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Enhanced Index Tracking with CVaR-Based Measures

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Abstract

The Enhanced Index Tracking Problem (EITP) calls for the determination of an optimal portfolio of assets with the bi-objective of maximizing the excess return of the portfolio above a benchmark and, simultaneously, minimizing the tracking error. The EITP is capturing a growing attention among academics, both for its practical relevance and for the scientific challenges that its study, as a multi-objective problem, poses. Several optimization models have been proposed in the literature, where the tracking error is measured in terms of standard deviation or in linear form using, for instance, the mean absolute deviation. More recently, reward-risk optimization measures, like the Omega ratio, have been adopted for the EITP. On the other side, shortfall or quantile risk measures have nowadays gained an established popularity in a variety of financial applications. In this paper, we propose a class of bi-criteria optimization models for the EITP, where risk is measured using the Weighted multiple Conditional Value-at-Risk (WCVaR). The WCVaR is defined as a weighted combination of multiple CVaR measures, and thus allows a more detailed risk aversion modeling compared to the use of a single CVaR measure. The application of the WCVaR to the EITP is analyzed, both theoretically and empirically. Through extensive computational experiments, the performance of the optimal portfolios selected by means of the proposed optimization models is compared, both in-sample and, more importantly, out-of-sample, to the one of the portfolios obtained using another recent optimization model taken from the literature.

Key words. Enhanced Index Tracking, Quantile Risk Measures, Conditional Value-at-Risk, Mean-Risk Models, Risk-Reward Ratios, Linear Programming.

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1 Introduction

In finance, the expression *index funds* identifies funds management strategies that have the objective of tracking the performance of a specific market index (the so-called *benchmark*), attempting to match, as much as possible, its returns. This investment strategy, usually called indexing or *index tracking*, is a passive form of fund management where the manager has a low degree of flexibility, and the fund is expected to reproduce the performance of the benchmark by properly choosing a representative selection of securities. The index tracking problem aims at minimizing a function, called the *tracking error*, which measures how closely the portfolio mimics the performance of the benchmark. Several authors studied the index tracking problem, proposing different optimization models, mainly based on different formulations of the tracking error, and solution methods. We refer to Sant'Anna *et al.* [33] for a recent overview of the relevant literature on the index tracking problem.

The term *enhanced index tracking* refers to an investment strategy that, while still attempting to track the market index, is designed to find a portfolio that slightly outperforms the benchmark. In other words, the manager of an enhanced index fund enjoys a little of leeway, trying to achieve a higher return than the benchmark but incurring into a minimal additional risk, as measured by the tracking error. The *Enhanced Index Tracking Problem (EITP)* aims at minimizing the tracking error, while simultaneously maximizing the excess return above the benchmark. A number of studies highlights that the amount invested in enhanced index funds steadily increased in the last three decades. The figures reported in Ahmed and Nanda [1] indicate that a sharp increase occurred in the middle of the '90s in both the number of enhanced index funds available and the total net assets under enhanced fund management. The same trend is pointed out by Jorion [12] who reports the outcomes of a survey conducted among fund managers of US institutional tax-exempt assets which indicate that, from 1994 to 2000, enhanced index funds have grown from USD 33 to USD 365 billions, which is a ten-fold factor! The same author also claims that, over the same period, passively managed funds have grown slower than enhanced index funds. Koshizuka *et al.* [14] mention that in the Tokyo stock exchange a significant amount of funds is managed by enhanced index tracking approaches. The growing popularity of the enhanced index funds is not only experienced in mature financial markets, but also in emerging markets. Weng and Wang [39] report a substantial increase in the importance of enhanced index funds in the Chinese market from 2008 to 2015. It is not surprising that, given this increasing spread of enhanced index funds, the topic is attracting a growing attention by the academic community, although the number of papers addressing the EITP, compared to the ones on the index tracking problem, is still limited and almost all the contributions appeared in the literature only in the last decade. Indeed, the first formalization of the EITP is due, to the best of our knowledge, to the paper by Beasley *et al.* [3], and most of the research proposing optimization models or solution methods for this problem dates from 2005. We refer to Canakgoz and Beasley [6] for an overview of the early literature on the EITP, and to Guastaroba *et al.* [10] for a review of the recent research. For this reason, we concentrate our literature review on the additional articles not included in the above references, and briefly mention here only some of the most relevant papers to the present research. In particular, based on approximated stochastic dominance conditions, Bruni *et al.* [4] formulate the EITP as a Linear Programming (LP) model with an exponential number of constraints. The formulation is solved using a separation procedure for the latter family of constraints. Along similar lines, Sharma *et al.* [34] propose an LP model for the EITP that aims at maximizing the mean portfolio return subject to constraints that limit the violation of the second order stochastic dominance criterion. Kwon and Wu [15] propose a mixed-integer second order cone programming formulation for the EITP, that maximizes the expected portfolio return subject to a limit on the portfolio risk and a bound on the tracking error. The portfolio risk is measured using the standard deviation of the portfolio returns, whereas the tracking error is defined as the standard deviation of the excess return of the portfolio from the benchmark. Their model includes also a cardinality constraint and buy-in thresholds, i.e., lower and upper bounds on the portfolio weights. Finally, the authors devise a robust counterpart of the above model. Computational results are given by solving both models with Gurobi.

We mentioned above that, at its core, the EITP has a bi-objective nature, like any other mean-risk portfolio optimization model. Despite this observation, very few authors address explicitly the EITP as a bi-objective optimization problem. Among them, it is worth citing the paper by Li *et al.* [16] where the EITP is formulated as a bi-objective mixed-integer non-linear optimization model that minimizes the tracking error, given by the downside standard deviation of the portfolio return from the benchmark, and maximizes the portfolio excess return. Their model includes, among other features, a cardinality constraint and buy-in thresholds, and is solved by means of an immunity-based multi-objective algorithm. Filippi *et al.* [9] cast the EITP as a bi-objective mixed-integer LP model which maximizes the excess return of the portfolio over the benchmark, and minimizes the tracking error, here defined as the absolute deviation between the portfolio and benchmark values. The authors devise a bi-objective heuristic framework for its solution. Bruni *et al.* [5] model the EITP as a bi-objective linear program that maximizes the average excess return of the portfolio over the benchmark, and minimizes the maximum downside deviation of the portfolio return from the market index. Like any other multi-objective approach, the methods devised in the former papers do not provide a single optimal solution, but rather a set of (Pareto) optimal solutions, or a set of near-optimal solutions if the solution method used is a heuristic. As a consequence, these approaches provide the decision maker with a, possibly wide, range of alternative solutions. However, this could be seen as a drawback instead of a point of strength, since they leave the choice of the specific solution to implement to the subjectivity of the decision maker. To overcome the above limit, some authors propose to cast the two objective functions of the EITP as a single objective expressed as a reward-risk ratio. In general terms, these ratios are performance measures that compare the expected returns of an investment (i.e., the reward) to the amount of risk undertaken to achieve these returns, and stem from the observation that there exists an inherent trade-off between the risk and the return of an investment. Nowadays, reward-risk ratios like the Sharpe ratio (see Sharpe [36]) and the Sortino ratio (see Sortino and Price [37]) are widely used to evaluate, compare and rank different investment strategies. To the best of our knowledge, Meade and Beasley [22] are the first ones attempting to use a reward-risk ratio in the context of enhanced indexation. The authors introduce a non-linear optimization model, based on the maximization of a modified Sortino ratio, and solve it by means of a genetic algorithm. However, the non-linearity of this model may represent an undesirable limitation to its use in financial practice, especially when portfolios have to meet several side constraints (such as cardinality constraints or buy-in thresholds) or when large-scale instances have to be solved since, in most cases, the inclusion of these features requires the introduction of binary and integer variables (see the survey by Mansini *et al.* [20]). Based on this observation, Guastaroba *et al.* [10] introduce two mathematical formulations for the EITP based on the Omega ratio. The Omega ratio is a performance measure introduced by Keating and Shadwick [13] which, broadly speaking, can be defined as the ratio between the expected value of the profits, defined as the portfolio returns over a predetermined target τ, and the expected value of the losses, that are the portfolio returns below τ. The first formulation introduced in Guastaroba *et al.* [10] applies a standard definition of the Omega ratio, computing the ratio with respect to a given target, whereas the second model, called the Extended Omega Ratio model, formulates the Omega ratio with respect to a random target. The authors show that both formulations, despite being non-linear in nature, can be transformed into LP models.

The computational results point out that the portfolios selected by the Extended Omega Ratio model consistently outperform, in term of out-of-sample performance, those optimized with the former model.

Since their introduction, quantile risk measures have had a crucial impact on the developments of new risk measures in finance. *Conditional Value-at-Risk* (*CVaR*), which is known also as Mean Excess Loss, Expected Shortfall, Worst Conditional Expectation, or Tail VaR, is one of such measures. The name CVaR was introduced in Rockafellar and Uryasev [30] where the risk measure is developed for continuous distributions, and later extended to general distributions (i.e., with a possibly discontinuous distribution function) in Rockafellar and Uryasev [31]. A relevant advantage of the CVaR is that for discrete random variables, i.e., when probabilities can be represented by using scenarios rather than densities, it can be optimized by means of LP methods. The success of the CVaR as a measure of risk is related to the theoretical properties it satisfies and to some practical considerations that make it attractive also among practitioners. From a theoretical point of view, the CVaR is a coherent risk measure as shown in Pflug [28] (see Artzner *et al.* [2] for the definition of coherent risk measures) and is consistent with the second-degree stochastic dominance as detailed in Ogryczak and Ruszczyński [27]. From a practical viewpoint, it is a *downside* risk measure in the sense that it does not penalize upside deviations, which are deviations of the portfolio returns above a given target and that any rational investor perceives as profits. Mansini *et al.* [19] suggest that the concept of CVaR can be extended to improve the risk averse modeling capabilities of the measure. Indeed, the authors show that a more detailed risk aversion modeling can be achieved by considering simultaneously multiple CVaR measures, each one specified by a given tolerance level, and then combining them together, as a weighted sum, into a single risk measure. The resulting measure is called the *Weighted multiple CVaR* (*WCVAR*) and is, obviously, LP computable.

Finally, it is worth mentioning the recent paper by Sharma *et al.* [35] where the concepts of Omega ratio optimization and CVaR are combined together in the general context of portfolio optimization, and, hence, not directly related to the EITP addressed in the current paper. Particularly, Sharma *et al.* [35] reformulate the original Omega ratio by computing the target τ (which is predetermined in its classical form) as the CVaR of a benchmark market portfolio.

Contributions. The paper provides several contributions. We introduce a theoretical framework for risk-reward ratio models, and employ it in the context of the EITP. We propose a novel class of bi-criteria optimization models expressed in terms of risk-reward ratios, where the risk measure is based on the WCVaR. More specifically, the WCVaR is a safety measure and, hence, it has to be maximized. In the proposed optimization models, we consider its deviation risk counterpart, the so-called weighted conditional drawdown measure. Following the findings reported in Guastaroba *et al.* [10], the class of optimization models introduced here is also designed with respect to a random target. We show that the resulting formulation, non-linear in nature, can be reformulated as an LP model. To validate the performance of the optimal portfolios selected by the proposed formulation, we conducted extensive computational experiments on benchmark instances taken from the literature, and compare their out-of-sample behavior with that of the portfolios constructed solving a reformulation of the Extended Omega Ratio model introduced in Guastaroba *et al.* [10]. Indeed, in the current paper, we express the Extended Omega Ratio model in terms of a risk-reward minimization, rather than a reward-risk maximization as it was originally proposed in [10]. Since, at least theoretically, the risk measure at the denominator of the reward-risk ratio can take null value, Guastaroba *et al.* [10] introduced additional constraints to guarantee its positivity and keep the problem always solvable. Our reformulation avoids such modeling issues. Despite the extensive experiments carried out, the outcomes do not seem to clearly favor one model over the others. On the other side, the results indicate a quite satisfactory ex-post performance of the optimal portfolios: all the optimal portfolios track very closely the behavior of the benchmark over the out-of-sample period, often achieving better returns.

Structure of the paper. The remainder of the paper is organized as follows. In Section 2, we introduce the basic notation and some preliminary concepts that will be used throughout the rest of the paper. Section 3 is devoted to the introduction of the mathematical formulation for the EITP based on the WCVAR. Computational experiments are reported in Section 4, where an extensive evaluation and comparison of the out-of-sample performance of the optimal portfolios is provided. Finally, some concluding remarks are drawn in Section 5.

2 Basic notation and preliminary concepts

This section is devoted to the introduction of some basic concepts and notation required to introduce the optimization models presented in the following section.

2.1 Basic notation

We consider an investor whose aim is to optimally select a portfolio of securities and hold it until the end of a specific investment horizon, i.e., the investor follows a so-called *buy-and-hold* strategy. Let $J = \{1, 2, \dots, n\}$ be the set of securities available for the investment. For each security $j \in J$, its rate of return is represented by a random variable (r.v.) R_j with a given mean $\mu_j = \mathbb{E}\{R_j\}$. Let $\mathbf{x} = (x_j)_{j=1,\dots,n}$ be the vector of decision variables x_j representing the shares (weights) that define a portfolio of securities. In any feasible portfolio the weights must sum to one, i.e., $\sum_{j=1}^{n} x_j = 1$, and short sales are not allowed, i.e., $x_j \geq 0$ for $j = 1, \dots, n$. Such basic constraints form a feasible set $\mathcal{P}$. Each portfolio $\mathbf{x}$ defines a corresponding r.v. $R_{\mathbf{x}} = \sum_{j=1}^{n} R_j x_j$ that represents the portfolio rate of return. The mean rate of return for portfolio $\mathbf{x}$ is given as $\mu(R_{\mathbf{x}}) = \mathbb{E}\{R_{\mathbf{x}}\} = \sum_{j=1}^{n} \mu_j x_j$. We consider T scenarios, each one with probability p_t, where $t = 1, \dots, T$. We assume that for each r.v. R_j its realization r_{jt} under scenario t is known and that, for each security j, with $j \in J$, its mean rate of return is computed as $\mu_j = \sum_{t=1}^{T} r_{jt} p_t$. The realization of the portfolio rate of return $R_{\mathbf{x}}$ under scenario t is given by $y_t = \sum_{j=1}^{n} r_{jt} x_j$. Although the optimization models that we are going to describe remain valid for any arbitrary set of scenarios or probability distribution function, we assume that the T scenarios are treated as equally probable, i.e., we set $p_t = 1/T$ for $t = 1, \dots, T$, and that these scenarios are represented by historical data observed on a stock exchange market.

Regarding the benchmark, we denote the r.v. representing its rate of return as R^I, whereas its realization under scenario t is denoted as r_t^I, with $t = 1, \dots, T$, and its mean rate of return as $\mu^I = \sum_{t=1}^{T} r_t^I p_t$. In enhanced indexation, the investor is interested in determining an optimal portfolio that outperforms the rate of return of the benchmark. This situation can be modeled using as a target some reference r.v. $R^\alpha = R^I + \alpha$ rather than simply the benchmark rate of return R^I. In these terms, R^α represents the rate of return beating the benchmark by a given excess return equal to α. Its realization under scenario t is denoted as $r_t^\alpha = r_t^I + \alpha$, with $t = 1, \dots, T$, and mean rate of return $\mu^\alpha = \sum_{t=1}^{T} r_t^\alpha p_t$. Finally, in the following the notation $(.)_+$ will denote the non-negative part of a quantity, that is, $(Q)_+ = \max\{Q, 0\}$.

2.2 Risk, safety and ratio measures

In his cornerstone research, Markowitz [21] suggests to model portfolio optimization problems as mean-risk bi-criteria problems, where the *mean portfolio return* $\mu(R_{\mathbf{x}})$ is maximized and, simultaneously, a *risk measure* $\varrho(R_{\mathbf{x}})$ is minimized. In the original Markowitz model, the standard deviation was used as the risk measure. Since then a number of other deviation risk measures have been considered that, akin to the standard deviation, are shift-invariant (i.e., not affected by any shift of the outcome scale) and are equal to 0 if applied to a risk-free portfolio, while take positive values for any risky portfolio (see Mansini *et al.* [17] for further details). A relevant drawback of such risk measures is that they are not consistent with the stochastic dominance order paradigms (e.g., see Whitmore and Findlay [40]) or other axiomatic models of risk-averse preferences (e.g., see Rothschild and Stiglitz [32]) and risk measurement (e.g., see Artzner *et al.* [2]).

In stochastic dominance, uncertain returns (modeled as random variables) are confronted by pointwise comparison of some performance functions constructed from their distribution functions. The first performance function is defined as the right-continuous cumulative distribution function: $F^{(1)}_{R_{\mathbf{x}}}(\eta) = F_{R_{\mathbf{x}}}(\eta) = \mathbb{P}\{R_{\mathbf{x}} \leq \eta\}$ and defines the first-degree stochastic dominance. The second function is derived from the first as $F^{(2)}_{R_{\mathbf{x}}}(\eta) = \int_{-\infty}^{\eta} F_{R_{\mathbf{x}}}(\xi)\ d\xi$, and defines the Second-degree Stochastic Dominance (SSD). We say that portfolio $\mathbf{x}'$ dominates $\mathbf{x}''$ under the SSD criterion (denoted as $\mathbf{x}' \succ_{SSD} \mathbf{x}''$), if $F^{(2)}_{R_{\mathbf{x}'}}(\eta) \leq F^{(2)}_{R_{\mathbf{x}''}}(\eta)$ for all η, with at least one strict inequality. The latter relation can be expressed in a weaker form, which claims that portfolio $\mathbf{x}'$ dominates $\mathbf{x}''$ under the weak SSD criterion ($\mathbf{x}' \succeq_{SSD} \mathbf{x}''$), if $F^{(2)}_{R_{\mathbf{x}'}}(\eta) \leq F^{(2)}_{R_{\mathbf{x}''}}(\eta)$ for all η. Furthermore, a feasible portfolio $\mathbf{x}' \in \mathcal{P}$ is said to be *SSD efficient* if there is no other feasible portfolio $\mathbf{x} \in \mathcal{P}$ such that $\mathbf{x} \succ_{SSD} \mathbf{x}'$. The concept of stochastic dominance relates the notion of risk to a possible failure of achieving some targets. As shown by Ogryczak and Ruszczyński [25], values of function $F^{(2)}_{R_{\mathbf{x}}}$, used to define the SSD relation can also be presented as the mean below-target deviations (also called the first-order lower partial moment), that is, we can write $F^{(2)}_{R_{\mathbf{x}}}(\eta) = \mathbb{E}\{(\eta - R_{\mathbf{x}})_+\}$. The latter is the simplest downside risk criterion that, when computed for a specific value τ, can be expressed as the mean below-target deviation τ:

$$\delta_\tau(R_{\mathbf{x}}) = \mathbb{E}\{(\tau - R_{\mathbf{x}})_+\} = F^{(2)}_{R_{\mathbf{x}}}(\tau). \tag{1}$$

For discrete rates of return represented by their realizations, function $\delta_\tau(R_{\mathbf{x}})$ is, when minimized, LP computable. Other portfolio performance measures have been introduced as safety measures to be, on the contrary, maximized. As a prominent example, we recall the worst realization studied by Young [41]. Mansini *et al.* [17] showed that for any risk measure $\varrho(R_{\mathbf{x}})$ a corresponding *safety measure* $\mu_\varrho(R_{\mathbf{x}}) = \mu(R_{\mathbf{x}}) - \varrho(R_{\mathbf{x}})$ can be defined and vice-versa. Compared to risk measures, safety measures may be consistent with the above-mentioned formal models of risk-averse preferences and risk measurement. Furthermore, a safety measure is said to be *risk relevant* if for any risky portfolio its value is smaller than the value it takes for a risk-free portfolio having the same mean rate of return.

We say that a safety measure $\mu_\varrho(R_{\mathbf{x}})$ is *SSD consistent* (or that the risk measure $\varrho(R_{\mathbf{x}})$ is SSD safety consistent) if $\mathbf{x}' \succeq_{SSD} \mathbf{x}''$ implies that $\mu_\varrho(R_{\mathbf{x}'}) \geq \mu_\varrho(R_{\mathbf{x}''})$. If a safety measure is SSD consistent then, except for portfolios with identical values of $\mu(R_{\mathbf{x}})$ and $\mu_\varrho(R_{\mathbf{x}})$ (and thereby of $\varrho(R_{\mathbf{x}})$), every efficient solution of the following bi-criteria problem:

$$\max\{(\mu(R_{\mathbf{x}}), \mu_\varrho(R_{\mathbf{x}}))\ :\ \mathbf{x} \in \mathcal{P}\}$$

is an SSD efficient portfolio (see Ogryczak and Ruszczyński [25] for further details).

The common approach used to tackle a Markowitz-type mean-risk model is to transform the objective of maximizing the mean portfolio return into constraint by imposing a minimum acceptable mean return μ_0, while minimizing the risk criterion. An alternative approach is to seek for a risky portfolio that offers the maximum increase of the portfolio mean return compared to a given target τ, per unit of risk incurred. Target τ is often represented by the mean return

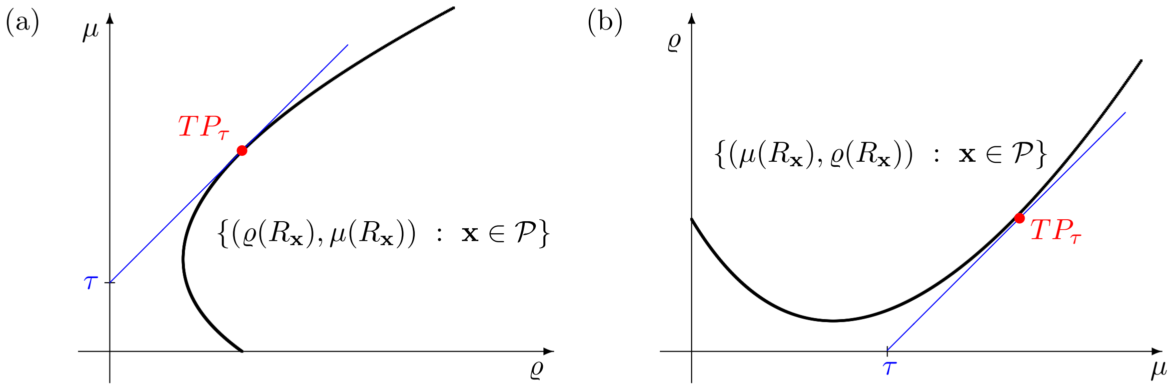


Figure 1: Tangency portfolio TP_τ by reward-risk ratio maximization (a) and by risk-reward ratio minimization (b).

of a risk-free asset. The latter approach leads to the following optimization problem expressed as a ratio:

$$\max \left\{ \frac{\mu(R_{\mathbf{x}}) - \tau}{\varrho(R_{\mathbf{x}})} \ : \ \mathbf{x} \in \mathcal{P} \right\}. \tag{2}$$

The optimal solution of problem (2) is usually called the tangency portfolio or the *market portfolio*. An intuitive explanation of the ratio optimization problem (2) is illustrated in Fig. 1(a). Mansini *et al.* [18] show that for LP computable risk measures, the reward-risk ratio optimization problem (2) can be converted into an LP form. When the risk-free rate of return r_0 is used instead of the target τ, ratio optimization problem (2) corresponds to the classical Tobin's model (cfr. [38]) of the modern portfolio theory, where the capital market line is the line drawn from the intercept corresponding to r_0 and that passes tangent to the mean-risk efficient frontier. Any point on this line provides the maximum return for each level of risk. The tangency portfolio TP_{r_0} is the portfolio of risky assets corresponding to the point where the capital market line is tangent to the efficient frontier.

Instead of the reward-risk ratio maximization (2), one may formulate the same problem in terms of risk-reward ratio minimization (see Fig. 1(b)) as follows:

$$\min \left\{ \frac{\varrho(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}}) - \tau} \ : \ \mathbf{x} \in \mathcal{P} \right\}. \tag{3}$$

Even though both ratio optimization models (2) and (3) are theoretically equivalent, the risk-reward formulation (3) enables an easier control of the denominator positivity by simply introducing the additional inequality $\mu(R_{\mathbf{x}}) - \tau \geq \varepsilon$, with $\varepsilon > 0$.

Note that two feasible portfolios having zero risk are both optimal to the risk-reward ratio model (3), even if they are characterized by different mean returns. This shortcoming can be regularized leading to the following formulation:

$$\min \left\{ \frac{\varrho(R_{\mathbf{x}}) + \varepsilon}{\mu(R_{\mathbf{x}}) - \tau} \ : \ \mu(R_{\mathbf{x}}) - \tau \geq \varepsilon, \ \mathbf{x} \in \mathcal{P} \right\}. \tag{4}$$

This regularization of the numerator is useful when for multiple portfolios the risk measure $\varrho(R_{\mathbf{x}})$ takes value equal to zero. In these cases, an optimal solution to problem (4) is the portfolio with the largest mean return. Obviously, the ε introduced in the numerator is, in principle, different from the one included in the problem constraint. However, to the sake of a simple exposition, we decided to use the same notation for both. Furthermore, the following theorem is valid.

Theorem 1 *Let $\mathbf{x}^0$ be an optimal portfolio to the risk-reward ratio optimization problem (4) that satisfies condition $\mu(R_{\mathbf{x}^0}) - \varrho(R_{\mathbf{x}^0}) \leq \tau$. For any deviation risk measure $\varrho(R_\mathbf{x})$, portfolio $\mathbf{x}^0$ is nondominated in terms of the bi-criteria mean-safety maximization $\max\{\mu(R_\mathbf{x}), \mu(R_\mathbf{x}) - \varrho(R_\mathbf{x})\}$, as well as in terms of the bi-criteria mean-risk optimization $\max\{\mu(R_\mathbf{x}), -\varrho(R_\mathbf{x})\}$.*

Proof. Suppose that there exists a feasible portfolio $\mathbf{x}$, i.e., $\mathbf{x} \in \mathcal{P}$ and $\mu(R_\mathbf{x}) - \tau \geq \varepsilon$, such that $\mu(R_\mathbf{x}) - \varrho(R_\mathbf{x}) \geq \mu(R_{\mathbf{x}^0}) - \varrho(R_{\mathbf{x}^0})$ and $\mu(R_\mathbf{x}) \geq \mu(R_{\mathbf{x}^0})$. Note that the objective function in problem (4) can be written as:

$$\frac{\varrho(R_\mathbf{x}) + \varepsilon}{\mu(R_\mathbf{x}) - \tau} = \frac{\tau - (\mu(R_\mathbf{x}) - \varrho(R_\mathbf{x})) + \varepsilon}{\mu(R_\mathbf{x}) - \tau} + 1.$$

Note that due to the optimality of $\mathbf{x}^0$ and the additional condition $\mu(R_{\mathbf{x}^0}) - \varrho(R_{\mathbf{x}^0}) \leq \tau$, in the above ratio both numerator and denominator are positive for solution $\mathbf{x}^0$, whereas the denominator is positive for any feasible portfolio $\mathbf{x}$. Hence, whenever $\mu(R_\mathbf{x}) - \varrho(R_\mathbf{x}) > \mu(R_{\mathbf{x}^0}) - \varrho(R_{\mathbf{x}^0})$ or $\mu(R_\mathbf{x}) > \mu(R_{\mathbf{x}^0})$, the following inequality holds:

$$\frac{\varrho(R_\mathbf{x}) + \varepsilon}{\mu(R_\mathbf{x}) - \tau} = \frac{\tau - (\mu(R_\mathbf{x}) - \varrho(R_\mathbf{x})) + \varepsilon}{\mu(R_\mathbf{x}) - \tau} + 1 < \frac{\tau - (\mu(R_{\mathbf{x}^0}) - \varrho(R_{\mathbf{x}^0})) + \varepsilon}{\mu(R_{\mathbf{x}^0}) - \tau} + 1 = \frac{\varrho(R_{\mathbf{x}^0}) + \varepsilon}{\mu(R_{\mathbf{x}^0}) - \tau}$$

which contradicts the optimality of $\mathbf{x}^0$. Therefore, $\mu(R_\mathbf{x}) = \mu(R_{\mathbf{x}^0})$ and $\varrho(R_\mathbf{x}) = \varrho(R_{\mathbf{x}^0})$, $\mathbf{x}$ is an equivalent optimal solution to (4), and portfolio $\mathbf{x}^0$ is nondominated in terms of the bi-criteria mean-safety maximization $\max\{\mu(R_\mathbf{x}), \mu(R_\mathbf{x}) - \varrho(R_\mathbf{x})\}$.

Finally, suppose a feasible portfolio $\mathbf{x}$ dominates $\mathbf{x}^0$ in terms of bi-criteria mean-risk optimization $\max\{\mu(R_\mathbf{x}), -\varrho(R_\mathbf{x})\}$, i.e., $\varrho(R_\mathbf{x}) \leq \varrho(R_{\mathbf{x}^0})$ and $\mu(R_\mathbf{x}) \geq \mu(R_{\mathbf{x}^0})$ with at least one strict inequality. Then, the two conditions $\mu(R_\mathbf{x}) - \varrho(R_\mathbf{x}) \geq \mu(R_{\mathbf{x}^0}) - \varrho(R_{\mathbf{x}^0})$ and $\mu(R_\mathbf{x}) \geq \mu(R_{\mathbf{x}^0})$ hold, with at least one strict inequality. Hence, optimal portfolio $\mathbf{x}^0$ is also nondominated in terms of bi-criteria mean-risk optimization $\max\{\mu(R_\mathbf{x}), -\varrho(R_\mathbf{x})\}$. $\square$

Note that condition $\mu(R_{\mathbf{x}^0}) - \varrho(R_{\mathbf{x}^0}) \leq \tau$ guaranteeing the efficiency of the optimal solution to the risk-reward ratio optimization problem (4) is equivalent to imposing that the value of ratio $(\varrho(R_{\mathbf{x}^0}) + \varepsilon)/(\mu(R_{\mathbf{x}^0}) - \tau)$ is greater than 1. Consequently, any risk-reward ratio model (4) is well-defined only if this condition is not violated.

Corollary 1 *Let $\mathbf{x}^0$ be an optimal portfolio to the risk-reward ratio optimization problem (4) that satisfies condition $\mu(R_{\mathbf{x}^0}) - \varrho(R_{\mathbf{x}^0}) \leq \tau$. For any deviation risk measure $\varrho(R_\mathbf{x})$ which is SSD safety consistent, i.e., $\mathbf{x}' \succeq_{SSD} \mathbf{x}'' \quad \Rightarrow \quad \mu(R_{\mathbf{x}'}) - \varrho(R_{\mathbf{x}'}) \geq \mu(R_{\mathbf{x}''}) - \varrho(R_{\mathbf{x}''})$, portfolio $\mathbf{x}^0$ is SSD nondominated with the exception of alternative optimal portfolios having the same values of mean return $\mu(R_{\mathbf{x}^0})$ and risk measure $\varrho(R_{\mathbf{x}^0})$.*

Proof. Suppose that there exists a feasible portfolio $\mathbf{x}$, i.e., $\mathbf{x} \in \mathcal{P}$ and $\mu(R_\mathbf{x}) \geq \tau + \varepsilon$, such that $\mathbf{x} \succeq_{SSD} \mathbf{x}^0$. This SSD relation implies that $\mu(R_\mathbf{x}) - \varrho(R_\mathbf{x}) \geq \mu(R_{\mathbf{x}^0}) - \varrho(R_{\mathbf{x}^0})$ and $\mu(R_\mathbf{x}) \geq \mu(R_{\mathbf{x}^0})$. Therefore, following Theorem 1, $\mathbf{x} \succeq_{SSD} \mathbf{x}^0$ implies that $\mu(R_\mathbf{x}) = \mu(R_{\mathbf{x}^0})$ and $\varrho(R_\mathbf{x}) = \varrho(R_{\mathbf{x}^0})$ and $\mathbf{x}$ is an alternative optimal solution to (4). $\square$

To apply directly the risk-reward ratio model (4) in the domain of the enhanced indexation, one should replace the target value τ with the mean rate of return μ^α, the latter as defined above in Section 2.1. As already mentioned, Guastaroba *et al.* [10] have shown that the performance of the portfolios selected by a ratio optimization model (in their paper the Omega ratio is expressed in terms of a reward-risk ratio model) can be significantly improved if the models are modified in order to take into consideration if the portfolio tracks, falls below or beats the benchmark under multiple scenarios. To this aim, one should formulate the risk-reward ratio model for a random benchmark return R^α, rather than for the mean rate of return μ^α. In other words, the optimization model is applied to the distribution of the difference $(R_{\mathbf{x}} - R^{\alpha})$, thus taking the following form:

$$\min \left\{ \frac{\varrho(R_{\mathbf{x}} - R^{\alpha}) + \varepsilon}{\mu(R_{\mathbf{x}} - R^{\alpha})} \ : \ \mu(R_{\mathbf{x}} - R^{\alpha}) \geq \varepsilon, \ \mathbf{x} \in \mathcal{P} \right\}. \tag{5}$$

Note that applying model (5) to the deterministic target τ, i.e., replacing $R^{\alpha} = \tau$, one gets exactly the standard risk-reward ratio model (4), as $\mu(R_{\mathbf{x}} - \tau) = \mu(R_{\mathbf{x}}) - \tau$ and for the deviation risk measure $\varrho(R_{\mathbf{x}} - \tau) = \varrho(R_{\mathbf{x}})$.

It is worth highlighting that, in the literature, some authors proposed ratio performance measures based on using the CVaR. In Appendix A, we discuss some of these ratio measures, and point out their similarities with the ones considered in the present paper.

2.3 Weighted CVaR risk measures

We consider the CVaR defined directly on the distribution of returns $R_{\mathbf{x}}$. Hence, the CVaR can be expressed by the following formula (see Ogryczak and Ruszczyński [27]):

$$M_{\beta}(R_{\mathbf{x}}) = \frac{1}{\beta} \int_0^{\beta} F_{R_{\mathbf{x}}}^{(-1)}(\xi) d\xi,$$

where $F_{R_{\mathbf{x}}}^{(-1)}$ is the quantile function for the portfolio return $R_{\mathbf{x}}$. It is defined as $F_{R_{\mathbf{x}}}^{(-1)}(\xi) = \inf\{\eta : F_{R_{\mathbf{x}}}(\eta) \geq \xi\}$ for $0 < \xi \leq 1$, i.e., the left-continuous inverse of the right-continuous cumulative distribution function $F_{R_{\mathbf{x}}}(\eta) = \mathbb{P}\{R_{\mathbf{x}} \leq \eta\}$. According to Mansini *et al.* [18], the CVaR can be classified as a safety measure and the corresponding (deviation) risk measure $\Delta_{\beta}(R_{\mathbf{x}}) = \mu(R_{\mathbf{x}}) - M_{\beta}(R_{\mathbf{x}})$ is called *conditional drawdown* (cfr. Ogryczak and Ruszczyński [26] and Chekhlov *et al.* [8]). For a discrete random variable represented by its realizations y_t, with $t = 1, \ldots, T$, both the CVaR $M_{\beta}(R_{\mathbf{x}})$ and its corresponding risk measure $\Delta_{\beta}(R_{\mathbf{x}})$ are LP computable as follows:

$$M_{\beta}(R_{\mathbf{x}}) = \max \left\{ \eta - \frac{1}{\beta} \sum_{t=1}^{T} p_t d_t \ : \ d_t \geq \eta - y_t, \quad d_t \geq 0, \ t = 1, \ldots, T \right\}, \tag{6}$$

and, for the risk measure:

$$\Delta_{\beta}(R_{\mathbf{x}}) = \min \left\{ \sum_{t=1}^{T} p_t y_t - \eta + \frac{1}{\beta} \sum_{t=1}^{T} p_t d_t \ : \ d_t \geq \eta - y_t, \quad d_t \geq 0, \ t = 1, \ldots, T \right\},$$

where η is an unbounded variable taking, at the optimum, the value of the β-quantile.

Although the CVaR is risk relevant for $0 < \beta < 1$, it represents only the mean within a part (tail) of the distribution of returns. Therefore, such a single criterion might present some limits when it is important to model various risk aversion preferences treating differently events that are more or less extreme. Aiming at enhancing its modeling capabilities, Mansini *et al.* [19] proposed to consider several, say m, tolerance levels $0 < \beta_1 < \beta_2 < \ldots < \beta_m < 1$ and combine together the corresponding CVaR measures $M_{\beta_k}(R_{\mathbf{x}})$, $k = 1, ..., m$. The weighted sum is used in [19] to combine the multiple CVaR criteria, leading to a weighted CVaR objective. The latter objective has been first introduced by Ogryczak [23] (although without using the name CVaR that was introduced later by Rockafellar and Uryasev [30]), where a portfolio optimization model based on historical data, and its LP computability was proven. The concept has been later extended to general decisions under risk by Ogryczak [24].

The *Weighted multiple CVaR* (WCVaR), as a weighted sum of several CVaR criteria combined by using positive (and normalized) weights, can be expressed as:

$$M_{\mathbf{w}}^{(m)}(R_{\mathbf{x}}) = \sum_{k=1}^{m} w_k M_{\beta_k}(R_{\mathbf{x}}), \quad \sum_{k=1}^{m} w_k = 1, \quad w_k > 0, \quad k = 1, \dots, m. \tag{7}$$

The corresponding deviation risk measure is the weighted sum of the $\Delta_{\beta_k}(R_{\mathbf{x}})$ measures, thus leading to the following form of the weighted conditional drawdown:

$$\Delta_{\mathbf{w}}^{(m)}(R_{\mathbf{x}}) = \mu(R_{\mathbf{x}}) - M_{\mathbf{w}}^{(m)}(R_{\mathbf{x}}) = \sum_{k=1}^{m} w_k \Delta_{\beta_k}(R_{\mathbf{x}}), \quad \sum_{k=1}^{m} w_k = 1, \quad w_k > 0, \quad k = 1, \dots, m. \tag{8}$$

Since, as mentioned above, the CVaR is coherent and SSD consistent, the same applies to the WCVaR. In particular, $\mathbf{x}' \succeq_{SSD} \mathbf{x}''$ implies that $M_{\mathbf{w}}^{(m)}(R_{\mathbf{x}'}) \geq M_{\mathbf{w}}^{(m)}(R_{\mathbf{x}''})$ (see Ogryczak and Ruszczyński [26]).

For returns represented by their realizations, the WCVaR measures are LP computable and can be represented by the following LP problems:

$$\begin{aligned} M_{\mathbf{w}}^{(m)}(R_{\mathbf{x}}) = \quad & \max \left\{ \sum_{k=1}^{m} w_k \eta_k - \sum_{k=1}^{m} \frac{w_k}{\beta_k} \sum_{t=1}^{T} p_t d_{tk} \right\}, \\ \text{s.t.} \quad & d_{tk} \geq q_k - y_t, \quad d_{tk} \geq 0 \quad t = 1, \dots, T; \ k = 1, \dots, m, \end{aligned} \tag{9}$$

and, for the weighted conditional drawdown:

$$\begin{aligned} \Delta_{\mathbf{w}}^{(m)}(R_{\mathbf{x}}) = \quad & \min \left\{ \sum_{t=1}^{T} p_t y_t - \sum_{k=1}^{m} w_k \eta_k + \sum_{k=1}^{m} \frac{w_k}{\beta_k} \sum_{t=1}^{T} p_t d_{tk} \right\} \\ \text{s.t.} \quad & d_{tk} \geq \eta_k - y_t, \quad d_{tk} \geq 0 \quad t = 1, \dots, T; \ k = 1, \dots, m, \end{aligned}$$

where η_k, with $k = 1, \dots, m$, are unbounded variables taking, at the optimum, the values of the corresponding β_k-quantiles. Note that model (9) with $m = 1$ corresponds to the standard single-criterion CVaR model, whereas using $m > 1$ and various settings of positive weights w_k enables to model a wide variety of risk averse preferences. Mansini *et al.* [19] identified two main classes of WCVaR measures, that primarily differ for the set of weights w_k used. Based on the results reported in their paper, we will use the Tail WCVaR, which is built as an approximation to the tail Gini measure. In more details, given a grid of m tolerance levels $0 < \beta_1 < \dots < \beta_k < \dots < \beta_m = \beta$, in the Tail WCVaR one may define the weights according to the following formulas:

$$w_k = \tfrac{\beta_k(\beta_{k+1} - \beta_{k-1})}{\beta^2} \quad k = 1, \dots, m-1, \text{ and } w_m = \tfrac{\beta_m(\beta_m - \beta_{m-1})}{\beta^2}, \tag{10}$$

where $\beta_0 = 0$. To the sake of brevity, in the remainder of the paper we will refer to the Tail WCVaR measure with weights defined as in (10) simply as WCVaR.

3 Optimization models for the Enhanced Index Tracking Problem

The present section is devoted to the introduction of the optimization models tested in the computational experiments. The model described in Section 3.1 is the risk-reward version of the formulation devised in Guastaroba *et al.* [10] and based on the Omega ratio. To the sake of brevity, we simply derive the LP formulation of our model and highlight the differences compared to the model in [10]. In the following Section 3.2, we introduce in detail the new optimization model based on the WCVaR.

3.1 Extended Omega ratio model

In its standard form, the Omega ratio is defined as the ratio between the expected value of the profits and the expected value of the losses where, for a predetermined threshold τ, portfolio returns over the target τ are considered as profits, whereas returns below the threshold are considered as losses. Ogryczak and Ruszczyński [25] prove that for any target value τ the following chain of equalities holds:

$$\mathbb{E}\{(R_{\mathbf{x}} - \tau)_+\} = \mu(R_{\mathbf{x}}) - (\tau - \mathbb{E}\{(\tau - R_{\mathbf{x}})_+\}) = \mu(R_{\mathbf{x}}) - \tau + \delta_\tau(R_{\mathbf{x}}), \tag{11}$$

where the last equality is related to the definition of mean below-target deviation expressed in (1). Thus, we can formulate the (standard) Omega ratio as follows:

$$\Omega(\tau, R_{\mathbf{x}}) = \frac{\mathbb{E}\{(R_{\mathbf{x}} - \tau)_+\}}{\mathbb{E}\{(\tau - R_{\mathbf{x}})_+\}} = \frac{\mu(R_{\mathbf{x}}) - \tau + \delta_\tau(R_{\mathbf{x}})}{\delta_\tau(R_{\mathbf{x}})} = 1 + \frac{\mu(R_{\mathbf{x}}) - \tau}{\delta_\tau(R_{\mathbf{x}})}.$$

Hence, the maximization of the above Omega ratio, with the additional restriction requiring $\mu(R_{\mathbf{x}}) - \tau \geq \varepsilon$, is equivalent to the minimization of the mean below-target deviation ratio $\frac{\delta_\tau(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}})-\tau}$. Note that restriction $\mu(R_{\mathbf{x}}) - \tau \geq \varepsilon$ along with (11) imply that $\mathbb{E}\{(R_{\mathbf{x}} - \tau)_+\} > \mathbb{E}\{(\tau - R_{\mathbf{x}})_+\}$, thus limiting the Omega ratio to take only values greater than 1. Actually, the mean below-target deviation $\delta_\tau(R_{\mathbf{x}})$ is not shift-invariant and, thereby, it is not a deviation risk measure. Instead of $\delta_\tau(R_{\mathbf{x}})$, one can consider the safety measure $\mu_{\delta_\tau}(R_{\mathbf{x}}) = \mathbb{E}\{\min\{R_{\mathbf{x}}, \tau\}\} = \tau - \delta_\tau(R_{\mathbf{x}})$, and then use the corresponding deviation risk measure $\varrho_\tau(R_{\mathbf{x}}) = \mu(R_{\mathbf{x}}) - \mu_{\delta_\tau}(R_{\mathbf{x}}) = \mu(R_{\mathbf{x}}) - \tau + \delta_\tau(R_{\mathbf{x}})$. Note that for any τ, the SSD relation $\mathbf{x}' \succeq_{SSD} \mathbf{x}''$ implies that $\delta_\tau(R_{\mathbf{x}'}) \leq \delta_\tau(R_{\mathbf{x}''})$ and, consequently, $\tau - \delta_\tau(R_{\mathbf{x}'}) \geq \tau - \delta_\tau(R_{\mathbf{x}''})$. The latter inequality guarantees the SSD consistency of the safety measure $\mu_{\delta_\tau}(R_{\mathbf{x}})$.

By replacing $\delta_\tau(R_{\mathbf{x}})$ with $\varrho_\tau(R_{\mathbf{x}})$, one can re-formulate the risk-reward ratio and obtain the following result:

$$\frac{\varrho_\tau(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}}) - \tau} = \frac{\mu(R_{\mathbf{x}}) - \tau + \delta_\tau(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}}) - \tau} = 1 + \frac{\delta_\tau(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}}) - \tau}. \tag{12}$$

Hence, the minimization of the risk-reward ratio $\frac{\varrho_\tau(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}})-\tau}$, with the additional restriction requiring $\mu(R_{\mathbf{x}}) - \tau \geq \varepsilon$, is equivalent to the minimization of the mean below-target deviation ratio $\frac{\delta_\tau(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}})-\tau}$ and, in turn, to the maximization of the Omega ratio. Indeed:

$$\frac{\varrho_\tau(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}}) - \tau} - 1 = \frac{\delta_\tau(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}}) - \tau} = \frac{1}{\Omega(\tau, R_{\mathbf{x}}) - 1}.$$

One can apply directly the risk-reward ratio $\frac{\varrho_\tau(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}})-\tau}$ in the domain of enhanced indexation by simply replacing τ with μ_α. Nevertheless, the findings reported by Guastaroba *et al.* [10] point out that, as mentioned before, a better performance can be achieved formulating the ratio with respect to the random target R^α, instead of a deterministic value τ. Replacing in the numerator of model (5) $\varrho(R_{\mathbf{x}} - R^\alpha)$ with the deviation risk measure $\varrho_0(R_{\mathbf{x}} - R^\alpha)$, one obtains the following problem:

$$\min \left\{ \frac{\varrho_0(R_{\mathbf{x}} - R^\alpha) + \varepsilon}{\mu(R_{\mathbf{x}} - R^\alpha)} \ : \ \mu(R_{\mathbf{x}} - R^\alpha) \geq \varepsilon, \ \mathbf{x} \in \mathcal{P} \right\}. \tag{13}$$

Note that the following equalities hold: $\varrho_0(R_{\mathbf{x}}) = \mu(R_{\mathbf{x}}) + \delta_0(R_{\mathbf{x}}) = \mathbb{E}\{(R_{\mathbf{x}})_+\}$, where the first equality is related to the above definition of the deviation risk measure, whereas the second is due to (11). The deviation risk measure $\varrho_0(R_{\mathbf{x}})$ is SSD safety consistent.

Additionally, note that $\mu(R_{\mathbf{x}}) - \varrho_0(R_{\mathbf{x}}) = -\delta_0(R_{\mathbf{x}}) \leq 0$. Hence, applying Theorem 1 and Corollary 1 to the distribution of the difference $(R_{\mathbf{x}} - R^\alpha)$ with $\tau = 0$, one gets the following corollary.

Corollary 2 *Let* $\mathbf{x}^0$ *be an optimal solution to the risk-reward optimization problem (13). Then, portfolio* $\mathbf{x}^0$ *is nondominated in terms of bi-criteria mean-safety maximization* $\max\{\mu(R_{\mathbf{x}} - R^\alpha), -\delta_0(R_{\mathbf{x}} - R^\alpha)\}$*, and is SSD nondominated with the exception of alternative optimal portfolios having the same values of mean return* $\mu(R_{\mathbf{x}^0} - R^\alpha)$ *and risk measure* $\varrho_0(R_{\mathbf{x}} - R^\alpha)$.

Furthermore, given the equivalence described by the chain of equalities (12), we can express the minimization of the extended Omega ratio as follows:

$$\min\left\{\frac{\delta_0(R_{\mathbf{x}} - R^\alpha) + \varepsilon}{\mu(R_{\mathbf{x}} - R^\alpha)} \ : \ \mu(R_{\mathbf{x}} - R^\alpha) \geq \varepsilon, \ \mathbf{x} \in \mathcal{P}\right\}. \tag{14}$$

For security returns described by discrete random variables having, for each security j, with $j \in J$, realization r_{jt} under scenario t, with $t = 1, \ldots, T$, one obtains the following non-linear optimization model:

$$\begin{aligned}
\min \quad & \frac{z_1 + \varepsilon}{z - \mu^\alpha} && (15)\\
\text{s.t.} \quad & z - \mu^\alpha \geq \varepsilon && (16)\\
& \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \ldots, n && (17)\\
& \sum_{j=1}^{n} \mu_j x_j = z && (18)\\
& \sum_{j=1}^{n} r_{jt} x_j = y_t \quad \text{for } t = 1, \ldots, T && (19)\\
& \sum_{t=1}^{T} p_t d_t = z_1 && (20)\\
& d_t \geq r_t^\alpha - y_t, \quad d_t \geq 0 \quad \text{for } t = 1, \ldots, T. && (21)
\end{aligned}$$

Objective function (15) minimizes the risk-reward ratio in (14), whereas constraint (16) imposes the positivity of the ratio denominator. Constraints (17) ensure that in any feasible portfolio the sum of the non-negative weights must be equal to one. Constraint (18) defines z as the mean portfolio rate of return, whereas for each scenario t, with $t = 1, \ldots, T$, constraint (19) defines the corresponding realization of the portfolio rate of return y_t. Moreover, in each scenario t, with $t = 1, \ldots, T$, constraint (21), along with (20) and objective function (15), forces the non-negative variable d_t to take value equal to $\max\{r_t^\alpha - y_t, 0\}$. As a consequence, constraint (20) defines variable z_1, minimized in objective function (15), as the mean below-target deviation.

Compared to the optimization model proposed in Guastaroba *et al.* [10], the main difference is that the objective function in model (15)-(21) is expressed in terms of a risk-reward minimization, rather than a reward-risk maximization. Although this modification is conceptually of minor importance, it avoids the introduction of additional constraints and auxiliary binary variables to deal with those critical situations where the risk measure at the denominator of the reward-risk ratio may take null value (see Guastaroba *et al.* [10]).

The non-linear optimization model (15)-(21) can be linearized using the Charnes-Cooper transformation introduced in [7]. Specifically, we apply the following substitutions $v_0 = 1/(z - \mu^\alpha)$, $v_1 = z_1/(z - \mu^\alpha)$, $v = z/(z - \mu^\alpha)$, $\tilde{x_j} = x_j/(z - \mu^\alpha)$, $\tilde{d}_t = d_t/(z - \mu^\alpha)$ and $\tilde{y_t} = y_t/(z - \mu^\alpha)$, divide all the constraints by $(z - \mu^\alpha)$, and add the constraint required by the transformation.

The resulting formulation is the following LP model:

$$\begin{array}{ll}
\min & v_1 + \varepsilon v_0 \\
\text{s.t.} & v - \mu^\alpha v_0 = 1 \\
& v_0 \leq 1/\varepsilon \\
& \sum_{j=1}^{n} \tilde{x}_j = v_0, \quad \tilde{x}_j \geq 0 \quad \text{for } j = 1, \ldots, n \\
& \sum_{j=1}^{n} \mu_j \tilde{x}_j = v \\
& \sum_{j=1}^{n} r_{jt} \tilde{x}_j = \tilde{y}_t \quad \text{for } t = 1, \ldots, T \\
& \sum_{t=1}^{T} p_t \tilde{d}_t = v_1 \\
& \tilde{d}_t \geq r_t^\alpha v_0 - \tilde{y}_t, \quad \tilde{d}_t \geq 0 \quad \text{for } t = 1, \ldots, T,
\end{array}$$

where the first constraint is a transformed form of the substitution $v_0 = 1/(z - \mu^\alpha)$ whose introduction is required by the Charnes-Cooper transformation. A more compact formulation can be obtained eliminating variables $\tilde{y}_t$, v, v_0, and v_1, which are defined by equations, leading to the following LP formulation:

$$\begin{array}{lll}
\text{(EOR model)} & \min & \sum_{t=1}^{T} p_t \tilde{d}_t + \varepsilon \sum_{j=1}^{n} \tilde{x}_j \\
& \text{s.t.} & \sum_{j=1}^{n} \tilde{x}_j \leq \frac{1}{\varepsilon}, \ \tilde{x}_j \geq 0 \quad j = 1, \ldots, n \\
& & \sum_{j=1}^{n} (\mu_j - \mu^\alpha) \tilde{x}_j = 1 \\
& & \tilde{d}_t \geq \sum_{j=1}^{n} (r_t^\alpha - r_{jt}) \tilde{x}_j, \ \tilde{d}_t \geq 0 \quad t = 1, \ldots, T.
\end{array} \tag{22}$$

After solving the transformed EOR model (22), the original values of variables x_j can be determined dividing $\tilde{x}_j$ by $\sum_{j=1}^{n} \tilde{x}_j$.

3.2 Extended WCVaR ratio model

As a consequence of Theorem 1, risk-reward ratio models are well-defined for deviation type risk measures. Hence, in a CVaR-based risk-reward ratio model one must use the deviation risk measure $\Delta_{\mathbf{w}}^{(m)}(R_{\mathbf{x}})$, i.e., the weighted conditional drawdown, instead of directly the WCVaR $M_{\mathbf{w}}^{(m)}(R_{\mathbf{x}})$. Therefore, the risk-reward ratio model for the EITP based on the WCVaR is the following:

$$\min \left\{ \frac{\Delta_{\mathbf{w}}^{(m)}(R_{\mathbf{x}} - R^\alpha) + \varepsilon}{\mu(R_{\mathbf{x}} - R^\alpha)} \ : \ \mu(R_{\mathbf{x}} - R^\alpha) \geq \varepsilon, \mathbf{x} \in \mathcal{P} \right\}, \tag{23}$$

where we replaced $\varrho(R_{\mathbf{x}} - R^\alpha)$ in the numerator of (5) with $\Delta_{\mathbf{w}}^{(m)}(R_{\mathbf{x}} - R^\alpha)$ as defined in (8).

Since the weighted conditional drawdown $\Delta_{\mathbf{w}}^{(m)}$ is a SSD safety consistent, applying Theorem 1 and Corollary 1 to the distribution of the difference $(R_{\mathbf{x}} - R^\alpha)$ with $\tau = 0$, one gets the following corollary.

Corollary 3 *Let $\mathbf{x}^0$ be an optimal solution to the risk-reward optimization problem (23) that satisfies condition $\mu(R_{\mathbf{x}^0} - R^\alpha) - \Delta_{\mathbf{w}}^{(m)}(R_{\mathbf{x}^0} - R^\alpha) \leq 0$. Then, portfolio $\mathbf{x}^0$ is nondominated in terms of bi-criteria mean-safety maximization $\max\{\mu(R_{\mathbf{x}} - R^\alpha), \mu(R_{\mathbf{x}} - R^\alpha) - \Delta_{\mathbf{w}}^{(m)}(R_{\mathbf{x}} - R^\alpha)\}$, and is SSD nondominated with the exception of alternative optimal portfolios having the same values of mean return $\mu(R_{\mathbf{x}^0} - R^\alpha)$ and risk measure $\Delta_{\mathbf{w}}^{(m)}(R_{\mathbf{x}^0} - R^\alpha)$.*

Under the assumption of security returns described by discrete random variables having, for each security j, with $j \in J$, realization r_{jt} under scenario t, with $t = 1, \ldots, T$, one obtains the following non-linear optimization model:

$$\begin{aligned}
\min \quad & \frac{z - \mu^\alpha - z_1 + \varepsilon}{z - \mu^\alpha} && (24)\\
\text{s.t.} \quad & (16) - (19) \\
& \sum_{k=1}^{m} w_k \eta_k - \sum_{k=1}^{m} \frac{w_k}{\beta_k} \sum_{t=1}^{T} p_t d_{tk} = z_1 && (25)\\
& d_{tk} \geq \eta_k - y_t + r_t^\alpha, \quad d_{tk} \geq 0 \quad \text{for } t = 1, \ldots, T;\; k = 1, \ldots, m. && (26)
\end{aligned}$$

Objective function (24) minimizes the risk-reward ratio in (23). Particularly, the numerator represents the deviation risk measure $\Delta_{\mathbf{w}}^{(m)}(R_{\mathbf{x}})$, and can be obtained by using the first equality in (8) and the definition of WCVaR in (7). Subsequently, the definition of CVaR, expressed in (6), is applied, for each k, with $k = 1, \ldots, m$, to the distribution of the difference $(R_{\mathbf{x}} - R^\alpha)$. Finally, constraint (25), along with (26), defines variable z_1 that, when maximized, represents the WCVaR measure $M_{\mathbf{w}}^{(m)}(R_{\mathbf{x}} - R^\alpha)$.

Also the non linear optimization model (24)–(26) can be linearized applying the following substitutions: $v_0 = 1/(z - \mu^\alpha)$, $v_1 = z_1/(z - \mu^\alpha)$, $v = z/(z - \mu^\alpha)$, $\tilde{x}_j = x_j/(z - \mu^\alpha)$, $\tilde{d}_{tk} = d_{tk}/(z - \mu^\alpha)$, $\tilde{\eta}_k = \eta_k/(z - \mu^\alpha)$, and $\tilde{y}_t = y_t/(z - \mu^\alpha)$, dividing all the constraints by $(z - \mu^\alpha)$, and adding the constraint required by the Charnes-Cooper transformation, leading to the following LP formulation:

$$\begin{aligned}
\min \quad & v - v_1 + (\varepsilon - \mu^\alpha) v_0 \\
\text{s.t.} \quad & v - \mu^\alpha v_0 = 1 \\
& v_0 \leq 1/\varepsilon \\
& \sum_{j=1}^{n} \tilde{x}_j = v_0, \quad \tilde{x}_j \geq 0 \quad \text{for } j = 1, \ldots, n \\
& \sum_{j=1}^{n} \mu_j \tilde{x}_j = v \\
& \sum_{j=1}^{n} r_{jt} \tilde{x}_j = \tilde{y}_t \quad \text{for } t = 1, \ldots, T \\
& \sum_{k=1}^{m} w_k \tilde{\eta}_k - \sum_{k=1}^{m} \frac{w_k}{\beta_k} \sum_{t=1}^{T} p_t \tilde{d}_{tk} = v_1 \\
& \tilde{d}_{tk} \geq \tilde{\eta}_k - \tilde{y}_t + r_t^\alpha v_0, \;\; \tilde{d}_{tk} \geq 0 \quad t = 1, \ldots, T;\; k = 1, \ldots, m,
\end{aligned}$$

where the first constraint is a transformed form of the substitution $v_0 = 1/(z - \mu^\alpha)$. After eliminating variables $\tilde{y}_t$, v, v_0, and v_1, which are defined by equations, one obtains the following more compact LP formulation:

$$
\begin{aligned}
\text{(EWCVaR model)}\ \min\quad & \sum_{j=1}^{n}(\mu_j - \mu^\alpha + \varepsilon)\tilde{x}_j - \sum_{k=1}^{m} w_k \tilde{\eta}_k + \sum_{k=1}^{m} \frac{w_k}{\beta_k} \sum_{t=1}^{T} p_t \tilde{d}_{tk} \\
\text{s.t.}\quad & \sum_{j=1}^{n} \tilde{x}_j \leq \frac{1}{\varepsilon},\ \tilde{x}_j \geq 0 \quad j = 1,\dots,n \\
& \sum_{j=1}^{n}(\mu_j - \mu^\alpha)\tilde{x}_j = 1 \\
& \tilde{d}_{tk} \geq \tilde{\eta}_k - \sum_{j=1}^{n}(r_{jt} - r_t^\alpha)\tilde{x}_j,\ \tilde{d}_{tk} \geq 0 \quad t = 1,\dots,T;\ k = 1,\dots,m.
\end{aligned}
\tag{27}
$$

As for the EOR model, after solving the transformed EWCVaR model (27), the original values of x_j can be determined dividing $\tilde{x}_j$ by $\sum_{j=1}^{n} \tilde{x}_j$.

4 Experimental analysis

This section is dedicated to the presentation and discussion of the computational experiments. They were conducted on a PC Intel XEON with 3.33 GHz 64-bit processor, 12 GB of RAM, and Windows 7 64-bit as Operating System. Optimization models were implemented in Java, compiled within NetBeans 8.0.2, and solved by means of CPLEX 12.6. After preliminary experiments, we decided to force CPLEX to not perform any cutting plane pass when solving the root node (parameter CutPass=-1). We used default values for all the other CPLEX parameters.

The results discussion is organized as follows. In Section 4.1, we briefly describe the instances and the optimization models we solved in the computational experiments, whereas in Section 4.2 we report on the in-sample characteristics of the optimal portfolios and provide an extensive validation of their out-of-sample performance.

4.1 Data sets and tested optimization models

In the computational experiments, we used the two data sets tested in Guastaroba *et al.* [10] with some small differences, as described below. To make the paper self-contained, we provide here a brief description of these instances and refer to the above paper for any further detail.

The first data set was introduced in Guastaroba *et al.* [11] and, from the name of the authors, is referred to as data set *GMS*. No change has been made to this data set. The set consists of 4 instances created from historical rates of return of the 100 securities composing the FTSE 100 Index. These instances were intentionally selected to span four different market trends. In particular, the first instance, hereafter called *GMS-UU*, considers an increasing trend of the benchmark (i.e., the market index is moving *Up*) in both the in-sample and the out-of-sample period. The second instance, from now on referred to as *GMS-UD*, has an increasing trend of the benchmark in the in-sample period and a decreasing one (i.e., it is moving *Down*) in the out-of-sample period. The third instance, henceforth called *GMS-DU*, is characterized by a decreasing trend in the in-sample period and by an increasing one in the out-of-sample period. Finally, the last instance, referred to as *GMS-DD* in the following, is characterized by a decreasing trend in both the in-sample and the out-of-sample periods.

Guastaroba *et al.* [10] used a second data set, which was generated from the 8 benchmark instances for the index tracking problem currently belonging to the OR-Library (available at `http://people.brunel.ac.uk/~mastjjb/jeb/orlib/indtrackinfo.html`). These instances consider the securities included in eight different stock market indices: the Hang Seng market index (related to the Hong Kong stock exchange market), the DAX 100 (Germany), the FTSE 100 (United Kingdom), the S&P 100 (USA), the Nikkei 225 (Japan), the S&P 500 (USA), the Russell 2000 (USA) and the Russell 3000 (USA). The number of securities included in these instances ranges from 31, composing the Hang Seng index, to 2151, composing the Russell 3000 index. We found that in the two largest instances there were some securities achieving extremely large weekly returns (even larger than 1000 %) in one or very few observations. Since rates of return of this magnitude have a strong impact on the average return of a security, even if realized in very few observations, we decided to remove the related security from the instance. In the following, this modified data set is called *ORL*, and each instance is referred to as ORL-ITβ, $\beta = 1, \ldots, 8$. Eventually, we removed two securities from both the ORL-IT7 and the ORL-IT8 instances.

Each of the above instances comprises 2 years of in-sample weekly observations (i.e., 104 scenarios) and 1 year of out-of-sample ones (i.e., 52 realizations). For each instance, the optimal portfolio composition is first decided by solving one of the optimization models described in the following and using the in-sample 104 scenarios. Then, the performance of the portfolios is evaluated by observing their behaviors over the 52 weeks following the date of portfolio selection.

The optimization models that we considered in our computational experiments are the following. To provide some insights on the effectiveness of the EWCVaR model (27), we solved it using four different sets of values for the tolerance levels $\{\beta_k\}_{k=1,\ldots,m}$. More specifically, the first model considers two tolerance levels (i.e., $m = 2$) equal to $\beta_1 = 0.05$ and $\beta_2 = 0.25$, respectively. This model is henceforth referred to as *EWCVaR(.05, .25)*. The second model, from now on denoted as *EWCVaR(.05, .25, .50)*, is based on the choice of three tolerance levels (i.e., $m = 3$). We set these three values equal to $\beta_1 = 0.05$, $\beta_2 = 0.25$, and $\beta_3 = 0.50$, respectively. The remaining two models consider only one tolerance level (i.e., $m = 1$). These models are hereafter called *ECVaR(.05)* and *ECVaR(.50)*, since they correspond to setting the tolerance level β_1 equal to 0.05 and 0.50, respectively. For each of the above models, weights w_k were computed according to (10). Finally, as a basis for comparison with the literature, we also solved the EOR model (22) on the aforementioned instances.

As mentioned above, the EWCVaR model (27) is valid only if the value of the ratio (23) is not smaller than 1. To guarantee that this condition is satisfied, we devised the following pre-processing procedure to choose the value of α to use in the experiments. For each instance, we solved separately each of the aforementioned EWCVaR models, starting with an initial value of α equal to 0. Then, we solved iteratively that optimization model increasing the value of α by 1 % (on yearly basis), as long as ratio (23) took a value smaller than 1. Finally, the maximum among the final value taken by α over all the models is chosen, to guarantee that the above condition is satisfied for any of the tested EWCVaR models. Computing times to solve the EWCVaR models, and hence to carry out the above pre-processing procedure, are negligible, as described in the following section.

Table 1 summarizes the main characteristics of all the tested instances, including the average in-sample return of the benchmark (column with header $\mu^I\%$) and the value of α used in the experiments (column $\alpha\%$). For the sake of readability, we expressed the latter two values in percentage and on yearly basis, even though they are expressed on weekly basis in the instances.

All instances in the two data sets are publicly available on the website of the Operational Research Group at the University of Brescia (`http://or-brescia.unibs.it`), in section "Benchmark Instances".

Table 1: The main characteristics of the tested data sets.

Data Set	**Instance**	**Benchmark**	n	T	μ^I%	α%
GMS	GMS-UU	FTSE 100	100	104	15.61	5.10
	GMS-UD	FTSE 100	100	104	17.39	7.21
	GMS-DU	FTSE 100	100	104	-21.15	4.06
	GMS-DD	FTSE 100	100	104	-11.81	10.45
ORL	ORL-IT1	Hang Seng	31	104	48.60	0.00
	ORL-IT2	DAX 100	85	104	7.16	3.03
	ORL-IT3	FTSE 100	89	104	14.20	8.28
	ORL-IT4	S&P 100	98	104	6.46	6.15
	ORL-IT5	Nikkei 225	225	104	-0.88	10.45
	ORL-IT6	S&P 500	457	104	26.07	24.42
	ORL-IT7	Russell 2000	1316	104	9.22	74.07
	ORL-IT8	Russell 3000	2149	104	23.36	59.30

4.2 Comparing the performance of the optimal portfolios

In Tables 2 and 3, we provide some in-sample and out-of-sample statistics summarizing the computational results obtained by solving all the tested models with the GMS and ORL data sets, respectively. Both tables have the same structure, and the meaning of each column header is as follows. Regarding the in-sample, we report the following statistics:

- ✓ *Div.*: the number of securities selected in the optimal portfolio;
- ✓ *Min %*: the minimum portfolio share (in percentage);
- ✓ *Max %*: the maximum portfolio share (in percentage).

On the other hand, as out-of-sample statistics we report the following ones:

- ✓ $y_t > r_t^I$ %: the number of weeks, divided by 52 and in percentage, that the portfolio rate of return has outperformed the benchmark in the out-of-sample period;
- ✓ r_{av} %: the average portfolio return on yearly basis (in percentage);
- ✓ *Excess Ret. %*: the out-of-sample average excess return of the portfolio over the benchmark, on yearly basis and in percentage. It is computed as [r_{av}] - [average benchmark return];
- ✓ *s-std*: the downside semi-standard deviation of the portfolio return compared to the benchmark return, computed as $\sqrt{\frac{1}{52}\sum_{t=1}^{52}(y_t - r_t^I)_-^2}$;
- ✓ *Sortino Index*: the average excess return divided by the semi-standard deviation s-std.

The above statistics provide a synthetic and clear assessment of both in-sample main characteristics and out-of-sample performance of the optimal portfolios. In both tables, for each instance, we highlighted in bold the model(s) that achieved the best value of the Sortino index (the larger, the better). As already mentioned, computing times required to optimally solve the tested models are always negligible (in the order of fractions of a second), and thus they have not been reported here. On the other side, one can note that for both the EOR model (22) and the EWCVaR model (27) the number of variables and constraints increase with the number of scenarios. Hence, finding an optimal solution for these models may become computationally challenging when the number of scenarios employed is very large. In Appendix B, we show that, taking advantage of LP duality, one can obtain more computationally efficient formulations to use with a large number of scenarios. Finally, to evaluate and easily compare the out-of-sample performance of the optimal portfolios over time, we plot in Figures 2–4 the ex-post cumulative returns yielded by all the selected portfolios and the respective benchmark in each of the 12 tested instances.

Table 2: Optimal portfolios: In-sample and out-of-sample statistics for the GMS data set.

		In-Sample			**Out-of-Sample**				
Instance	Model	Div.	Min %	Max %	$y_t > r_t^I$ %	r_{av} %	Excess Ret. %	s-std	Sortino Index
GMS-UU	***EOR***	***32***	***0.27***	***9.86***	***53.85***	***46.08***	***0.16***	***0.0068***	***0.0031***
	EWCVaR(.05, .25)	30	0.16	12.12	53.85	44.07	-1.85	0.0069	-0.0358
	EWCVaR(.05, .25, .50)	31	0.32	13.31	53.85	43.15	-2.77	0.0079	-0.0468
	ECVaR(.05)	33	0.08	10.82	53.85	42.84	-3.08	0.0067	-0.0615
	ECVaR(.50)	34	0.10	14.33	55.77	43.62	-2.30	0.0079	-0.0389
GMS-UD	EOR	34	0.26	9.75	61.54	1.39	9.93	0.0041	0.4835
	EWCVaR(.05, .25)	36	0.33	10.76	65.38	1.73	10.27	0.0043	0.4749
	EWCVaR(.05, .25, .50)	34	0.19	10.56	65.38	-0.18	8.36	0.0043	0.3940
	ECVaR(.05)	***38***	***0.21***	***10.18***	***67.31***	***4.45***	***12.99***	***0.0044***	***0.5819***
	ECVaR(.50)	35	0.14	8.91	65.38	-2.39	6.15	0.0041	0.3070
GMS-DU	EOR	31	0.14	10.50	48.08	31.55	-2.64	0.0034	-0.1133
	EWCVaR(.05, .25)	33	0.14	10.11	53.85	33.04	-1.15	0.0034	-0.0491
	EWCVaR(.05, .25, .50)	35	0.05	10.45	46.15	32.25	-1.94	0.0032	-0.0877
	ECVaR(.05)	37	0.13	10.19	53.85	32.41	-1.78	0.0030	-0.0851
	ECVaR(.50)	***32***	***0.10***	***10.40***	***53.85***	***33.59***	***-0.60***	***0.0030***	***-0.0283***
GMS-DD	EOR	34	0.06	10.33	53.85	-20.53	2.40	0.0058	0.1019
	EWCVaR(.05, .25)	***33***	***0.17***	***9.76***	***53.85***	***-19.86***	***3.07***	***0.0061***	***0.1228***
	EWCVaR(.05, .25, .50)	33	0.08	9.80	55.77	-21.03	1.90	0.0058	0.0802
	ECVaR(.05)	34	0.10	8.22	55.77	-20.94	1.99	0.0052	0.0932
	ECVaR(.50)	36	0.13	10.93	59.62	-20.87	2.06	0.0058	0.0869

Table 2 summarizes the results for the GMS data set. Looking at the in-sample portfolio diversification (column Div.), it is evident that all the portfolios have a similar cardinality. Also the minimum and maximum portfolio shares (columns Min % and Max %, respectively) take very similar values in all the instances, with the only exclusion of instance GMS-UU, where a slightly larger deviation among the values of the maximum share can be identified. Note that the maximum portfolio share never exceeds the 14.5%, indicating that in all the optimal portfolios the budget available has been, in a broad sense, well-diversified among the securities. As far as the out-of-sample performance is considered, after analyzing the figures reported in Table 2 and the cumulative returns depicted in Figure 2, one can conclude that all the optimal portfolios perform similarly and well: they closely mimic the benchmark, often outperform it (even if not for the entire out-of-sample period), and show a limited performance deviation between each others. Some differences are evident for instance GMS-UD. In this case, all the optimized portfolios clearly outperform the benchmark, although differently. In more details, Figure 2(b) shows that the optimal portfolios selected by models EOR and EWCVaR(.05, .25, .50) achieve the highest cumulative returns in the first part of the ex-post period, whereas they are clearly outperformed by the portfolio selected by model ECVaR(.05) in the last part of the ex-post period. Analyzing more in depth the figures reported in Table 2, one can also notice that for the GMS-UU instance the portfolio selected with the EOR model is the one that yielded the best ex-post cumulative return, and the only one that achieved a (slightly) positive average excess return (see column Excess Ret. %).

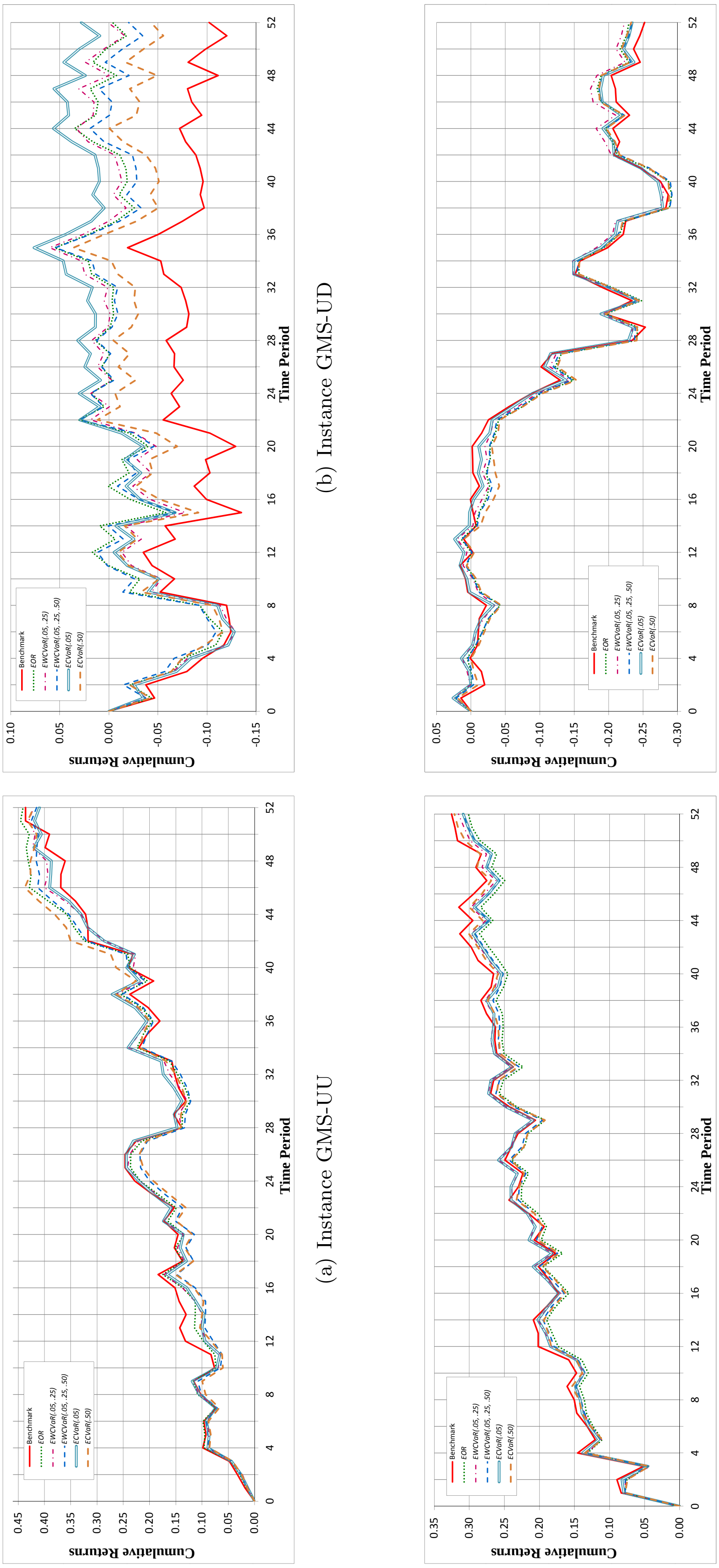
We now turn our attention to the results regarding the ORL data set, which are summarized in Table 3 and illustrated in Figures 3 and 4. As far as the four smallest instances of this data set are considered, Figures 3(a)-3(d) show that all the optimal portfolios replicate quite closely the ex-post behavior of their benchmark. In more details, regarding instances ORL-IT1 through

Table 3: Optimal portfolios: In-sample and out-of-sample statistics for the ORL data set.

		In-Sample			**Out-of-Sample**				
Instance	Model	Div.	Min %	Max %	$y_t > r_t^I$ %	r_{av} %	Excess Ret. %	s-std	Sortino Index
ORL-IT1	EOR	25	0.24	16.52	59.62	-13.06	2.86	0.0027	0.2383
	EWCVaR(.05, .25)	25	0.31	15.37	55.77	-13.29	2.64	0.0026	0.2251
	EWCVaR(.05, .25, .50)	25	0.12	16.19	61.54	-13.04	2.89	0.0026	0.2498
	ECVaR(.05)	26	0.35	15.35	48.08	-14.19	1.73	0.0025	0.1584
	ECVaR(.50)	***25***	***0.09***	***15.90***	***61.54***	***-12.30***	***3.62***	***0.0024***	***0.3437***
ORL-IT2	EOR	45	0.05	8.88	63.46	2.08	1.73	0.0017	0.1955
	EWCVaR(.05, .25)	51	0.08	9.85	61.54	1.47	1.12	0.0020	0.1072
	EWCVaR(.05, .25, .50)	48	0.14	8.24	61.54	2.24	1.90	0.0015	0.2468
	ECVaR(.05)	51	0.08	9.85	61.54	1.47	1.12	0.0020	0.1072
	ECVaR(.50)	***47***	***0.10***	***8.42***	***59.62***	***2.34***	***1.99***	***0.0013***	***0.2931***
ORL-IT3	EOR	47	0.10	6.35	51.92	-5.13	1.40	0.0026	0.1117
	EWCVaR(.05, .25)	46	0.08	7.22	50.00	-6.31	0.23	0.0029	0.0160
	EWCVaR(.05, .25, .50)	46	0.06	7.46	48.08	-5.73	0.80	0.0030	0.0552
	ECVaR(.05)	46	0.08	7.22	50.00	-6.31	0.23	0.0029	0.0160
	ECVaR(.50)	***45***	***0.05***	***5.98***	***48.08***	***-4.37***	***2.16***	***0.0025***	***0.1786***
ORL-IT4	EOR	47	0.05	6.29	46.15	5.90	0.17	0.0023	0.0133
	EWCVaR(.05, .25)	54	0.08	5.15	53.85	5.49	-0.24	0.0020	-0.0223
	EWCVaR(.05, .25, .50)	***50***	***0.05***	***5.62***	***53.85***	***6.02***	***0.29***	***0.0021***	***0.0258***
	ECVaR(.05)	54	0.08	5.15	53.85	5.49	-0.24	0.0020	-0.0223
	ECVaR(.50)	46	0.08	8.98	48.08	5.07	-0.66	0.0028	-0.0453
ORL-IT5	EOR	57	0.08	9.04	46.15	-15.06	-3.99	0.0033	-0.2684
	EWCVaR(.05, .25)	***67***	***0.08***	***7.42***	***44.23***	***-13.80***	***-2.74***	***0.0026***	***-0.2311***
	EWCVaR(.05, .25, .50)	62	0.06	7.93	46.15	-14.68	-3.61	0.0030	-0.2627
	ECVaR(.05)	***67***	***0.08***	***7.42***	***44.23***	***-13.80***	***-2.74***	***0.0026***	***-0.2311***
	ECVaR(.50)	58	0.05	8.34	46.15	-15.11	-4.04	0.0032	-0.2752
ORL-IT6	***EOR***	***57***	***0.10***	***5.71***	***57.69***	***28.34***	***5.44***	***0.0044***	***0.1924***
	EWCVaR(.05, .25)	60	0.05	5.87	55.77	27.55	4.65	0.0049	0.1474
	EWCVaR(.05, .25, .50)	54	0.11	6.34	53.85	27.06	4.16	0.0049	0.1311
	ECVaR(.05)	60	0.05	5.87	55.77	27.55	4.65	0.0049	0.1474
	ECVaR(.50)	59	0.08	6.26	53.85	23.39	0.49	0.0055	0.0139
ORL-IT7	***EOR***	***74***	***0.09***	***5.56***	***61.54***	***29.45***	***16.41***	***0.0066***	***0.3993***
	EWCVaR(.05, .25)	77	0.05	5.03	57.69	25.25	12.22	0.0068	0.2908
	EWCVaR(.05, .25, .50)	77	0.05	5.03	57.69	25.25	12.22	0.0068	0.2908
	ECVaR(.05)	77	0.05	5.03	57.69	25.25	12.22	0.0068	0.2908
	ECVaR(.50)	72	0.07	4.55	55.77	27.51	14.48	0.0080	0.2906
ORL-IT8	EOR	59	0.05	10.39	50.00	38.69	16.84	0.0090	0.2772
	EWCVaR(.05, .25)	***59***	***0.05***	***10.03***	***61.54***	***50.83***	***28.98***	***0.0095***	***0.4348***
	EWCVaR(.05, .25, .50)	57	0.08	10.60	59.62	43.89	22.03	0.0092	0.3498
	ECVaR(.05)	***59***	***0.05***	***10.03***	***61.54***	***50.83***	***28.98***	***0.0095***	***0.4348***
	ECVaR(.50)	60	0.05	10.15	59.62	39.87	18.02	0.0085	0.3126

Table 4: Optimization models: A summary of their out-of-sample rankings.

Model	**# First**	**# Second**	**# Third**	**# Fourth**	**# Last**
EOR	3	4	2	1	2
EWCVaR(.05, .25)	3	4	2	3	0
EWCVaR(.05, .25, .50)	1	3	3	4	1
ECVaR(.05)	3	2	3	2	2
ECVaR(.50)	4	0	1	2	5



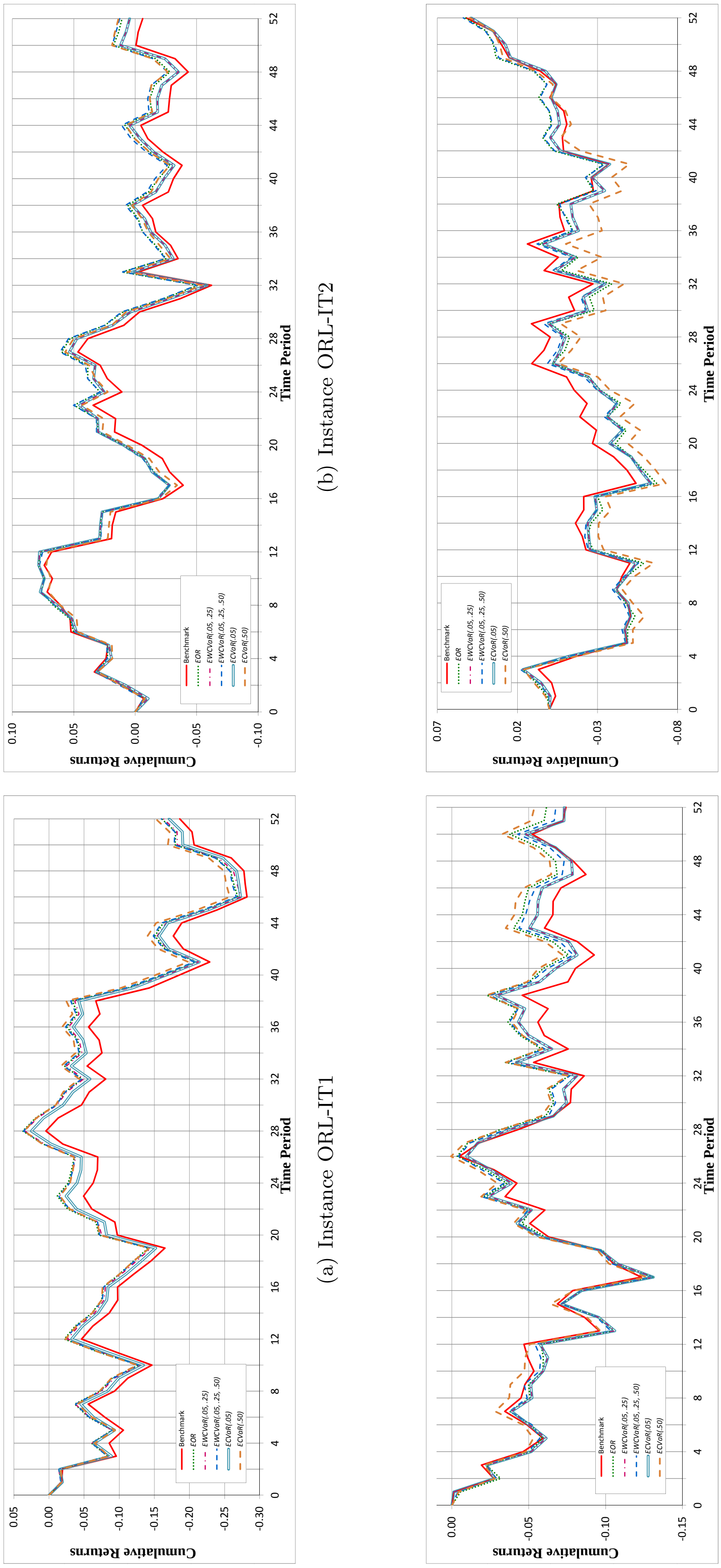
(a) Instance GMS-UU

(b) Instance GMS-UD

(c) Instance GMS-DU

(d) Instance GMS-DD

Figure 2: Out-of-sample cumulative returns: A comparison among the optimization models and the benchmark on the GMS data set.



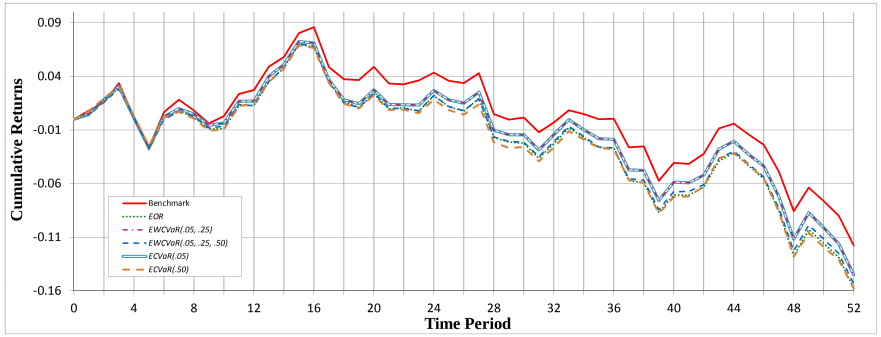
(a) Instance ORL-IT1

(b) Instance ORL-IT2

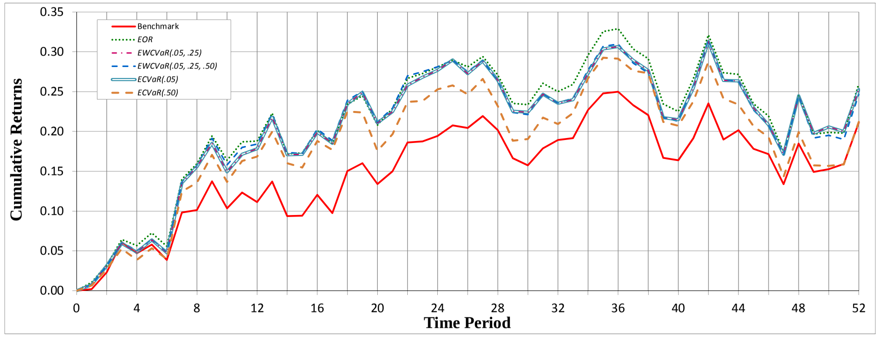
(c) Instance ORL-IT3

(d) Instance ORL-IT4

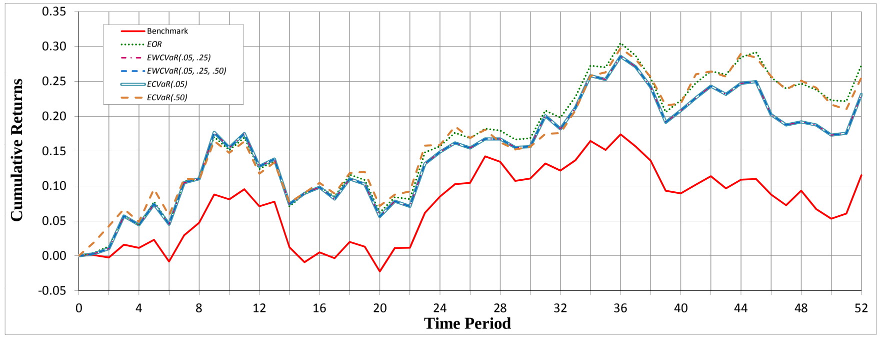
Figure 3: Out-of-sample cumulative returns: A comparison among the optimization models and the benchmark on the 4 smallest ORL instances.



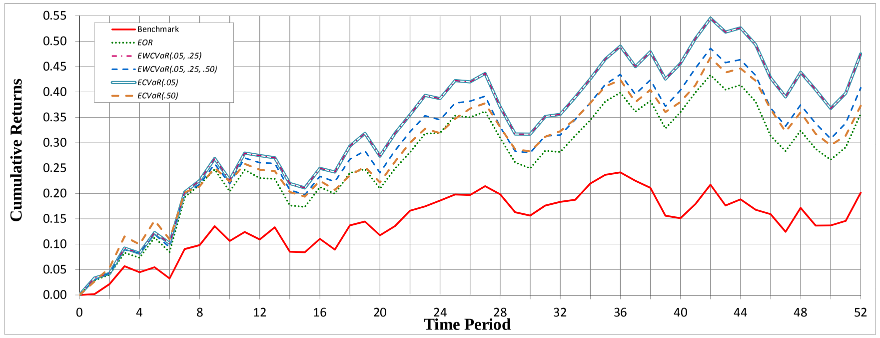
(a) Instance ORL-IT5



(b) Instance ORL-IT6



(c) Instance ORL-IT7



(d) Instance ORL-IT8

Figure 4: Out-of-sample cumulative returns: A comparison among the optimization models and the benchmark on the 4 largest ORL instances.

ORL-IT3, the portfolios that perform best are the ones obtained solving model ECVaR(.50), achieving an average excess return that ranges from 1.99% to 3.62%, with values of statistic s-std slightly smaller than the ones of the other optimal portfolios. Conversely, the portfolio selected by the ECVaR(.50) model is the one that performs worst in instance ORL-IT4 (compare the values of the Sortino index). For this instance, the only portfolios that achieved a positive excess return are the ones determined by solving the EOR and, in particular, the EWCVaR(.05, .25, .50) models. Regarding instance ORL-IT5, the ex-post cumulative returns yielded by the benchmark are always better than the ones achieved by the optimized portfolios, although the differences are not very large (the average excess return of the portfolios ranges from -2.74% to -4.04%). In this instance, the portfolio selected using the EWCVaR(.05, .25) and the ECVaR(.05) models is the one that looses less compared to the benchmark. In the three largest-scale instances of the ORL data set, all the optimized portfolios outperform considerably the benchmark (see Figures 4(b)-4(d)). Regarding instance ORL-IT6, the portfolio selected by model EOR provides the best out-of-sample results, achieving an average excess return equal to 5.44%, beating the 57.69% of times (out of the 52 ex-post observations) the return yielded by the benchmark, and with the smallest downside risk (statistic s-std takes a value approximately equal to 0.0044). Nevertheless, the performance of the portfolio obtained by the ECVaR(.05) and the EWCVaR(.05,.25) models is only slightly worse, achieving an average excess return roughly equal to 4.65%, beating the benchmark return 55.77% of the times, and with a downside risk around 0.0049. As far as instance ORL-IT7 is considered, the portfolio constructed by solving the EOR model achieves the best ex-post performance: it yields an average excess return of approximately 16.41% and a value of the Sortino index equal to 0.3993. It is worth noting that, with the exception of the ECVaR(.50) model, all the CVaR-based models select the same optimal portfolio. Finally, regarding instance ORL-IT8, both the ECVaR(.05) and the EWCVaR(.05, .25) models find the same optimal portfolio. More importantly, the latter portfolio is the one performing best ex-post, yielding an average excess return roughly equal to 28.98% (which is considerably larger than that achieved by the other portfolios), and being only slightly riskier (compare the figures reported in column s-std). Note that the portfolio with the worst value of the Sortino index for instance ORL-IT8 is the one selected by the EOR model. It is worth highlighting that in both the ORL-IT7 and ORL-IT8 instances, all the optimal portfolios largely outperform the benchmark over the entire out-of-sample period, yielding much larger cumulative returns than the ones achieved by the market index. A similar finding also occurs in instance ORL-IT6 for most of the optimal portfolios, with the exception of the portfolio obtained solving model ECVaR(.50). Actually, the latter clearly outperforms the benchmark over most of the out-of-sample period, but achieving similar cumulative returns towards the end of the period. Interesting enough, although we treat differently some more or less extreme events, for several instances the ECVaR(.05) and ECVaR(.05, 0.25) models find the same optimal portfolios, perhaps indicating that in these cases a larger number of in-sample observations, or further levels of diversification and weight setting might help.

Summarizing the previous discussion, the experimental results that we conducted indicate that no optimization model shows a clear dominance over the others. Indeed, considering all the instances we tested, it is not possible to determine a “winning model”, nor a “losing model”. It is, however, possible to determine the following general guidelines. Firstly, optimization models can be a valuable tool to support investment decisions. Indeed, it is worth noting that in 10 out of the 12 instances, at least one optimal portfolio outperforms ex-post the respective benchmark in terms of average return yielded. More interestingly, in 8 out of the 12 instances all the optimal portfolios outperform their benchmark. Secondly, analyzing in more details the results reported in Table 2, one can notice that the instances where most of (or even all) the optimal portfolios yielded a negative excess return are those when the market trend is increasing out-of-sample (i.e., instances GMS-UU and GMS-DU). Considering that in these two instances the optimal portfolios yielded an average return at least equal to 42.84% and 31.55%, respectively, we believe that these are situations where for an investor it is, from a practical perspective, less relevant to outperform a benchmark. On the other side, this becomes crucial when the market trend is decreasing ex-post, as for instances GMS-UD and GMS-DD. Note that, in these two cases, all the optimal portfolios yielded a positive excess return, at least equal to 6.15% and to 1.90%, respectively. To provide some further insights into the performance of the optimal portfolios, Table 4 summarizes the ranking of the five models according to the Sortino index values. More precisely, this table reports the number of times (out of the 12 instances) that the portfolio selected by each optimization model was ranked from the first to the fifth position, based on the Sortino index. Although, the ECVaR(.50) model, with a value of 4, achieved the highest number of times the first position, it slips back to the worst performance when one considers the cumulative sum of the first two positions (the total number remains equal to 4, which is the same result attained by model EWCVaR(.05, .25, .50)). It is worth noting that the ECVaR(.50) model is also the one with the highest number of times in the last position. If one considers the sum of first and second positions, models EOR and EWCVaR(.05, .25) are the best ones with a cumulative sum equal to 7. Analyzing the figures in more details, one can notice that the EWCVaR(.05, .25) model has never been in the last position, whereas the EOR model attained twice the last position, hence making the former preferable to the latter. Although the EWCVaR(.05, .25) and the ECVaR(.05) models often select the same optimal portfolios, the rankings indicate that in the remaining instances model EWCVaR(.05, .25) performs better than model ECVaR(.05). Finally, model EWCVaR(.05, .25, .50) is the more conservative, awarding the first position and the last one only once each, hence ranking in the middle positions for most of the instances.

5 Conclusions

In recent years, shortfall or quantile risk measures have been playing a central role in financial applications. The Conditional Value-at-Risk (CVaR) is one of such measures. In this paper, we formulated the Enhanced Index Tracking Problem based on using Weighted CVaR (WCVaR) measures, which are defined as combinations of a few CVaR measures thus allowing a more detailed risk aversion modeling while preserving the simplicity of the CVaR. More precisely, we used the weighted conditional drawdown measure, corresponding to the WCVaR, to formulate the problem as a class of risk-reward ratio optimization models which, using standard linearization techniques, can be reformulated in terms of LP solvable models. The performance of the portfolios optimized by means of these models has been compared to the one of the portfolios constructed using the Extended Omega Ratio (EOR) model presented in Guastaroba *et al.* [10] and here reformulated as a risk-reward model. All optimization models were solved by using CPLEX.

We conducted extensive computational experiments on two different sets of benchmark instances, exploring different market trends both in-sample and out-of-sample. The results indicate that no optimization model clearly dominates all the others in terms of out-of-sample performance. On the other side, the results suggest that optimization models can represent, to an investor, a valuable quantitative tool to support investment decisions. All the optimal portfolios tracked very closely the behavior of the benchmark over the out-of-sample period, often achieving better average and cumulative returns.

Appendix A: Other CVaR-related ratio measures

In this section, we discuss some well-known ratio performance measures that are related to the CVaR, and therefore relevant to our research. In particular, we highlight in the following the similarities among these ratio measures and the ones used in the current paper. Consider the following risk-return ratio that uses a single CVaR measure:

$$DDR_\beta(\tau, R_{\mathbf{x}}) = \frac{\Delta_\beta(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}}) - \tau} \quad \rightarrow \quad \min .$$

Following Theorem 1 and Corollary 1, one gets a well-defined model for $\mu(R_{\mathbf{x}}) - \tau > 0$ and SSD consistent for $M_\beta(R_{\mathbf{x}}) = \mu(R_{\mathbf{x}}) - \Delta_\beta(R_{\mathbf{x}}) \le \tau$. Rachev *et al.* [29] introduced a CVaR-related ratio measure called the Stable-Tail Adjusted Return Ratio (in short, STARR) defined as:

$$STARR_\beta(\tau, R_{\mathbf{x}}) = \frac{\mathbb{E}\{R_{\mathbf{x}} - \tau\}}{\tau - M_\beta(R_{\mathbf{x}})} = \frac{\mu(R_{\mathbf{x}}) - \tau}{\tau - M_\beta(R_{\mathbf{x}})} \quad \rightarrow \quad \max .$$

Note that the additional restriction $\tau - M_\beta(R_{\mathbf{x}}) > 0$ must be imposed, in addition to $\mu(R_{\mathbf{x}}) - \tau > 0$, to guarantee the positivity of the above ratio. Obviously, this ratio can be reformulated in terms of risk-reward ratio optimization as follows:

$$\overline{STARR}_\beta(\tau, R_{\mathbf{x}}) = \frac{\tau - M_\beta(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}}) - \tau} \quad \rightarrow \quad \min .$$

Actually, note that the following equalities hold:

$$\overline{STARR}_\beta(\tau, R_{\mathbf{x}}) = \frac{\tau - M_\beta(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}}) - \tau} = \frac{\tau - \mu(R_{\mathbf{x}}) + \Delta_\beta(R_{\mathbf{x}})}{\mu(R_{\mathbf{x}}) - \tau} = -1 + DDR_\beta(\tau, R_{\mathbf{x}}).$$

Hence, optimizing $STARR_\beta(\tau, R_{\mathbf{x}})$ is equivalent to optimizing $DDR_\beta(\tau, R_{\mathbf{x}})$, although the former requires the additional restriction $\tau - M_\beta(R_{\mathbf{x}}) > 0$.

A more general CVaR-related ratio measure, called the Rachev ratio (R-ratio), was introduced in [29]. It is defined as the ratio of the expected excess tail return above a certain threshold level (percentile of the right tail distribution), divided by the expected excess tail loss beyond another threshold level (percentile of the left tail distribution). In terms of the CVaR measure $M_\beta(R_{\mathbf{x}})$, the R-ratio can be expressed as follows:

$$RR_{(\beta_1,\beta_2)}(\tau, R_{\mathbf{x}}) = \frac{-M_{\beta_1}(\tau - R_{\mathbf{x}})}{-M_{\beta_2}(R_{\mathbf{x}} - \tau)} \quad \rightarrow \quad \max,$$

with parameters $0 < \beta_1, \beta_2 \le 1$. Note that in the special case obtained by setting $\beta_1 = 1$ and $\beta_2 = \beta$, the corresponding R-ratio reduces to the $STARR_\beta(\tau, R_{\mathbf{x}})$:

$$RR_{(1,\beta)}(\tau, R_{\mathbf{x}}) = \frac{-M_1(\tau - R_{\mathbf{x}})}{-M_\beta(R_{\mathbf{x}} - \tau)} = \frac{\mathbb{E}\{R_{\mathbf{x}} - \tau\}}{\tau - M_\beta(R_{\mathbf{x}})} = \frac{\mu(R_{\mathbf{x}}) - \tau}{\tau - M_\beta(R_{\mathbf{x}})} = STARR_\beta(\tau, R_{\mathbf{x}}).$$

Another interesting special case of the R-ratio is defined by setting $\beta_1 = 1-\beta$ and $\beta_2 = \beta$. In this case, the corresponding R-ratio is defined as the ratio of the expected excess tail return above a certain threshold level, divided by the expected excess tail loss beyond the complementary threshold level. Thus, it is a quantile form of the Omega ratio introduced in Keating and Shadwick [13], and used in the current paper as a basis for the EOR model. We show below that despite the $RR_{(1-\beta,\beta)}(\tau, R_{\mathbf{x}})$ does not represent $STARR_\beta(\tau, R_{\mathbf{x}})$, their maximization is equivalent.

In order to analyze the general R-ratio, let us recall the relations between upper (right) tail mean and the lower (left) tail mean of a distribution. Recall that $F_{R_{\mathbf{x}}}(\eta) = \mathbb{P}\{R_{\mathbf{x}} \leq \eta\}$ denotes the right-continuous cumulative distribution function of $R_{\mathbf{x}}$, whereas $F^{(-1)}_{R_{\mathbf{x}}}(\xi)$ is the corresponding quantile function defined as the left-continuous inverse of the cumulative distribution function $F_{R_{\mathbf{x}}}$. Similarly, let $\overline{F}_{R_{\mathbf{x}}}(\eta) = \mathbb{P}\{R_{\mathbf{x}} \geq \eta\}$, be the left-continuous right tail cumulative distribution function which, for any real value η, provides the probability of having returns larger than or equal to η. Then, let $\overline{F}^{(-1)}_{R_{\mathbf{x}}}$ denote the right tail quantile function defined as the left-continuous inverse of the right tail cumulative distribution function $\overline{F}_{R_{\mathbf{x}}}$, i.e., $\overline{F}^{(-1)}_{R_{\mathbf{x}}}(\xi) = \sup\{\eta : \overline{F}_{R_{\mathbf{x}}}(\eta) \geq \xi\}$, for $0 < \xi \leq 1$. Note that $\overline{F}^{(-1)}_{R_{\mathbf{x}}}(\xi) = F^{(-1)}_{R_{\mathbf{x}}}(1-\xi)$. Furthermore, the (convex) absolute Lorenz curve for any distribution may be viewed as an integrated quantile function:

$$F^{(-2)}_{R_{\mathbf{x}}}(\xi) = \int_0^{\xi} F^{(-1)}_{R_{\mathbf{x}}}(\alpha)d\alpha.$$

Alternatively, the upper (concave) absolute Lorenz curve may be used which integrates the right tail quantile function:

$$\overline{F}^{(-2)}_{R_{\mathbf{x}}}(\xi) = \int_0^{\xi} \overline{F}^{(-1)}_{R_{\mathbf{x}}}(\alpha)d\alpha.$$

Actually, both the classical (lower) and the upper absolute Lorenz curves are symmetric with respect to the diagonal line $\mu(R_{\mathbf{x}})\xi$, in the sense that the differences $\overline{F}^{(-2)}_{R_{\mathbf{x}}}(\xi) - \mu(R_{\mathbf{x}})\xi$ and $\mu(R_{\mathbf{x}})\xi - F^{(-2)}_{R_{\mathbf{x}}}(\xi)$ are equal for symmetric arguments ξ and $(1-\xi)$, i.e.:

$$\overline{F}^{(-2)}_{R_{\mathbf{x}}}(\xi) + F^{(-2)}_{R_{\mathbf{x}}}(1-\xi) = \mu(R_{\mathbf{x}}), \quad \text{for any } 0 \leq \xi \leq 1.$$

Therefore, one can write:

$$\begin{aligned} -M_{\beta_1}(\tau - R_{\mathbf{x}}) &= \frac{1}{\beta_1}\overline{F}^{(-2)}_{R_{\mathbf{x}}-\tau}(\beta_1) = \frac{1}{\beta_1}(\mu(R_{\mathbf{x}} - \tau) - F^{(-2)}_{R_{\mathbf{x}}-\tau}(1-\beta_1)) \\ &= \frac{1}{\beta_1}(\mu(R_{\mathbf{x}}) - \tau) - \frac{1-\beta_1}{\beta_1}M_{(1-\beta_1)}(R_{\mathbf{x}} - \tau). \end{aligned}$$

Consequently, the following equalities hold:

$$\begin{aligned} RR_{(\beta_1,\beta_2)}(\tau, R_{\mathbf{x}}) &= \frac{-M_{\beta_1}(\tau - R_{\mathbf{x}})}{-M_{\beta_2}(R_{\mathbf{x}} - \tau)} = \frac{1}{\beta_1}\frac{\mu(R_{\mathbf{x}}) - \tau}{\tau - M_{\beta_2}(R_{\mathbf{x}})} + \frac{1-\beta_1}{\beta_1}\frac{M_{(1-\beta_1)}(R_{\mathbf{x}} - \tau)}{M_{\beta_2}(R_{\mathbf{x}} - \tau)} \\ &= \frac{1}{\beta_1}STARR_{\beta_2}(\tau, R_{\mathbf{x}}) + \frac{1-\beta_1}{\beta_1}\frac{M_{(1-\beta_1)}(R_{\mathbf{x}} - \tau)}{M_{\beta_2}(R_{\mathbf{x}} - \tau)}. \end{aligned}$$

In particular, in the special case of $\beta_1 = 1 - \beta$ and $\beta_2 = \beta$, one can write:

$$RR_{(1-\beta,\beta)}(\tau, R_{\mathbf{x}}) = \frac{1}{1-\beta}STARR_{\beta}(\tau, R_{\mathbf{x}}) + \frac{\beta}{1-\beta},$$

which confirms that the maximization of $RR_{(1-\beta,\beta)}(\tau, R_{\mathbf{x}})$ is equivalent to the maximization of $STARR_{\beta}(\tau, R_{\mathbf{x}})$, and thereby to the minimization of $DDR_{\beta}(\tau, R_{\mathbf{x}})$. On the other hand, for the general case of $\beta_1 \neq 1 - \beta_2$ the optimization of $RR_{(\beta_1,\beta_2)}(\tau, R_{\mathbf{x}})$ cannot be linearized and it is rather difficult to implement and solve.

Since in the literature both ratios $STARR_{\beta}(\tau, R_{\mathbf{x}})$ and $RR_{(1-\beta,\beta)}(\tau, R_{\mathbf{x}})$ are usually formulated with respect to a predetermined target τ, we preferred to provide the analysis above in these terms. However, the analysis can be easily extended to a random target R^α, leading to the following ratios:

$$DDR_\beta(0, R_{\mathbf{x}} - R^\alpha) = \frac{\Delta_\beta(R_{\mathbf{x}} - R^\alpha)}{\mu(R_{\mathbf{x}} - R^\alpha)} \quad \rightarrow \quad \min,$$

$$\overline{STARR}_\beta(0, R_{\mathbf{x}} - R^\alpha) = \frac{-M_\beta(R_{\mathbf{x}} - R^\alpha)}{\mu(R_{\mathbf{x}} - R^\alpha)} \quad \rightarrow \quad \min,$$

and,

$$RR_{(\beta_1,\beta_2)}(0, R_{\mathbf{x}} - R^\alpha) = \frac{-M_{\beta_1}(R^\alpha - R_{\mathbf{x}})}{-M_{\beta_2}(R_{\mathbf{x}} - R^\alpha)} \quad \rightarrow \quad \max.$$

Appendix B: Dealing with a large number of scenarios

Note that the EOR model (22) has $n+T$ decision variables, and $T+2$ constraints. Our computing experiments indicate that state-of-the-art solvers can find an optimal solution to such a model in a short computing time when the number of scenarios is relatively small. Nevertheless, finding an optimal solution can become computationally challenging when the number of scenarios considered becomes (very) large. In these cases, one can take advantage of LP duality, and reformulate the LP model (22) in the following terms:

$$\begin{array}{ll} \max & q - h \\ \text{s.t.} & \displaystyle\sum_{t=1}^{T} (r_t^\alpha - r_{jt}) u_t + (\mu_j - \mu^\alpha) q - \varepsilon h \le \varepsilon \quad j = 1, \ldots, n \\ & h \ge 0, \;\; 0 \le u_t \le p_t \quad t = 1, \ldots, T. \end{array}$$

This model contains $2+T$ variables, i.e., variables q, h, and u_t, respectively. Note that the T constraints corresponding to variables $\tilde{d}_t$ from model (22) take the form of simple upper bounds on variables u_t, thus not affecting the problem complexity. Hence, the number of constraints in the above model is proportional to the number n of securities available for the investment. This guarantees a remarkable computational efficiency of the dual model even for a very large number of scenarios, i.e., $T >> n$.

In a similar way, the LP dual to the EWCVaR model (27) is the following:

$$\begin{array}{ll} \max & q - h \\ \text{s.t.} & \displaystyle\sum_{t=1}^{T} u_{tk} = w_k \quad k = 1, \ldots, m \\ & \displaystyle\sum_{t=1}^{T}\sum_{k=1}^{m} (r_{jt} - r_t^\alpha) u_{tk} + (\mu_j - \mu^\alpha) q - \varepsilon h \le (\mu_j - \mu^\alpha + \varepsilon) \quad j = 1, \ldots, n \\ & 0 \le u_{tk} \le \dfrac{p_t w_k}{\beta_k} \quad t = 1, \ldots, T; \; k = 1, \ldots, m \\ & h \ge 0. \end{array}$$

The latter model contains $2+mT$ variables, i.e., variables q, h, and u_{tk}, respectively, and the mT constraints corresponding to variables $\tilde{d}_{tk}$ from model (27) are simple upper bounds on u_{tk}. Thus, and similar to the dual above of the EOR model (22), the number of constraints in this optimization model is only proportional to the number n of securities available for the investment, and independent from the number of scenarios.

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