problem with appropriate monotonic weights (Kostreva and Ogryczak, 1999) but such a search process is usually difficult to control (Ogryczak, Śliwiński and Wierzbicki, 2003). Better controllability and the complete parameterization of nondominated solutions even for non-convex, discrete problems can be achieved with the direct use of the reference point methodology introduced by Wierzbicki (1982) and later extended leading to efficient implementations of the so-called aspiration/reservation based decision support (ARBDS) approach with many successful applications (Lewandowski and Wierzbicki, 1989). The ARBDS approach is an interactive technique allowing the DM to specify the requirements in terms of aspiration and reservation levels, i.e., by introducing acceptable and required values for several criteria. Depending on the specified aspiration and reservation levels, a special scalarizing achievement function is built which may be directly interpreted as expressing utility to be maximized. Maximization of the scalarizing achievement function generates an efficient solution to the multiple criteria problem. The solution is accepted by the DM or some modifications of the aspiration and reservation levels are introduced to continue the search for a better solution. The ARBDS approach provides a complete parameterization of the efficient set to multi-criteria optimization. Hence, when applying the ARBDS methodology to the ordered cumulated criteria in (35), one may generate all fairly efficient solutions of the original problem (1). Indeed, initial experiments with such an approach to the problem of network dimensioning with elastic traffic have confirmed the theoretical advantages of the method (Ogryczak and Wierzbicki, 2004).

Although defined with simple linear constraints the auxiliary conditions (37) introduces m^2 additional variables and inequalities into the original model. This may cause a serious computational burden for real-life problems containing numerous outcomes. In order to reduce the problem size one may attempt the restrict the number of criteria in the problem (35). Let us consider a sequence of indices $K = \{k_1, k_2, \ldots, k_q\}$, where $1 = k_1 < k_2 < \ldots < k_{q-1} < k_q = m$, and the corresponding restricted form of the multiple criteria model (35):

$$\max \{(\eta_{k_1}, \eta_{k_2}, \ldots, \eta_{k_q}) \ : \ \eta_k = \bar{\theta}_k(\mathbf{f}(\mathbf{x})) \quad \text{for } k \in K, \quad \mathbf{x} \in Q\} \tag{44}$$

with only $q < m$ criteria. Following Theorem 5, multiple criteria model (35) allows us to generate any fairly efficient solution of problem (1). Reducing the number of criteria we restrict these opportunities. Nevertheless, one may still generate reasonable compromise solutions. First of all the following assertion is valid.

Theorem 6 *If $\mathbf{x}^o$ is an efficient solution of the restricted problem (44), then it is an efficient (Pareto-optimal) solution of the multiple criteria problem (1) and it can be fairly dominated only by another efficient solution $\mathbf{x}'$ of (44) with exactly the same values of criteria: $\bar{\theta}_k(\mathbf{f}(\mathbf{x}')) = \bar{\theta}_k(\mathbf{f}(\mathbf{x}^o))$ for all $k \in K$.*

Proof. Suppose, there exists $\mathbf{x}' \in Q$ which dominates $\mathbf{x}^o$. This means, $y'_i = f_i(\mathbf{x}') \geq y^o_i = f_i(\mathbf{x}^0)$ for all $i \in I$ with at least one inequality strict. Hence, $\bar{\theta}_k(\mathbf{y}') \geq \bar{\theta}_k(\mathbf{y}^o)$ for all $k \in K$ and $\bar{\theta}_{k_q}(\mathbf{y}') > \bar{\theta}_{k_q}(\mathbf{y}^o)$ which contradicts efficiency of $\mathbf{x}^o$ within the restricted problem (44).

Suppose now that $\mathbf{x}' \in Q$ fairly dominates $\mathbf{x}^o$. Due to Theorem 5, this means that $\bar{\theta}_i(\mathbf{y}') \geq \bar{\theta}_i(\mathbf{y}^o)$ for all $i \in I$ with at least one inequality strict. Hence, $\bar{\theta}_k(\mathbf{y}') \geq \bar{\theta}_k(\mathbf{y}^o)$ for all $k \in K$ and any strict inequality would contradict efficiency of $\mathbf{y}^o$ within the restricted problem (44). Thus, $\bar{\theta}_k(\mathbf{y}') = \bar{\theta}_k(\mathbf{y}^o)$ for all $k \in K$ which completes the proof. □

It follows from Theorem 6 that while restricting the number of criteria in the multiple criteria model (35) we can essentially still expect reasonably fair efficient solution and only *unfairness* may be related to the distribution of outcomes within classes of skipped target values. In other words, we have guaranteed some rough fairness while it can be possibly improved by redistribution of outcomes within the intervals $(\theta_{k_j}(\mathbf{y}), \theta_{k_{j+1}}(\mathbf{y})]$ for $j = 1, 2, \ldots, q-1$.

5 Ordered targets

Vector $\bar{\Theta}(\mathbf{y})$ can be viewed graphically with the absolute Lorenz curve which can be mathematically formalized as follows. First, we introduce the right-continuous cumulative distribution function:

$$F_{\mathbf{y}}(\xi) = \sum_{i=1}^{m} \frac{1}{m}\delta_i(\xi) \quad \text{where} \quad \delta_i(\xi) = \begin{cases} 1 & \text{if } y_i \leq \xi \\ 0 & \text{otherwise} \end{cases}$$

which for any real value ξ provides the measure of outcomes smaller or equal to ξ. Next, we introduce the quantile function $F_{\mathbf{y}}^{(-1)}$ as the left-continuous inverse of the cumulative distribution function $F_{\mathbf{y}}$:

$$F_{\mathbf{y}}^{(-1)}(\nu) = \inf \{\xi : F_{\mathbf{y}}(\xi) \geq \nu\} \quad \text{for } 0 < \nu \leq 1.$$

Hence, $\theta_i(\mathbf{y}) = F_{\mathbf{y}}^{(-1)}(i/m)$. Further, by integrating $F_{\mathbf{y}}^{(-1)}$, one gets $F_{\mathbf{y}}^{(-2)}(0) = 0$ and

$$F_{\mathbf{y}}^{(-2)}(\nu) = \int_0^{\nu} F_{\mathbf{y}}^{(-1)}(\alpha)d\alpha \quad \text{for } 0 < \nu \leq 1$$

as the second order quantile function. Graphs of functions $F_{\mathbf{y}}^{(-2)}(\nu)$ (with respect to ν) take the form of convex curves (Fig. 4), the *absolute Lorenz curves (ALC)*. In our case of m outcomes, $F_{\mathbf{y}}^{(-2)}(i/m) = \frac{1}{m}\bar{\theta}_i(\mathbf{y})$ for $i = 1,\ldots,m$ and the absolute Lorenz curve is a piecewise linear curve connecting point (0,0) and points $(i/m, \bar{\theta}_i(\mathbf{y})/m)$ for $i = 1,\ldots,m$. Due to Theorem 5, a fairly dominated outcome vector is represented by the ALC below that for a dominating vector. Vector of equal outcomes is graphically represented as an ascent line and it obviously dominates nay unequal vector with the same mean. However, with the relation of fair dominance an outcome vector of large unequal outcomes may be preferred to an outcome vector with small equal outcomes. Fig. 4 presents the absolute Lorenz curves for outcome distributions of three vectors. One can easily see that vector of perfectly equal outcomes $\mathbf{y}'' = (3,3,3)$ dominates vector $\mathbf{y}' = (1,3,5)$, but it is further dominated by vector $\mathbf{y}''' = (5,4,9)$ of unequal larger outcomes.

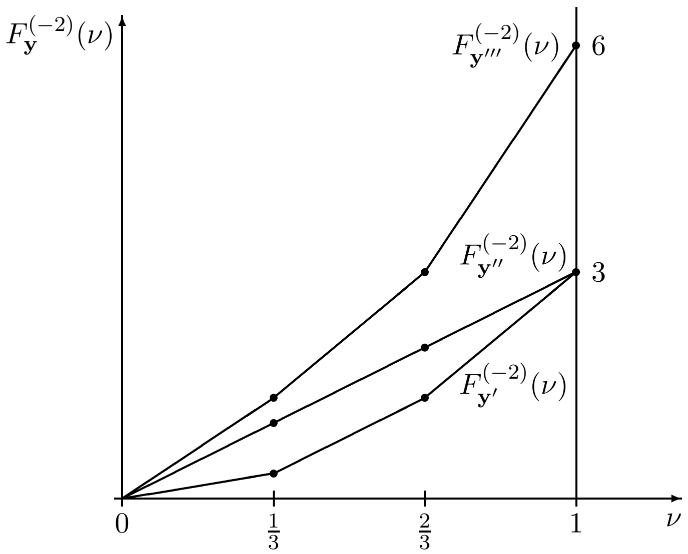

Figure 4: Fair dominance and the absolute Lorenz curves: $\mathbf{y}''' \succ_e \mathbf{y}'' \succ_e \mathbf{y}'$.

The fair dominance is equivalent to the pointwise inequalities of the entire absolute Lorenz curves

$$\mathbf{y}' \succeq_e \mathbf{y}'' \quad \Leftrightarrow \quad F_{\mathbf{y}'}^{(-2)}(\nu) \geq F_{\mathbf{y}''}^{(-2)}(\nu) \quad \text{for all } 0 \leq \nu \leq 1. \tag{45}$$

Nevertheless, the fair dominance is completely characterized by comparison of only m values $F_{\mathbf{y}}^{(-2)}(i/m)$ for $i = 1, \ldots, m$.

Recently, an intriguing duality relation between the second quantile function $F_{\mathbf{y}}^{(-2)}$ and the second cumulative distribution function $F_{\mathbf{y}}^{(2)}(\tau) = \int_{-\infty}^{\tau} F_{\mathbf{y}}(\xi) d\xi$ has been shown (Ogryczak and Ruszczyński, 2002). Namely, function $F_{\mathbf{y}}^{(-2)}$ is a conjugate (Rockafellar, 1970) of $F_{\mathbf{y}}^{(2)}$, i.e., for every $\nu \in [0, 1]$, one gets

$$F_{\mathbf{y}}^{(-2)}(\nu) = \sup_{\tau} \{\tau\nu - F_{\mathbf{y}}^{(2)}(\tau)\}. \tag{46}$$

It follows from the duality theory that one may completely characterize the fair dominance by the pointwise comparison the conjugate function $F^{(2)}$:

$$\mathbf{y}' \succeq_e \mathbf{y}'' \quad \Leftrightarrow \quad F_{\mathbf{y}'}^{(2)}(\tau) \le F_{\mathbf{y}''}^{(2)}(\tau) \quad \text{for all } \tau \in R. \tag{47}$$

In other words, the absolute Lorenz order is equivalent to the increasing concave order more commonly known as the Second Stochastic Dominance (SSD) relation (Müller and Stoyan, 2002).

The SSD relation was widely studied while comparing general distributions of uncertain outcome in the area of decision making under risk. Function $F_{\mathbf{y}}^{(2)}$, used to define the SSD relation, can also be presented as follows (Ogryczak and Ruszczyński, 1999):

$$F_{\mathbf{y}}^{(2)}(\tau) = \frac{1}{m} \sum_{i=1}^{m} (\tau - y_i)_+ \tag{48}$$

Hence, the SSD relation can be seen as a dominance for mean below-target deviations from all possible targets. Similar to absolute Lorenz curves, graphs of functions $F_{\mathbf{y}}^{(2)}$ define O-R curves (Ogryczak and Ruszczyński, 1999) $(\tau, F_{\mathbf{y}}^{(2)}(\tau))$ allowing to depict the SSD relation, and thereby the fair dominance of several outcome vectors $\mathbf{y}$. For distributions of m outcomes, we consider, the O-R curve is a piecewise linear convex curve with breakpoints at targets τ representing some coordinates of outcome vector $\mathbf{y}$. In the case of a perfectly outcome vector ($y_i = \mu(\mathbf{y}) \; \forall \, i$), the graph of $F_{\mathbf{y}}^{(2)}$ consists of two line segments: the axis τ for $\tau \le \mu(\mathbf{y})$ and the ascent line $\tau - \mu(\mathbf{y})$ for $\tau \ge \mu(\mathbf{y})$. Any unequal outcome vector with the same mean value $\mu(\mathbf{y})$ yields an O-R curve above (precisely, not below) these two line segments, including them for $\tau \le \min_i y_i$ and $\tau \ge \max_i y_i$, respectively. Fig. 5 presents the O-R curves for outcome distributions of three vectors compared with absolute Lorenz curves in Fig 4. Recall that vector of perfectly equal outcomes $\mathbf{y}'' = (3, 3, 3)$ dominates vector $\mathbf{y}' = (1, 3, 5)$, but it is further dominated by vector $\mathbf{y}''' = (5, 4, 9)$ of unequal larger outcomes. Note that according to (47), a fairly dominated outcome vector is represented by the O-R curve above that for a dominating vector.

For m-dimensional outcome vectors we consider, the entire $F_{\mathbf{y}}^{(2)}$ is completely defined by its values for at most m different targets representing values of several outcomes y_i while the remaining values follows from the linear interpolation. Nevertheless, these target values are dependent on specific outcome vectors and one cannot define any universal grid of targets allowing to define O-R curves for all possible outcome vectors. Therefore, in order to get a computational procedure one need either to aggregate mean shortages for infinite number of targets or to focus analysis on arbitrarily preselected finite grid of targets. The former turns out to lead us to the mean utility optimization models (15). Indeed, classical results of majorization theory (c.f., Marshall and Olkin, 1979) relate the mean utility comparison to the comparison of the weighted

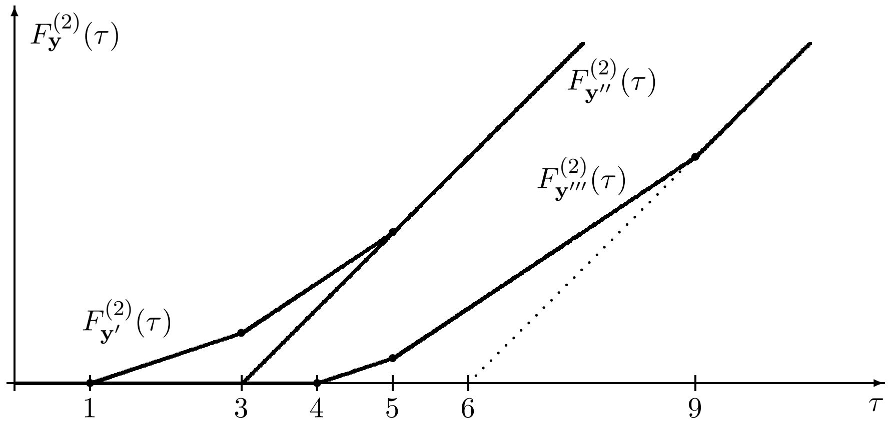

Figure 5: Fair dominance and the O-R curves: $\mathbf{y}''' \succ_e \mathbf{y}'' \succ_e \mathbf{y}'$.

mean shortages

$$\sum_{i=1}^{m} u(y'_i) \geq \sum_{i=1}^{m} u(y''_i) \quad \Leftrightarrow \quad \int_{-\infty}^{\infty} w(\xi) F^2_{\mathbf{y}'}(\xi) d\xi \leq \int_{-\infty}^{\infty} w(\xi) F^2_{\mathbf{y}''}(\xi) d\xi.$$

Actually, for bounded outcomes $a \leq y_i \leq b$ on has $\sum_{i=1}^{m} u(y_i) = \int_a^b u(\xi) dF_{\mathbf{y}}(\xi)$ and therefore, integrating twice by parts, one may easily verify that

$$\sum_{i=1}^{m} u(y'_i) - \sum_{i=1}^{m} u(y''_i) = \int_a^b u''(\xi) F^2_{\mathbf{y}'}(\xi) d\xi - \int_a^b u''(\xi) F^2_{\mathbf{y}''}(\xi) d\xi.$$

Hence, the maximization of a concave and increasing utility function u is equivalent to minimization of the weighted aggregation $\int_a^b w(\xi) F^2_{\mathbf{y}}(\xi) d\xi$ with positive weights $w(\xi) = -u''(\xi)$ (due to concavity of u).

In order to take advantages of the multiple criteria methodology one needs to focus on a finite set of target values. Let $V = \{v_1, v_2, \ldots, v_r\}$ (where $v_1 < v_2 < \cdots < v_r$) denote the set of target values for individual outcomes. For each target value we introduce function $h_k(\mathbf{y})$ expressing the partial (below target) cumulated outcomes

$$h_k(\mathbf{y}) = \sum_{i=1}^{m} \min\{y_i, v_k\} = m v_k - \sum_{i=1}^{m} (v_k - y_i)_+ = m(v_k - F^2_{\mathbf{y}}(v_k)) \tag{49}$$

Since the quantities $h_k(\mathbf{y})$ are complementary to $F^2_{\mathbf{y}}(v_k)$, following (47), we can seek fairly efficient allocation patterns using the standard multiple criteria optimization problem with r criteria:

$$\max \{(h_1(\mathbf{f}(\mathbf{x})), \ldots, h_r(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\}. \tag{50}$$

Theorem 7 *In the case of finite outcome set $\mathbf{f}(Q) \subset V^m$, a feasible allocation pattern $\mathbf{x} \in Q$ is a fair solution of problem (1), iff it is a Pareto-optimal solution of the multiple criteria problem (50).*

Proof. Let $\mathbf{f}(\mathbf{x}^o) \in Q$ be a Pareto-optimal solution of problem (50). Suppose, there exists $\mathbf{x}' \in Q$ such that $\mathbf{y}' = \mathbf{f}(\mathbf{x}')$ fairly dominates $\mathbf{y}^o = \mathbf{f}(\mathbf{x}^o)$. Due to (47), this means that

$m(\tau - F^2_{\mathbf{y}'}(\tau)) \geq m(\tau - F^2_{\mathbf{y}^o}(\tau))$ for all $\tau \in R$ with at least one inequality strict. Hence, $\hat{h}_k(\mathbf{y}') \geq \hat{h}_k(\mathbf{y}^o)$ for all $k = 1, \ldots, r$ and any strict inequality would contradict efficiency of $\mathbf{x}^o$ within the problem (50). Thus, $\hat{h}_k(\mathbf{x}') = \hat{h}_k(\mathbf{x}^o)$ and $F^2_{\mathbf{y}'}(v_k) = F^2_{\mathbf{y}^o}(v_k)$ for all $k = 1, \ldots, r$. Since $\mathbf{f}(Q) \subset V^m$, the piecewise linear functions $F^2_{\mathbf{y}'}(\tau)$ and $F^2_{\mathbf{y}''}(\tau)$ are completely defined by their values at targets v_k and thereby $F^2_{\mathbf{y}'}(\tau) = F^2_{\mathbf{y}^o}(\tau)$ for all $\tau \in R$.

Let $\mathbf{f}(\mathbf{x}^o) \in Q$ be a fairly efficient solution of problem (1). Suppose, $\mathbf{f}(\mathbf{x}^o)$ is not a Pareto-optimal solution of problem (50). This means, there exists $\mathbf{x}' \in Q$ such that $\mathbf{y}' = \mathbf{f}(\mathbf{x}')$ dominates $\mathbf{y}^o = \mathbf{f}(\mathbf{x}^o)$ in terms of criteria $h_k(\mathbf{y})$. Hence, $\hat{h}_k(\mathbf{y}') \geq \hat{h}_k(\mathbf{y}^o)$ for all $k = 1, \ldots, r$ with at least one inequality strict. Due to (49), this means that $F^2_{\mathbf{y}'}(v_k) \leq F^2_{\mathbf{y}^o}(v_k)$ for all $k = 1, \ldots, r$ with at least one strict inequality. Since $\mathbf{f}(Q) \subset V^m$, the piecewise linear functions $F^2_{\mathbf{y}'}(\tau)$ and $F^2_{\mathbf{y}''}(\tau)$ are completely defined by their values at targets v_k and thereby $F^2_{\mathbf{y}'}(\tau) \leq F^2_{\mathbf{y}^o}(\tau)$ for all $\tau \in R$ with at least one inequality strict. The latter contradicts fair efficiency of $\mathbf{x}^o$, which completes the proof. ☐

Corollary 3 *In the case of finite outcome set $\mathbf{f}(Q) \subset V^m$, the MMF solution (42) can be found by the standard lexicographic optimization with r criteria:*

$$\operatorname{lex\,max} \{(h_1(\mathbf{f}(\mathbf{x})), \ldots, h_r(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\}. \tag{51}$$

Note that multicriteria model (50) may be formulated as, similar to (38), LP expansion to the original resource allocation problem

$$\begin{aligned} &\max\ (h_1, h_2, \ldots, h_m) \\ &\text{subject to} \quad \mathbf{x} \in Q \\ &h_k = \sum_{i=1}^{m} t_{ik} \quad \text{for } k = 1, \ldots, r \\ &t_{ik} \leq f_i(\mathbf{x}), \quad t_{ik} \leq v_k \quad \text{for } i = 1, \ldots, m,\ k = 1, \ldots, r \end{aligned} \tag{52}$$

Although while the ordered outcomes model (38) has always m criteria defined by the number of outcomes, the latter model has r criteria according to the number of target values. Following Theorem 7, multiple criteria model (50) allows us to generate any fairly efficient allocation pattern, provided that the set of target values covers all possible outcome values. This requirement may be lead, in general, to very numerous set of criteria or even may make impossible to take full advantages of model (50). Nevertheless, there are many discrete resource allocation problems where the outcomes take values from a quite small grid of values. Actually, initial experiments with the MMF solution to discrete location problems (Ogryczak and Śliwiński, 2006) have shown much higher efficiency of the lexicographic optimization according to model (50) than the lexicographic optimization (43).

Certainly in many practical resource allocation problems one cannot consider target values covering all attainable outcomes. Reducing the number of criteria we restrict opportunities to generate all possible fair allocations. Nevertheless, one may still generate reasonable compromise solutions as the following assertion is valid.

Theorem 8 *If all attainable outcomes are upper bounded by the largest target v_r, then any $\mathbf{x}^o$ Pareto-optimal solution of problem (50) is also an efficient solution of the multiple criteria problem (1) and it can be fairly dominated only by another efficient solution $\mathbf{x}'$ of (50) with exactly the same values of criteria: $\hat{h}_k(\mathbf{f}(\mathbf{x}')) = \hat{h}_k(\mathbf{f}(\mathbf{x}^o))$ for all $k = 1, \ldots, r$.*

Proof. Suppose, there exists $\mathbf{x}' \in Q$ such that $\mathbf{y}' = \mathbf{f}(\mathbf{x}')$ dominates $\mathbf{y}^o = \mathbf{f}(\mathbf{x}^o)$. This means, $y'_i \geq y^o_i$ for all $i \in I$ with at least one inequality strict. Hence, $\hat{h}_k(\mathbf{y}') \geq \hat{h}_k(\mathbf{y}^o)$ for all $k = 1, \ldots, r$ and $\hat{h}_r(\mathbf{y}') > \hat{h}_r(\mathbf{y}^o)$ which contradicts efficiency of $\mathbf{x}^o$ within the restricted problem (50).

Suppose now that $\mathbf{y}' \in Q$ fairly dominates $\mathbf{y}^o$. Due to (47), this means that $m(\tau - F^2_{\mathbf{y}'}(\tau)) \geq m(\tau - F^2_{\mathbf{y}^o}(\tau))$ for all $\tau \in R$ with at least one inequality strict. Hence $\hat{h}_k(\mathbf{y}') \geq \hat{h}_k(\mathbf{y}^o)$ for all $k = 1, \ldots, r$ and any strict inequality would contradict efficiency of $\mathbf{x}^o$ within the problem (50). Thus, $\hat{h}_k(\mathbf{x}') = \hat{h}_k(\mathbf{x}^o)$ for all $k = 1, \ldots, r$. ◻

It follows from Theorem 8 that even the set of targets in the multiple criteria model (50) does not cover all possible outcome values, we can essentially still expect reasonably fair efficient solution and only *unfairness* may be related to the distribution of outcomes within classes of skipped targets. In other words, we have guaranteed some rough fairness while it can be possibly improved by redistribution of outcomes within the intervals $(v_k, v_{k+1}]$ for $k = 1, 2, \ldots, r-1$. Thus, we may generate various fairly efficient allocation patterns as Pareto-optimal solutions to the multiple criteria problem (50) with a reasonably small grid of target values. Various interactive multiple criteria techniques can be used for such an analysis including the reference point methods.

6 Conclusions

The problems of efficient and fair resource allocation arise in various systems which serve many users. Fairness is, essentially, an abstract socio-political concept that implies impartiality, justice and equity. Nevertheless, in operations research it was quantified with various solution concepts. In this paper we have demonstrated that these solution concept may be viewed as some specific approaches to multicriteria models while direct consideration of the multicriteria models themselves may allow for a better decision support methodology. The equitable optimization with the preference structure that complies with both the efficiency (Pareto-optimality) and with the Pigou-Dalton principle of transfers has been used to formalize the fair solution concepts. Bicriteria mean-equity models to search for a compromise fair and efficient allocations have been justified under some requirements and limits on the inequality measures to be used. Two alternative multiple criteria models equivalent to equitable optimization have been introduced thus allowing to generate a variety of fair and efficient resource allocation pattern by possible using of the reference point approaches.

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