Mathematical Programming Models for General Weighted OWA Criteria

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Abstract

The problem of aggregating multiple numerical attributes to form overall measure is of considerable importance in many disciplines. In location problems, classical approaches are based on minimization of the average distance (the median concept) or minimization of the maximum distance (the center concept) to the service facilities. The median solution concept is primarily concerned with the spatial efficiency while the center concept is focused on the spatial equity. The ordered weighted averaging (OWA) aggregation uses the preference weights assigned to the ordered values rather than to the specific attributes. It is widely recognized for location problems as a general model (the ordered median) allowing to build various solution concepts taking into account distribution of distances. Importance weights are used in location problems to express the client demand for a service thus defining the location decision output as distances distributed according to measures defined by the demand weights. The standard ordered median concept does not take into account such demand weights. They can be treated with the Weighted OWA (WOWA) aggregation though the importance weights make the WOWA concept much more complicated than the original OWA. In this paper we analyze mathematical programming models for location problems with the WOWA objective functions (the weighted ordered median).

Key words. Location, Ordered Median, OWA, WOWA, Linear Programming, Mixed Integer Linear Programming, Equity, Fairness.

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1 Introduction

Location analysis is a field of operations research with a long tradition, which deals with distribution of spatial units to meet specific objectives and requirements [1, 2]. It is widely applied in many domains of engineering, for example to design various kinds of networks (distribution, telecommunications). The key element in the location problems are utilities that express an abstract measure of distance between the suppliers and clients of the considered services. If the individual clients are independent of each other, in addition to the global efficiency, the distribution of distances plays an important role [3]. Justice (equity of distribution) becomes an additional criterion for assessing the resulting solution. This approach is especially important in decisions concerning the location of public facilities, for example hospitals, crisis management centers, schools [4], where clients (citizens) have the right to a fair public access in accordance with regulations.

Demand weights are used in location problems to express the client demand for a service thus defining the location decision output as distances distributed according to measures defined by the demand weights. Note that the model of such distribution weights allows us for a clear interpretation of demand weights as the client repetitions at the same place. Splitting a client into two clients sharing the demand at the same geographical point does not cause any change of the final distribution of distances. Therefore, the distribution model of weights is important to accommodate various demand coefficients in location problems.

Numerous models for the discrete location problem were developed. Many of them differ only in the aggregation function. It is immediately apparent when we take into account effect of the siting facilities on individuals or groups [5] and consider the multicriteria model with objectives corresponding to these individual evaluations (impacts) [6]. The most commonly used aggregation is based on the weighted mean, called the median concept, where positive importance weights p_i $(i = 1, \dots, m)$ are allocated to several clients

$$A_{\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} y_i p_i. \tag{1}$$

The weights are typically normalized to the total 1 ($\sum_{i=1}^{m} p_i = 1$). When all weights are equal we obtain simple arithmetic average. The average objective is equivalent to the total sum, which aims to global efficiency and it might discriminate isolated and low populated sites. To overcome these difficulties, especially when the equity of distribution is important, another popular approach, called the center concept [7], is used. This objective is independent of the demand weights and corresponds to the worst outcome (the situation of the client in the worst position)

$$M(\mathbf{y}) = \max_{i=1,\dots,m} y_i. \tag{2}$$

However, the center criterion might lead to substantial increase in total distance, and thus deteriorate global efficiency. Additionally considering only maximal distance limits the possibility to differentiate various feasible solutions [8] and may optimize only subsets of all criteria in some cases [9].

In various location problems some compromise between spatial efficiency and spatial equity is desired. This issue occurs even in public sector, where decision-maker has to reconcile social equity with economic efficiency [10]. Cent-dian [11] is an example of compromise approach, which corresponds to convex combination of median and center criteria. Unfortunately, it has limited ability to generate compromise solutions [12]. Apart from the center criterion, a wide range of different equity (inequality) measures were also analyzed [5]. But simple minimization of inequality measure can cause undesirable side effects. Thus the bicriteria mean-equity model is usually used, which minimizes both the average and the inequality measure. However, not all inequality measures are well suited for such models [13]. Another approach that tries to bring together efficiency and equity is the lexicographic center approach [6], which extends the classic center criterion. However, as it provides the most equitable (fair) efficient solution, its modelling flexibility is limited.

During the last decade a new type of objective function in location theory, called ordered median (OM) function has been developed and analyzed. It originates from early models [14, 15] of compensatory extensions of the lexicographic center approach, thus representing weighted sum of the ordered outcomes (distances). The ordered median location models were formulated for locations on networks [16], on the plane [17] and for general discrete location problems [18]. Some special classes of the ordered solution concepts such as k-centrum and conditional median were independently developed for location problems [19, 20, 21]. The general OM methodology was developed [22] unifying various location models. Exact and approximate solution methods were studied [23, 24, 25]. The OM objective function unifies and generalizes most common objective functions used in location theory. In fact, the ordered median function corresponds to the Ordered Weighted Averaging (OWA) aggregation, developed by Yager [26], with the non-negative preference weights. The OWA operator is a weighted average with weights allocated to the ordered distances (i.e. to the largest distance, the second largest and so on) rather than to the distances of specific clients. When applying the OWA aggregation to optimization problems with attributes modeled by variables the weighting of the ordered outcome values causes that the OWA operator is nonlinear even for linear programming (LP) formulation of the original constraints and criteria. Yager [27] has shown that the nature of the nonlinearity introduced by the ordering operations allows one to convert the OWA optimization into a mixed integer programming problem. In [28] there was shown that the OWA optimization with monotonic weights can be formed as a standard linear program of higher dimension, thus leading to efficient solution techniques for many related problems [29]. We compared different MILP formulation of OWA for any non-negative preference weights for OMP and examined the possibility of improving the computational performance by introducing various valid inequalities [30]. MILP formulations and valid inequalities for the OWA aggregation were also studied for different combinatorial optimization problems [31].

The OWA operator allows to model various aggregation functions from the maximum through the arithmetic mean to the minimum [32]. Thus, it enables modeling of various preferences from the optimistic to the pessimistic one. On the other hand, the OWA does not allow to allocate any demand weights to specific clients. Actually, the weighted mean (1) cannot be expressed in terms of the OWA aggregations. Typical ordered median model allows weighting of several clients only by straightforward rescaling of the distance values. However, the ordered median approach might be extended by the incorporation of the demand weights by rescaling accordingly clients measure within the distribution of distances as defined in the so-called Weighted OWA (WOWA) aggregation [33, 34]. The WOWA aggregation uses two sets of weights: the preferential (OWA type) weights and the demand (distribution measure) weights. Since its introduction, the WOWA operator has been successfully applied to many fields of decision making [35, 36, 37, 38] including metadata aggregation problems [39, 40] and implicit application to the portfolio optimization [41]. The WOWA operator covers as special cases both the standard weighted mean (the weighted median solution concept defined with the demand weights) in the case of equal all the preference weights, as well as the OWA average (the ordered median solution concept defined with the preferential weights) in the case of equal all the demand weights. Therefore applying the WOWA aggregation we can obtain solutions optimal in terms of the distance distribution expressed by the demand weights.

This paper studies basic properties of the Weighted Ordered Median Problem (WOMP) taking into account the demand weights following the WOWA aggregation rules. The WOWA operator is a particular case of the Choquet integral defined with quite a complicated formula. Nevertheless, linear programming formulations were introduced for optimization of the WOWA objective with monotonic preferential weights thus representing the equitable preferences. We proposed general MILP models of the WOWA aggregation for any non-negative preference weights. We examined the computational performance of WOMP and consider the possibility of improving it by introducing various additional constraints. The paper is organized as follows. Next section describes the location problem as the multiobjective optimization problem with objectives corresponding individual clients evaluations of the location schemes. It discusses the way the demand weights are included in the problem and their interpretation. Section 3 presents the background of the ordered operators, specifically the WOWA aggregation. In Section 4 we analyze mathematical programming formulations for the WOWA aggregation and their possible reinforcements with valid inequalities. Section 5 describes the computational experiments and analyzes the obtained results. In Section 6 we conclude with main observations and propose some future research steps.

2 Problem Description

We consider discrete location problem [42], which can also be defined as network location problem, where facilities are allowed to be placed only on vertices (or subset of vertices) of the underlying network [43]. We assume no capacity limit of facilities. There is given a set of m sites (e.g. clients) and a set of potential facility locations. Without loss of generality it can be assumed that these two sets are identical. We have to place n facilities ($n \leq m$) to satisfy demands from the clients. Then each client is assigned to the facility that meets its demand. The assignment is done in such a way to optimize a given objective function. The objective function is usually based on distances (costs) between the clients and the facilities. Because we consider unlimited capacities each client is assigned the closest facility. Formally the model can be expressed in the following form:

$$\min \quad (y_1, y_2, \ldots, y_m) \tag{3a}$$

$$\text{s.t.} \quad y_i = \sum_{j=1}^{m} c_{ij} x'_{ij} \qquad \text{for} \quad i = 1, 2, \ldots, m, \tag{3b}$$

$$\sum_{j=1}^{m} x_j = n, \tag{3c}$$

$$\sum_{j=1}^{m} x'_{ij} = 1 \qquad \text{for} \quad i = 1, 2, \ldots, m, \tag{3d}$$

$$x'_{ij} \leq x_j \qquad \text{for} \quad i, j = 1, 2, \ldots, m, \tag{3e}$$

$$x_j \in \{0, 1\} \qquad \text{for} \quad i, j = 1, 2, \ldots, m, \tag{3f}$$

$$x'_{ij} \geq 0 \qquad \text{for} \quad i, j = 1, 2, \ldots, m, \tag{3g}$$

where c_{ij} denotes the cost of satisfying total demand of client i from facility j. The main decisions are described by binary variables: x_j ($j = 1, 2, \ldots, m$) is equal to 1 if a facility is built at site j and equal to 0 otherwise. There are also binary variables that represent allocation decisions: x'_{ij} ($i, j = 1, 2, \ldots, m$) is equal to 1 if the demand of client i is satisfied by facility j and 0 otherwise. Due to lack of capacity restriction each client will be assigned to the closest facility and therefore variables x'_{ij} can be relaxed to continuous variables. The auxiliary variable y_i (3b) expresses the cost of satisfying the demand of client i. Constraint (3c) enforces that exactly n facilities are placed. The requirement that full demand of each client is satisfied is modeled with constraint (3d). Constraint (3e) ensures that the clients are assigned to the existing facilities. Thus constraints (3c)–(3g) defines the set of feasible solutions $\mathcal{F}$, which according to constraint (3b) is mapped into the set of attainable outcome (cost) vectors $\mathbf{y}$.

Further, for each client $i = 1, 2, \ldots, m$ there is also given weight p_i, which determines the demand for service. Thus problem (3) defines distribution of outcomes $y_i = f_i(\mathbf{x})$ with measures p_i for $i = 1, 2, \ldots, m$. Integer weights could be interpreted as client multiplication within one location, what preserves the distribution of outcomes. This allows to disaggregate the problem to basic form, where demand weights for all clients are equal to $p_i = 1$. Similarly, one can proceed with any rational weights. Such transformation is possible, but in practice usually causes significant increase in size of the problem (number of clients) and thus made the problem impossible to solve. Our approach can directly take into account the demands weights, without the need for disaggregation.

It is worth mentioning that it is also possible to use weights to scale the distance, thus to define the outcomes as $y_i = p_i f_i(\mathbf{x})$ for $i = 1, 2, \ldots, m$ with a uniform distribution (with single client at each site). This approach is substantially different from the optimization of the outcomes distribution with the weights p_i. In practice, the distance scaling may be implemented within the individual objective functions f_i. It leads to an equivalent problem without explicit weights but with suitably transformed distances (multiplied by weights). Therefore, such application of the weights is not specifically addressed in the work, as it can be solved by the basic formulation of the location problem.

3 The Ordered Weighted Averages

Let $\mathbf{w} = (w_1, \ldots, w_m)$ be a weighting vector of dimension m such that $w_i \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^{m} w_i = 1$. The OWA aggregation of attributes $\mathbf{y} = (y_1, \ldots, y_m)$, as introduced by Yager [26], can be mathematically formalized as follows. First, we introduce the ordering map $\Theta : R^m \to R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \geq \theta_2(\mathbf{y}) \geq \cdots \geq \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, \ldots, m$. Further, we sum the weighted ordered values $\Theta(\mathbf{y})$, i.e. the OWA aggregation takes the following form:

$$A_{\mathbf{w}}(\mathbf{y}) = \sum_{i=1}^{m} w_i \theta_i(\mathbf{y}). \tag{4}$$

The OWA aggregation (4) allows one to model various aggregation functions from the maximum ($w_1 = 1$, $w_i = 0$ for $i = 2, \ldots, m$) through the arithmetic mean ($w_i = 1/m$ for $i = 1, \ldots, m$) to the minimum ($w_m = 1$, $w_i = 0$ for $i = 1, \ldots, m-1$). In the case of decreasing weights $w_1 \geq w_2 \geq \ldots \geq w_m$, the OWA aggregation is a convex function thus, when minimized it models the so-called fairness [44] or equitable preferences [13]. The latter are important for many locations problems related to public facilities and thus requiring modeling the equity preferences. On the other hand, the weighted mean (1) aggregation, the standard criterion of the median location problems, cannot be expressed as an OWA aggregation. Actually, the OWA aggregations are symmetric with respect to the individual attributes and they do not allow to represent any importance weights allocated to specific attributes, except of possible scaling of outcome values (distances) [22].

Importance weighted averaging is a central task in decision problems of many kinds and the ordered averaging model enables one to introduce importance weights to affect criteria importance by rescaling accordingly its measure within the distribution of achievements as defined in the so-called Weighted OWA (WOWA) aggregation [33]. Let $\mathbf{w} = (w_1, \ldots, w_m)$ be OWA weights and let $\mathbf{p} = (p_1, \ldots, p_m)$ be an additional importance weighting vector such that $p_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^m p_i = 1$. The corresponding WOWA aggregation of outcomes $\mathbf{y} = (y_1, \ldots, y_m)$ is defined as follows [33]:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} \omega_i \theta_i(\mathbf{y}), \tag{5}$$

where the weights ω_i are defined as

$$\omega_i = w^*(\sum_{k\leq i} p_{\tau(k)}) - w^*(\sum_{k<i} p_{\tau(k)}) \tag{6}$$

with w^* a monotone increasing function that interpolates points $(\frac{i}{m}, \sum_{k\leq i} w_k)$ together with the point (0.0) and τ representing the ordering permutation for $\mathbf{y}$ (i.e. $y_{\tau(i)} = \theta_i(\mathbf{y})$). Function w^* is required to be a straight line whenever the points can be interpolated in this way. Due to this requirement, the WOWA aggregation covers the standard weighted mean (1) with weights p_i as a special case of equal preference weights ($w_i = 1/m$ for $i = 1, \ldots, m$). Actually, one may use only the simplest case of linear interpolation thus leading to the piecewise linear function $w^*(\alpha) = \int_0^\alpha g(\xi)\ d\xi$ with the stepwise generating function

$$g(\xi) = m w_k \quad \text{for } (k-1)/m < \xi \leq k/m, \quad k = 1, \ldots, m. \tag{7}$$

When introducing breakpoints $\beta_i = \sum_{k\leq i} p_{\tau(k)}$ and $\beta_0 = 0$, weights ω_i can be expressed as $\omega_i = \int_0^{\beta_i} g(\xi)\ d\xi - \int_0^{\beta_{i-1}} g(\xi)\ d\xi = \int_{\beta_{i-1}}^{\beta_i} g(\xi)\ d\xi$ and the entire WOWA aggregation as

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} \theta_i(\mathbf{y}) \int_{\beta_{i-1}}^{\beta_i} g(\xi)\ d\xi = \int_0^1 g(\xi) F_{\mathbf{y}}^{(-1)}(\xi)\ d\xi, \tag{8}$$

where $F_{\mathbf{y}}^{(-1)}$ is the stepwise function $F_{\mathbf{y}}^{(-1)}(\xi) = \theta_i(\mathbf{y})$ for $\beta_{i-1} < \xi \leq \beta_i$. Function $F_{\mathbf{y}}^{(-1)}$ can also be mathematically recognized as the left-continuous inverse of the left-continuous right tail cumulative distribution function (cdf):

$$F_{\mathbf{y}}(d) = \sum_{i\in I} p_i \delta_i(d) \quad \text{where} \quad \delta_i(d) = \begin{cases} 1 & \text{if } y_i \geq d \\ 0 & \text{otherwise} \end{cases}, \tag{9}$$

which for any real (outcome) value d provides the measure of outcomes greater or equal to d. That means, $F_{\mathbf{y}}^{(-1)}$ is the quantile function defined as

$$F_{\mathbf{y}}^{(-1)}(\xi) = \sup\ \{\eta : F_{\mathbf{y}}(\eta) \geq \xi\} \quad \text{for } 0 < \xi \leq 1. \tag{10}$$

Using the stepwise generation function (7) within the general WOWA formula (8) leads us to the following expression of the WOWA aggregation:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \int_0^1 g(\xi) F_{\mathbf{y}}^{(-1)}(\xi)\ d\xi = \sum_{k=1}^{m} w_k m \int_{(k-1)/m}^{k/m} F_{\mathbf{y}}^{(-1)}(\xi)\ d\xi. \tag{11}$$

We will treat formula (11) as a formal definition of the WOWA aggregation of m-dimensional outcomes $\mathbf{y}$ defined by m-dimensional importance weights $\mathbf{p}$ and preferential weights $\mathbf{w}$. Note that quantities $m\int_{(k-1)/m}^{k/m} F_{\mathbf{y}}^{(-1)}(\xi)\, d\xi$ express the conditional means within the corresponding quantiles $(k-1)/m$ and k/m. In the case of equal importance weights $p_i = 1/m$, formula (11) represents the standard definition of the OWA aggregation (4), since $F_{\mathbf{y}}^{(-1)}(\xi) = \theta_k(\mathbf{y})$ for $(k-1)/m \le \xi < k/m$. Although formula (11) allows one to express general WOWA aggregations by using the preferential weights to redefine $F_{\mathbf{y}}^{(-1)}(\xi) = \theta_k(\mathbf{y})$ accordingly.

Example 1. To illustrate the concept of the WOWA aggregation let us consider a location problem with 5 sites ($m = 5$) and the normalized demand weights $\mathbf{p} = (0.1, 0.2, 0.2, 0.4, 0.1)$. Thus the demand needs of the second and third clients are twice the demand of the first client, and the fourth client has four times bigger demand than the first one (the demand needs of the fifth and first clients are equal). Furthermore, assume the preference weights $\mathbf{w} = (0.4, 0.3, 0.15, 0.1, 0.05)$.

Let us consider a feasible solution with the cost (distance) vector $\mathbf{y} = (1, 3, 2, 4, 5)$. First, we focus on the classic definition of the WOWA aggregation given by formula (5). On the basis of vector $\mathbf{w}$ we can determine function w^*, which according to definition interpolates points $(0, 0)$, $(1/5, 0.4)$, $(2/5, 0.7)$, $(3/5, 0.85)$, $(4/5, 0.95)$, $(1, 1)$. Assuming the simplest piecewise linear function, it can be expressed as

$$w^*(\pi) = \begin{cases} \frac{0{,}4}{0{,}2}\pi & \text{for} \quad 0 \le \pi \le 0{,}2, \\ 0{,}4 + \frac{0{,}3}{0{,}2}(\pi - 0{,}2) & \text{for} \quad 0{,}2 < \pi \le 0{,}4, \\ 0{,}7 + \frac{0{,}15}{0{,}2}(\pi - 0{,}4) & \text{for} \quad 0{,}4 < \pi \le 0{,}6, \\ 0{,}85 + \frac{0{,}1}{0{,}2}(\pi - 0{,}6) & \text{for} \quad 0{,}6 < \pi \le 0{,}8, \\ 0{,}95 + \frac{0{,}05}{0{,}2}(\pi - 0{,}8) & \text{for} \quad 0{,}8 < \pi \le 1. \end{cases}$$

Now, we can calculate weights ω_i ($i = 1, \ldots, m$) according to formula (6). Namely:

$$\begin{aligned} &\omega_1 = w^*(p_5) = w^*(0{,}1) = 0{,}2, \\ &\omega_2 = w^*(p_5 + p_4) - w^*(p_5) = 0{,}775 - 0{,}2 = 0{,}575, \\ &\omega_3 = w^*(p_5 + p_4 + p_2) - w^*(p_5 + p_4) = 0{,}9 - 0{,}775 = 0{,}125, \\ &\omega_4 = w^*(p_5 + p_4 + p_2 + p_3) - w^*(p_5 + p_4 + p_2) = 0{,}975 - 0{,}9 = 0{,}075, \\ &\omega_5 = w^*(p_5 + p_4 + p_2 + p_3 + p_1) - w^*(p_5 + p_4 + p_2 + p_3) = 1 - 0{,}975 = 0{,}025. \end{aligned}$$

The concept of calculating weights ω_i is also presented in Figure 1. Finally, according to formula (5) the WOWA operator value is $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = 0.2 \cdot 5 + 0.575 \cdot 4 + 0.125 \cdot 3 + 0.075 \cdot 2 + 0.025 \cdot 1 = 3.85$.

Next, we exemplify the alternative procedure for calculating the WOWA aggregation value according to formula (11). For the given cost vector $\mathbf{y}$ and the demand weights $\mathbf{p}$ we can determine the cumulative distribution function (9) and the corresponding quantile function (10):

$$F_{\mathbf{y}}(d) = \begin{cases} 1 & \text{dla} \quad d \le 1, \\ 0{,}9 & \text{for} \quad 1 < d \le 2, \\ 0{,}7 & \text{for} \quad 2 < d \le 3, \\ 0{,}5 & \text{for} \quad 3 < d \le 4, \\ 0{,}1 & \text{for} \quad 4 < d \le 5, \\ 0 & \text{for} \quad d > 5, \end{cases} \qquad F_{\mathbf{y}}^{(-1)}(\xi) = \begin{cases} 5 & \text{for} \quad 0 < \xi \le 0{,}1, \\ 4 & \text{for} \quad 0{,}1 < \xi \le 0{,}5, \\ 3 & \text{for} \quad 0{,}5 < \xi \le 0{,}7, \\ 2 & \text{for} \quad 0{,}7 < \xi \le 0{,}9, \\ 1 & \text{for} \quad 0{,}9 < \xi \le 1. \end{cases}$$

Then, based on the quantile function, we can calculate the averages of the ordered cost vector for the consecutive equal demand portions of $1/m$. The averages correspond to the integrals

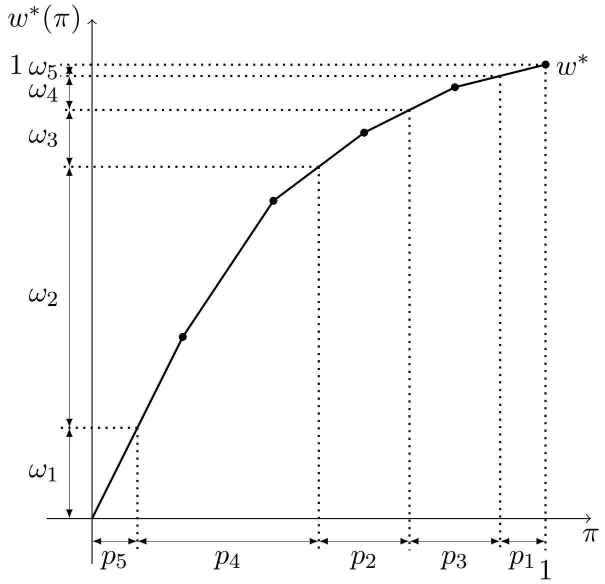

Figure 1: Concept of WOWA weights ω_i for Example 1

$\int_{(k-1)/5}^{k/5} F_{\mathbf{y}}^{(-1)}(\xi)\ d\xi$ for $k = 1, \ldots, 5$:

$$\int_0^{1/5} F_{\mathbf{y}}^{(-1)}(\xi)\ d\xi = 0.1 \cdot 5 + 0.1 \cdot 4 = 0.9,$$
$$\int_{1/5}^{2/5} F_{\mathbf{y}}^{(-1)}(\xi)\ d\xi = 0.2 \cdot 4 = 0.8,$$
$$\int_{2/5}^{3/5} F_{\mathbf{y}}^{(-1)}(\xi)\ d\xi = 0.1 \cdot 4 + 0.1 \cdot 3 = 0.7,$$
$$\int_{3/5}^{4/5} F_{\mathbf{y}}^{(-1)}(\xi)\ d\xi = 0.1 \cdot 3 + 0.1 \cdot 2 = 0.5,$$
$$\int_{4/5}^{1} F_{\mathbf{y}}^{(-1)}(\xi)\ d\xi = 0.1 \cdot 2 + 0.1 \cdot 1 = 0.3.$$

Figure 2 presents the quantile function $F_{\mathbf{y}}^{(-1)}(\xi)$, based on which the integrals were calculated. Finally, according to formula (11), the WOWA value is $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = 5 \cdot (0.4 \cdot 0.9 + 0.3 \cdot 0.8 + 0.15 \cdot 0.7 + 0.1 \cdot 0.5 + 0.05 \cdot 0.3) = 3.85$.

One can also consider slightly different way to compute the WOWA aggregation value according to formula (11), which boils down to different calculation of the integrals $\int_{(k-1)/5}^{k/5} F_{\mathbf{y}}^{(-1)}(\xi)\ d\xi$. It requires a transformation of the cost vector, after which each cost corresponds to the equal demand portion, which is a divisor of $1/m$. It is possible, when there is a common divisor for all demand weights p_i $(i = 1, \ldots, m)$ and value $1/m$. A sufficient condition is that all demand weights and value $1/m$ are rational — such assumption seems reasonable in case of the location problems. The transformation leads to replication of cost components of $\mathbf{y}$ according to the demand weights $\mathbf{p}$ (precisely, each component y_i is repeated the number of times the determined portion is lower than p_i for $i = 1, \ldots, m$). The cost vector from the example problem can be transformed into new vector, where individual cost corresponds to 0.1 portion of demand. Consequently, we get a new cost vector $\check{\mathbf{y}} = (1, 3, 3, 2, 2, 4, 4, 4, 4, 5)$ with 10 components (the second and third

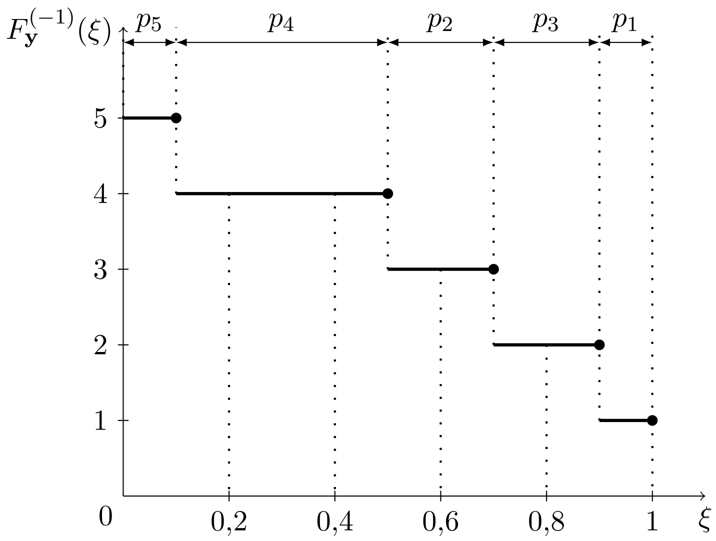

Figure 2: Quantile function $F_{\mathbf{y}}^{(-1)}(\xi)$ for Example 1

component was doubled and the fourth component was repeated four times). The preference weights $\mathbf{w}$ can be now applied to the new cost vector $\check{\mathbf{y}}$ such that the first weight is multiplied by the average of the two largest costs, the second weight by the average of the next two largest costs, and so on. Finally, we get $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = 0.4 \cdot 4.5 + 0.3 \cdot 4 + 0.15 \cdot 3.5 + 0.1 \cdot 2.5 + 0.05 \cdot 1.5 = 3.85$.

When in (11) the left-tail integrals are used rather than those on intervals one gets

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{m} m w_k (L(\mathbf{y}, \mathbf{p}, \frac{k}{m}) - L(\mathbf{y}, \mathbf{p}, \frac{k-1}{m})), \tag{12}$$

where $L(\mathbf{y}, \mathbf{p}, \alpha)$ is defined by left-tail integrating of quantile function $F_{\mathbf{y}}^{(-1)}$, i.e.

$$L(\mathbf{y}, \mathbf{p}, 0) = 0 \quad \text{and} \quad L(\mathbf{y}, \mathbf{p}, \alpha) = \int_0^{\alpha} F_{\mathbf{y}}^{(-1)}(\xi) d\xi \quad \text{for } 0 < \alpha \leq 1. \tag{13}$$

In particular, $L(\mathbf{y}, \mathbf{p}, 1) = \int_0^1 F_{\mathbf{y}}^{(-1)}(\xi) d\xi = A_{\mathbf{p}}(\mathbf{y})$. Graphs of functions $L(\mathbf{y}, \mathbf{p}, \alpha)$ (with respect to α) are concave curves, the so-called (upper) absolute Lorenz curves [13]. In the case of equal importance weights $p_i = 1/m$ thus representing the standard OWA aggregation, one gets $L(\mathbf{y}, \mathbf{p}, \frac{k}{m}) = \frac{1}{m} \sum_{i=1}^{k} \theta_i(\mathbf{y})$ and formula (12) reduces to (4). Similar to the OWA aggregation, in the case of decreasing OWA weights $w_1 \geq w_2 \geq \ldots \geq w_m$, the WOWA aggregation is a convex function thus modeling equitable preferences when minimized.

4 MILP Models for WOWA Optimization

4.1 Basic models

Following formula (12), the WOWA aggregation may be expressed as

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{m} m w_k (L(\mathbf{y}, \mathbf{p}, \frac{k}{m}) - L(\mathbf{y}, \mathbf{p}, \frac{k-1}{m})) = \sum_{k=1}^{m} w'_k L(\mathbf{y}, \mathbf{p}, \frac{k}{m}), \tag{14}$$

where $w'_m = mw_m$, $w'_k = m(w_k - w_{k+1})$. Due to formula (13), values of function $L(\mathbf{y}, \mathbf{p}, \alpha)$ for any $0 \le \alpha \le 1$ can be found by optimization:

$$L(\mathbf{y}, \mathbf{p}, \alpha) = \max_{u_i} \{\sum_{i=1}^{m} y_i u_i : \sum_{i=1}^{m} u_i = \alpha, \quad 0 \le u_i \le p_i \quad \forall\ i\ \}. \tag{15}$$

The above problem is an LP for a given outcome vector $\mathbf{y}$ while it becomes nonlinear for $\mathbf{y}$ being a vector of variables. This difficulty can be overcome by taking advantages of the LP dual to (15). Introducing dual variable t corresponding to the equation $\sum_{i=1}^{m} u_i = \alpha$ and variables d_i corresponding to upper bounds on u_i one gets the following LP dual of problem (15):

$$L(\mathbf{y}, \mathbf{p}, \alpha) = \min_{t, d_i} \{\alpha t + \sum_{i=1}^{m} p_i d_i : t + d_i \ge y_i,\ d_i \ge 0 \quad \forall\ i\} \tag{16a}$$

$$= \min_{t} \{\alpha t + \sum_{i=1}^{m} p_i \max\{y_i - t, 0\}\ \}, \tag{16b}$$

where the optimal value $\bar{t}$ is the α-quantile of distribution of values y_i with respect to the measures p_i. Equation (16a) enables the following statement.

Proposition 1. *For any vector $\mathbf{y}$, value ϱ fulfills inequality $L(\mathbf{y}, \mathbf{p}, \alpha) \le \varrho$ if and only if there exist t and d_i $(i = 1, \ldots, m)$ such that*

$$\alpha t + \sum_{i=1}^{m} p_i d_i \le \varrho \quad \text{and} \quad t + d_i \ge y_i,\ d_i \ge 0 \quad \forall\ i.$$

Consider minimization of the WOWA aggregation

$$\min\{A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) :\ \mathbf{y} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} \in \mathcal{F}\}. \tag{17}$$

Note that in the case of equitable WOWA aggregation specified by decreasing weights $w_1 \ge w_2 \ge \ldots \ge w_m$, following (14) the WOWA aggregation takes the form

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{m} w'_k L(\mathbf{y}, \mathbf{p}, \frac{k}{m})$$

with positive weights w'_k. Therefore, the following assertion can be proven.

Proposition 2. *In the case of WOWA aggregation defined by decreasing weights $w_1 \ge w_2 \ge \ldots \ge w_m$, optimization problem (17) may be expressed as the following problem with auxiliary linear inequalities:*

$$\begin{aligned}
\min_{\varrho_k, t_k, d_{ik}, y_i} \quad & \sum_{k=1}^{m} w'_k \varrho_k \\
\text{s.t.} \quad & \frac{k}{m} t_k + \sum_{i=1}^{m} p_i d_{ik} \le \varrho_k && \text{for} \quad k = 1, \ldots, m, \\
& t_k + d_{ik} \ge y_i,\ d_{ik} \ge 0 && \text{for} \quad i, k = 1, \ldots, m, \\
& \mathbf{y} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} \in \mathcal{F}.
\end{aligned}$$

Model from Proposition 2 is further depicted as MWLP.

In general case of WOWA with non-monotonic weights w_i, one may get negative coefficient w' in formula (14). Therefore, one cannot rely on minimization of only upper bounds ϱ_k as in Proposition 2. For negative coefficients one needs to use lower bounds on the corresponding Lorenz terms.

Following (16b) and taking into account that optimal value $\bar{t}$ is the corresponding quantile thus one of the values y_i we get that $L(\mathbf{y}, \mathbf{p}, \alpha) \geq \varrho$ if and only if

$$\varrho \leq \alpha y_{i'} + \sum_{i=1}^{m} p_i \max\{y_i - y_{i'}, 0\} \qquad \text{for } i' = 1, \ldots, m.$$

Proposition 3. *For any vector* $\mathbf{y}$*, value* ϱ *fulfills inequality* $L(\mathbf{y}, \mathbf{p}, \alpha) \geq \varrho$ *if and only if there exist* $z_{ii'}$ *and* $\bar{d}_{ii'}$ $(i', i = 1, \ldots, m)$ *such that*

$$\begin{aligned}
&\varrho \leq \alpha y_{i'} + \sum_{\substack{i=1 \\ i \neq i'}}^{m} p_i \bar{d}_{ii'} && \textit{for} \quad i' = 1, \ldots, m, \\
&\bar{d}_{ii'} \leq y_i - y_{i'} + M z_{ii'} && \textit{for} \quad i' \neq i = 1, \ldots, m, \\
&\bar{d}_{ii'} \leq M(1 - z_{ii'}) && \textit{for} \quad i' \neq i = 1, \ldots, m, \\
&z_{ii'} \in \{0, 1\} && \textit{for} \quad i' \neq i = 1, \ldots, m.
\end{aligned}$$

This allows us to form a MILP model for general WOWA optimization.

Proposition 4. *In the case of any WOWA aggregation, optimization problem (17) may be expressed as the following problem with auxiliary linear inequalities and binary variables:*

$$\min_{\varrho_k, t_k, d_{ik}, y_i, \bar{d}_{ii'}, z_{ii'}} \quad \sum_{k=1}^{m} w'_k \varrho_k \tag{18a}$$

$$\textit{s.t.} \quad \frac{k}{m} t_k + \sum_{i=1}^{m} p_i d_{ik} \leq \varrho_k \qquad \textit{for} \quad k = 1, \ldots, m, \tag{18b}$$

$$t_k + d_{ik} \geq y_i, \ d_{ik} \geq 0 \qquad \textit{for} \quad i, k = 1, \ldots, m, \tag{18c}$$

$$\varrho_k \leq \frac{k}{m} y_{i'} + \sum_{\substack{i=1 \\ i \neq i'}}^{m} p_i \bar{d}_{ii'} \qquad \textit{for} \quad i', k = 1, \ldots, m, \tag{18d}$$

$$\bar{d}_{ii'} \leq y_i - y_{i'} + M z_{ii'} \qquad \textit{for} \quad i' \neq i = 1, \ldots, m, \tag{18e}$$

$$\bar{d}_{ii'} \leq M(1 - z_{ii'}) \qquad \textit{for} \quad i' \neq i = 1, \ldots, m, \tag{18f}$$

$$z_{ii'} \in \{0, 1\} \qquad \textit{for} \quad i' \neq i = 1, \ldots, m, \tag{18g}$$

$$\mathbf{y} = \mathbf{f}(\mathbf{x}), \ \mathbf{x} \in \mathcal{F}. \tag{18h}$$

All constraints (18b)–(18h) together represent a valid MILP model for general WOWA optimization. However, there is no need to use both upper and lower bound constraints for all k. From the model one may eliminate the corresponding upper constraints (18b)–(18c) in case of $w'_k < 0$ and the corresponding lower constraint (18d) for $w'_k \geq 0$. Constraints (18e)–(18g) may be skipped only in the case of all $w'_k \geq 0$ (equitable WOWA). This model is further depicted as MW1.

Binary variables $z_{ii'}$ with constraints (18e)–(18g) represent pairwise comparisons of values y_i and $y_{i'}$. Exactly, $z_{ii'} = 1$ when $y_i < y_{i'}$ and $z_{ii'} = 0$ otherwise. Number of binary variables and constraints may be reduced by taking advantages of the symmetry for variables $\bar{d}_{ii'}$ and $\bar{d}_{i'i}$. Indeed, constraints (18e)–(18g) can be replaced with the following

$$\bar{d}_{ii'} \leq y_i - y_{i'} + Mz_{ii'} \quad \text{for} \quad i', i = 1, \dots, m; i < i', \tag{19a}$$
$$\bar{d}_{ii'} \leq M(1 - z_{ii'}) \quad \text{for} \quad i', i = 1, \dots, m; i < i', \tag{19b}$$
$$\bar{d}_{ii'} \leq y_i - y_{i'} + \bar{d}_{i'i} \quad \text{for} \quad i', i = 1, \dots, m; i > i', \tag{19c}$$
$$z_{ii'} \in \{0, 1\} \quad \text{for} \quad i', i = 1, \dots, m; i < i'. \tag{19d}$$

Nevertheless, the model still remains very complex with numerous variables and constraints. This model is further depicted as MW2.

4.2 Valid inequalities

In this section we propose some valid inequalities for the WOWA optimization models.

First, for MILP models we consider simple additional constraint on variables $\bar{d}_{ii'}$, which should be non-negative, that is,

$$\bar{d}_{ii'} \geq 0 \quad \text{for} \quad i, i' = 1, 2, \dots, m. \tag{20}$$

Next, we may notice that the linear constraints on $\bar{d}_{ii'}$ variables may be additionally strengthen by adding several transitivity relations on pairwise comparisons. The transitivity relations means that when $y_i < y_{i'}$ and $y_{i'} < y_{i''}$ then $y_i < y_{i''}$. Such dependency is equivalent to the following constraint

$$z_{ii''} \geq z_{ii'} + z_{i'i''} - 1 \quad \text{for} \quad i, i', i'' = 1, 2, \dots, m; i < i' < i''. \tag{21}$$

Constraint (21) can be regarded as a lower bound for binary variables arising from the transitivity relations. Similarly, one can add upper bound, which corresponds to the following relationship: if $y_i \geq y_{i'}$ and $y_{i'} \geq y_{i''}$ then $y_i \geq y_{i''}$. The equivalent constraint can be stated as

$$z_{ii''} \leq z_{ii'} + z_{i'i''} \quad \text{for} \quad i, i', i'' = 1, 2, \dots, m; i < i' < i''. \tag{22}$$

It should be emphasized, however, that the transitivity relation generates huge number of constraints, which on the other hand may have an adverse effect on the computational performance of MILP models.

We also consider restrictions on the value of the function $L(\mathbf{y}, \mathbf{p}, \alpha)$. Let us look at the maximum increment of $L(\mathbf{y}, \mathbf{p}, \alpha)$. For a given k the $L(\mathbf{y}, \mathbf{p}, k/m)$ represents a weighted average of k/m largest cost $\mathbf{y}$ with weights distribution given by $\mathbf{p}$. Keeping in mind the non-increasing order of the outcomes, the difference between the weighted averages of the $(k+1)/m$ and k/m greatest cost may be, at most, such as between the weighted averages of the k/m and $(k-1)/m$ largest costs for each $k = 2, 3, \dots, m-1$. Taking into account the limiting case for $k = 1$, these restrictions can be expressed as follows:

$$\begin{aligned} &\rho_{k+1} \leq 2\rho_k - \rho_{k-1} \qquad \text{for} \quad k = 2, \dots, m-1, \\ &\rho_2 \leq 2\rho_1. \end{aligned} \tag{23}$$

We may also impose the lower bound on the function $L(\mathbf{y}, \mathbf{p}, \alpha)$. Due to the non-increasing order of the outcomes, for a given k the value of $L(\mathbf{y}, \mathbf{p}, k/m)$ is at least k/m-th part of the whole, that is, the value of the function $L(\mathbf{y}, \mathbf{p}, 1)$, which is the weighted average of all outcomes for $k = 1, 2, \dots, m$. Formally, this restriction can be stated as follows

$$\rho_k \geq \frac{k}{m} \sum_{i=1}^{m} p_i y_i \quad \text{for} \quad k = 1, 2, \dots, m. \tag{24}$$

Constraints (23) and (24) can also be applied for model MWLP.

To clarify, the last four constraints are further depicted as:

- c_1 — constraint (21),
- c_2 — constraint (22),
- c_3 — constraint (23),
- c_4 — constraint (24).

5 Computational Tests

5.1 Experimental procedure

The experimental procedure has been analogous to that presented in [24]. In order to check the computational performance of the presented models and their different formulations, we have applied them to various instances of the location problem. To generate various instances we have considered some parameters characterizing the location problem and have determined their sets of possible values. Then, based on combinations of these parameters various instances of the location problems have been defined. We have considered the following parameters: the number of sites (locations) m, the number of facilities to be placed n and the type of problem defined by the vector of preference weights $\mathbf{w}$ in the WOWA aggregation. Besides these, we have also generated additional parameter $\mathbf{p}$ corresponding to the demand requirements.

The number of sites is very important parameter because, in fact, it determines the size of the problem. Seven cases of sites number are considered: $m \in \{8, 10, 12, 15, 20, 25, 30\}$. Due to computational complexity MILP formulations are tested on smaller sizes, and LP formulation on all sizes.

The second parameter, the number of facilities, is defined as proportional to the problem size (m value). Following cases are examined: $n = \lceil \frac{m}{4} \rceil$, $n = \lceil \frac{m}{3} \rceil$, $n = \lceil \frac{m}{2} \rceil$, $n = \lceil \frac{m}{2} + 1 \rceil$, where $\lceil a \rceil$ is the smallest integer value not smaller than a.

Type of problem corresponds to objective function, which is defined by the preference weighting vector $\mathbf{w}$. This vector determines the structure and thus the complexity of the problem. We consider 12 problem types (the same as in [30]), which are described in Table 1 with respect to the number of criteria m and the number of facilities n. By $\lceil a \rceil$ we denote the smallest integer not less than a, and by $\lfloor a \rfloor$ the largest integer not greater than a. The first eight problems appear in literature [25]. The n-median and n-center are the most popular objective functions in multicriteria optimization. The k-centra and $k_1 + k_2$-trimmed mean are less popular but also known in the field. We can identify problems T5–T8 as artificial and we use them particularly to test the computational efficiency for choppy weighting vector. As the last four types T9–T12 we consider problems with monotonic weights. Depending on the type of monotonicity they are simpler (T9, T11 with decreasing weights) or harder (T10, T12 with increasing weights) problems. These problem types can be treated as extended versions of $\min \max$ (T9, T11) and $\min \min$ (T10, T12) objective functions, respectively.

The demand weights $\mathbf{p}$ have been generated according to Zipf distribution, which is primarily associated with the distribution of the words frequency in text corpora [45]. But according to [46], in his work [47] Zipf also referred to the population size of cities, and the relationship in this area had also been noticed by Auerbach [48]. Thus, it seems justified to use this distribution for the location problems. Zipf distribution is also present in other domains, for example in the distribution of company sizes [49], for which the location model may also be applied. Although

Table 1: Problem types defined by the weighting vector $\mathbf{w}$ with respect to the number of criteria m and the number of facilities n.

type	name/description	weighting vector $\mathbf{w}$	parameters
T1	n-median	$(\underbrace{1,\ldots,1}_{m})$	
T2	n-center	$(1,\underbrace{0,\ldots,0}_{m-1})$	
T3	k-centra	$(\underbrace{1,\ldots,1}_{k},0,\ldots,0)$	$k=\lfloor\frac{m}{3}\rfloor$
T4	k_1+k_2-trimmed mean	$(\underbrace{0,\ldots,0}_{k_1},1,\ldots,1,\underbrace{0,\ldots,0}_{k_2})$	$k_1=\lceil\frac{m}{10}\rceil$, $k_2=\lceil n+\frac{m}{10}\rceil$
T5	Alternating 0's and 1's, beginning with 1.	$(1,0,1,0,1,0,\ldots)$	
T6	Alternating 0's and 1's, beginning with 0.	$(0,1,0,1,0,1,\ldots)$	
T7	Repeating the sequence $(1,1,0)$.	$(1,1,0,1,1,0,\ldots)$	
T8	Repeating the sequence $(1,0,0)$.	$(1,0,0,1,0,0,\ldots)$	
T9	Beginning with m and decreasing by 1.	$(m,m-1,\ldots,2,1)$	
T10	Such as T9, but in reverse order (increasing).	$(1,2,\ldots,m-1,m)$	
T11	Beginning with $3m$ and decreasing in a piecewise linear manner, k weights by 3, next k weights by 2 and rest by 1.	$(3m,\underbrace{3(m-1),\ldots,3(m-k)}_{k},$ $\underbrace{3(m-k)-2,\ldots,3(m-k)-2k}_{k},$ $3m-5k-1,3m-5k-2,\ldots)$	$k=\lfloor\frac{m}{3}\rfloor$
T12	Such as T11, but in reverse order (increasing).	$(\ldots,3m-5k-2,3m-5k-1,$ $\underbrace{3(m-k)-2k,\ldots,3(m-k)-2}_{k},$ $\underbrace{3(m-k),\ldots,3(m-1)}_{k},3m)$	$k=\lfloor\frac{m}{3}\rfloor$

recent works [46], which are based on more accurate data (for the smaller towns), show that the distribution of city sizes corresponds rather to log-normal distribution, the authors of these works admit at the same time that for some of the largest cities it practically coincides with the Zipf distribution. Such approximation seems to be sufficient for the purposes of our study.

According to Zipf distribution the size of any object (phenomenon) is inversely proportional to its rank, when ordering the objects from the biggest to the smallest ones. Formally, it means $p_i \sim 1/i^b$, where p_i is the size of an object in the i-th ranking position. The exponent b is very close to 1 and for the sake of simplicity it is usually assumed that $b = 1$ (this assumption is also adopted in this paper). We also presume that location indexes correspond to the positions in ranking, thus the locations are ordered by decreasing demand size. To summarize, the normalized demand weights can be expressed as

$$p_i = \frac{1}{i \sum_{k=1}^{m} \frac{1}{k}} \quad \text{for } i = 1, \dots, m.$$

For each size case we have generated 15 cost matrices with zeros on the main diagonal and the remaining entries randomly generated from a discrete uniform distribution in the interval $[1, 100]$. These matrices have been assigned to each combination of the parameters with corresponding problem size. Thus we have received a set of test problem instances.

Additionally, we have also checked the performance of model MWLP on problems with 100 and 200 locations from OR-library[1] [50].

The experimental procedure has been implemented in C++ and IBM ILOG CPLEX Optimization Studio (including the solver CPLEX) version 12.4 [51] has been used to solve optimization problems. Computational experiments have been performed on a machine with the Intel Core2 Duo 2.53 GHz (mobile) and 3 GB of RAM.

5.2 Results

A time limit of 600 seconds has been imposed as the maximum solution time for a single instance of the location problem. Obtained results are presented below. Upper index in front of the time is the number of instances of the 15 that exceeded the time limit. In cases where all 15 instances exceeded the time limit, minus sign is placed. Minus sign is also used for instances from OR-library that exceeded the time limit.

5.2.1 Non-increasing preference weights

When the preference weights w_k ($k = 1, \dots, m$) are non-increasing, then all weights w'_k ($k = 1, \dots, m$) are non-negative, and thus both models MW1 and MW2 reduce to the linear model MWLP.

We have carried out computational tests to check the performance of the linear model as well as the influence of valid inequalities c_3 and c_4. Detailed results for $m = 25$ and $m = 30$ are presented in Table 2. Figure 3 shows results for the 30 locations averaged over variants of facilities number (besides averaging over cost matrix instances).

Table 2: Average solution times [s] for linear formulation MWLP (with and without valid inequalities)

Problem			CPU[s]			
type	m	n	MWLP	c_3	c_4	c_3c_4
T1	25	7	0.02	0.05	0.03	0.14

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[1] http://people.brunel.ac.uk/ mastjjb/jeb/info.html

Problem				CPU[s]		
type	m	n	MWLP	c_3	c_4	c_3c_4
		9	0.02	0.04	0.02	0.13
		13	0.01	0.02	0.01	0.08
		14	0.02	0.02	0.01	0.08
	30	8	0.04	0.12	0.05	0.26
		10	0.04	0.08	0.04	0.23
		15	0.02	0.03	0.02	0.13
		16	0.02	0.03	0.02	0.13
T2	25	7	0.83	1.54	0.60	1.67
		9	0.80	1.31	0.48	1.39
		13	0.68	1.04	0.29	0.98
		14	0.56	0.84	0.23	0.89
	30	8	1.81	4.00	1.60	4.55
		10	1.78	3.73	1.23	3.75
		15	1.29	1.91	0.57	2.11
		16	1.05	1.84	0.45	1.76
T3	25	7	0.51	2.88	0.39	2.86
		9	0.51	2.87	0.34	2.80
		13	0.21	1.17	0.16	1.13
		14	0.12	0.75	0.11	0.72
	30	8	0.81	5.82	0.69	5.67
		10	0.81	5.32	0.60	5.02
		15	0.25	1.98	0.26	1.97
		16	0.21	1.92	0.19	1.85
T9	25	7	0.46	0.62	0.55	0.67
		9	0.41	0.54	0.48	0.64
		13	0.12	0.20	0.18	0.24
		14	0.09	0.16	0.15	0.17
	30	8	0.73	0.97	0.84	1.05
		10	0.56	0.84	0.74	0.95
		15	0.14	0.25	0.24	0.32
		16	0.12	0.22	0.22	0.28
T11	25	7	0.33	0.43	0.38	0.47
		9	0.32	0.42	0.38	0.48
		13	0.10	0.15	0.15	0.20
		14	0.08	0.13	0.13	0.16
	30	8	0.50	0.68	0.63	0.81
		10	0.50	0.67	0.59	0.81
		15	0.13	0.21	0.22	0.29
		16	0.11	0.20	0.20	0.27

As one can see, model MWLP copes quite well with solving problems up to 30 locations. The longest times concerns n-center problems and are about a few seconds. For other types of problems with non-increasing weights the solution times are shorter, reaching the shortest values for n-median problems. Constraint c_3 worsens the solution times for all types of problems under consideration. It is significant deterioration, even several times, in case of problems of types T1–T3. This applies to both cases where c_3 is the only valid inequality, as well as the formulation with two valid inequalities. The situation is slightly different for constraint c_4. For problems of types T1, T9, T11 impact of this constraint is negative, but to a much lesser extent than that of constraint c_3. On the contrary, for problems of types T2 and T3 constraint c_4 allows us to achieve shorter solution times. However, the differences are rather small, especially for the problem of type TC3.

The results obtained for the problems up to 30 locations suggest that valid inequalities for linear model of WOWA does not allow for significant performance improvement, and sometimes they may cause considerable deterioration of solution times. Although constraint c_4 improves the solution times for two problem types, the reduction is relatively minor. Constraint c_4 is a lower bound for the upper limit of function $L(\mathbf{y}, \mathbf{p}, \alpha)$, and thus its direction is consistent with constraints in basic linear formulation of WOWA. For this reason it seems that c_4 does not complicate too much the structure of a set of feasible solutions, and has a positive impact in solving algorithm for some problems. On the other hand, it appears that constraint c_3 makes the structure of the feasible set much harder, which consequently leads to worse solution times.

We have also checked the performance for larger problems. The performance of basic linear model and impact of constraint c_4 have been examined by solving problems with 100 and 200 locations from OR-library. The results are shown in Table 3.

Table 3: Solution times [s] for problems from OR-library ($m \in \{100, 200\}$) by linear formulation MWLP (with and without valid inequality) (minus sign for the problems that were not solved within 600 s time limit)

Problem				CPU[s]	
type	name	m	n	MWLP	c_4
T1	pmed1	100	5	0.56	0.63
	pmed2		10	0.29	0.37
	pmed3		10	0.34	0.42
	pmed4		20	0.22	0.33
	pmed5		33	0.22	0.29
	pmed6	200	5	17.31	24.16
	pmed7		10	2.29	2.82
	pmed8		20	1.93	2.68
	pmed9		40	1.72	2.5
	pmed10		67	1.38	2.17
T2	pmed1	100	5	–	–
	pmed2		10	–	–
	pmed3		10	–	–
	pmed4		20	–	–
	pmed5		33	–	–
	pmed6	200	5	–	–
	pmed7		10	–	–
	pmed8		20	–	–
	pmed9		40	–	–
	pmed10		67	–	–
T3	pmed1	100	5	27.01	30.56
	pmed2		10	–	–
	pmed3		10	88.1	99.58
	pmed4		20	–	–
	pmed5		33	36.05	62.73
	pmed6	200	5	–	–
	pmed7		10	–	–
	pmed8		20	–	–
	pmed9		40	–	–
	pmed10		67	274.44	255.76
T9	pmed1	100	5	24.48	38.28
	pmed2		10	81.17	100.87
	pmed3		10	35.03	44.78
	pmed4		20	35.37	53.77
	pmed5		33	4.06	16.87
	pmed6	200	5	–	–
	pmed7		10	–	–
	pmed8		20	–	–
	pmed9		40	–	–
	pmed10		67	–	–

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Problem				CPU[s]	
type	name	m	n	MWLP	c_4
T11	pmed1	100	5	7.44	13.54
	pmed2		10	15.98	37.85
	pmed3		10	22.52	30.16
	pmed4		20	17.44	31.36
	pmed5		33	4.09	13.27
	pmed6	200	5	–	–
	pmed7		10	–	–
	pmed8		20	–	–
	pmed9		40	–	–
	pmed10		67	576.76	–

It can be seen that except for n-median problems (T1), only a few problems with 200 locations have been solved within the limit of 600 s. Considering the problems with 100 locations, the worst results relate to problems of types T2 and T3. We have failed to solve any problem of type T2, and have managed to solve 3 of 5 problems of type T3. As we can remember, for these types of problems constraint c_4 has shortened slightly the solution time in case of small problems (up to 30 sites). However, results for bigger instances do not confirm this observations, because for solved problems of type T3 the solution times are longer in case of formulation with constraint c_4. It suggests that valid inequalities in linear model of WOWA usually makes the problems harder to solve.

5.2.2 Non-monotonic and non-decreasing preference weights

When the preference weights are non-decreasing or non-monotonic the binary variables are required in the model. This leads to MILP models, whose computational complexity is significantly greater than LP models. We have tested computational performance of MILP models on problems with 8 and 10 locations. First, two MILP models have been compared taking into account one simple valid inequality, so the following four formulations have been examined:

- $MW1_1$ — model (18a)–(18h),
- $MW1_2$ — model (18a)–(18h) with constraint (20),
- $MW2_1$ — model (18a)–(18c), (18h), (19a)–(19d),
- $MW2_2$ — model (18a)–(18c) , (18h), (19a)–(19d) with constraint (20).

Next, for the best of above formulations we have considered four more complex valid inequalities c_1–c_4.

First, let us compare model MW1 with MW2 and investigate the effect of constraint (20). The full results are presented in Table 4. Figure 4 shows the results for problems with 10 locations, averaged additionally over variants of facilities number. Clearly, model MW2 achieves significantly shorter times than model MW1. The differences depend on the problem type, but in many cases they exceed the order of magnitude. It applies to both versions of the model, ie. with and without constraint (20). This constraint in itself also shortens the solution times in most cases, although its impact is lower, at most it halves the solution times. On the other hand, it takes a long time to solve even such small problems (8, 10 locations), specially problems of types T4–T8. Model MW1 did not solve many problems with 10 locations within the time limit. Model MW2 did better, although it did not manage to solve all the problems either, especially instances with larger number of facilities. Somewhat comforting is the fact that only problems of type T4 seem to be of substantial practical importance, due to the nature of the preference weights. It is also surprising that problems of types T10 and T12, with the increasing preference weights, are relatively easy to solve. Perhaps it is due to all non-zero preference weights in these problem types.

Table 4: Average solution times [s] for MILP models

Problem			CPU[s]			
type	m	n	$MW1_1$	$MW1_2$	$MW2_1$	$MW2_2$
TC4	8	2	2.99	1.41	0.63	0.39
		3	18.01	6.29	1.45	0.83
		4	76.19	26.70	3.38	1.97
		5	^{10}516,71	^{6}364,93	11.80	6.64
	10	3	78.17	59.77	8.54	4.12
		4	^{2}141,83	^{1}156,31	24.45	8.07
		5	^{14}594,06	^{11}549,54	131.12	26.22
		6	–	–	^{6}431,12	^{1}176,46
TC5	8	2	5.08	4.01	1.25	0.84
		3	5.50	4.56	1.24	0.86
		4	5.48	3.84	1.38	0.86
		5	4.37	4.25	0.92	0.75
	10	3	141.67	95.26	23.90	19.01
		4	172.59	158.08	23.52	21.34
		5	^{1}168,07	^{2}201,03	23.54	17.27
		6	^{2}188,8	^{3}189,2	47.05	16.54
TC6	8	2	13.68	11.85	2.71	1.78
		3	36.35	23.83	4.45	3.66
		4	178.59	63.66	7.18	6.38
		5	^{7}391,16	138.80	8.32	8.30
	10	3	^{12}594,46	^{10}544,71	101.00	63.31
		4	–	^{13}580,75	^{1}187,92	119.40
		5	–	–	^{3}290,17	199.29
		6	–	–	^{13}570,7	^{2}403,8
TC7	8	2	1.29	0.94	0.62	0.35
		3	0.88	0.80	0.45	0.30
		4	0.70	0.65	0.29	0.23
		5	0.48	0.44	0.21	0.18
	10	3	21.46	13.62	5.38	2.76
		4	42.25	11.66	4.59	2.04
		5	18.36	9.71	2.91	1.95
		6	6.17	5.90	2.69	2.23
TC8	8	2	6.01	2.60	1.06	0.79
		3	5.81	6.80	1.18	0.78
		4	1.92	2.24	0.69	0.56
		5	3.45	2.56	0.61	0.65
	10	3	^{4}274,03	^{2}217,87	23.41	17.24
		4	^{4}248,97	^{4}249,46	25.13	16.98
		5	^{3}206,18	^{3}194,49	20.74	18.27
		6	^{3}224,4	^{1}121,01	19.66	18.30
TC10	8	2	1.19	0.75	0.50	0.28
		3	0.68	0.54	0.26	0.22
		4	0.48	0.40	0.22	0.19
		5	0.27	0.27	0.17	0.14
	10	3	6.62	4.61	1.19	0.77
		4	4.84	3.25	1.24	0.78
		5	2.90	1.77	0.78	0.58
		6	1.54	1.08	0.49	0.37
TC12	8	2	0.51	0.44	0.35	0.21
		3	0.26	0.33	0.19	0.14
		4	0.22	0.27	0.17	0.12

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Problem			CPU[s]			
type	m	n	$MW1_1$	$MW1_2$	$MW2_1$	$MW2_2$
		5	0.15	0.19	0.11	0.10
	10	3	1.25	1.02	0.59	0.43
		4	0.87	0.86	0.44	0.35
		5	0.56	0.58	0.30	0.28
		6	0.34	0.50	0.20	0.24

In view of the low computational performance of the MILP models for WOWA, we have considered possibility of its improvement by more complex valid inequalities. For this purpose we have selected model $MW2_2$, which obtained the best results. Then each of previously proposed valid inequalities c_1–c_4 have been added to this model and using such extended formulation we have applied it to the test problems. We have considered 6 formulations: 4 with a single valid inequality and 2 with two valid inequalities. Detailed results are presented in Table 5. Figures 5 and 6 show the solution times for problems with 10 locations additionally averaged over variants of facilities number. Symbols of valid inequalities are used to sign models $MW2_2$ extended by corresponding constraints.

Table 5: Average solution times [s] for model $MW2_2$ with valid inequalities

Problem			CPU[s]						
type	m	n	$MW2_2$	c_1	c_2	c_1c_2	c_3	c_4	c_3c_4
TC4	8	2	0.39	0.43	0.38	0.43	0.63	0.37	0.72
		3	0.83	0.83	0.85	0.90	1.53	0.89	1.78
		4	1.97	1.61	1.51	1.24	4.07	2.02	3.46
		5	6.64	4.67	4.22	1.87	5.28	7.16	4.89
	10	3	4.12	4.47	3.94	3.49	8.77	4.10	10.61
		4	8.07	9.58	10.21	9.13	16.45	7.47	19.38
		5	26.22	24.26	22.19	14.17	31.81	32.67	47.29
		6	[1]176.46	77.86	82.20	32.88	108.25	[1]195.52	113.34
TC5	8	2	0.84	0.84	0.87	0.90	0.30	0.90	0.33
		3	0.86	1.04	0.99	0.97	0.30	0.87	0.31
		4	0.86	0.97	0.98	0.94	0.43	0.91	0.36
		5	0.75	0.89	0.81	0.71	0.29	0.78	0.19
	10	3	19.01	18.99	17.92	18.46	1.28	18.43	1.42
		4	21.34	21.03	17.61	21.93	1.74	24.67	1.47
		5	17.27	20.61	21.38	19.18	1.91	18.34	1.78
		6	16.54	20.83	21.27	20.92	1.31	23.20	1.19
TC6	8	2	1.78	1.64	1.59	1.63	0.83	1.85	1.20
		3	3.66	2.37	2.57	2.02	1.95	3.12	2.05
		4	6.38	4.39	5.14	2.47	2.84	5.92	3.45
		5	8.30	4.99	4.85	2.60	3.17	8.63	3.05
	10	3	63.31	55.66	51.80	42.82	15.67	75.97	16.24
		4	119.40	69.93	80.91	50.29	27.57	121.48	38.25
		5	199.29	111.63	119.99	67.19	66.10	205.66	89.80
		6	[2]403.8	199.69	200.33	93.59	150.59	[6]472.57	215.57
TC7	8	2	0.35	0.35	0.34	0.39	0.23	0.33	0.26
		3	0.30	0.30	0.30	0.36	0.17	0.30	0.17
		4	0.23	0.30	0.27	0.27	0.14	0.25	0.16
		5	0.18	0.20	0.21	0.25	0.13	0.18	0.14
	10	3	2.76	3.37	2.98	3.32	0.69	2.56	0.73
		4	2.04	2.70	2.51	2.59	0.53	2.47	0.58
		5	1.95	2.11	1.91	2.27	0.46	1.98	0.42
		6	2.23	1.91	1.86	1.84	0.35	1.88	0.34
TC8	8	2	0.79	0.67	0.69	0.71	0.47	0.72	0.54
		3	0.78	0.75	0.75	0.69	0.32	0.73	0.33

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Problem type	m	n	MW2$_2$	c$_1$	c$_2$	c$_1$c$_2$	c$_3$	c$_4$	c$_3$c$_4$
		4	0.56	0.64	0.57	0.56	0.19	0.68	0.21
		5	0.65	0.62	0.60	0.60	0.22	0.63	0.24
	10	3	17.24	20.07	20.17	20.59	6.73	20.32	7.08
		4	16.98	19.22	17.00	17.79	4.91	16.83	5.61
		5	18.27	16.24	16.42	18.08	3.17	16.85	2.94
		6	18.30	17.96	18.92	20.03	1.61	20.33	2.17
TC10	8	2	0.28	0.28	0.28	0.31	0.30	0.29	0.32
		3	0.22	0.22	0.22	0.26	0.21	0.23	0.23
		4	0.19	0.21	0.21	0.25	0.18	0.19	0.19
		5	0.14	0.16	0.16	0.18	0.14	0.14	0.14
	10	3	0.77	0.98	1.01	1.01	0.86	0.88	0.88
		4	0.78	0.94	0.93	0.97	0.80	0.78	0.84
		5	0.58	0.66	0.67	0.73	0.58	0.56	0.57
		6	0.37	0.38	0.42	0.57	0.36	0.34	0.36
TC12	8	2	0.21	0.23	0.24	0.26	0.21	0.21	0.21
		3	0.14	0.19	0.15	0.19	0.17	0.16	0.17
		4	0.12	0.15	0.17	0.16	0.12	0.13	0.13
		5	0.10	0.13	0.11	0.14	0.10	0.12	0.12
	10	3	0.43	0.48	0.49	0.56	0.41	0.43	0.45
		4	0.35	0.41	0.47	0.52	0.35	0.37	0.38
		5	0.28	0.33	0.34	0.43	0.27	0.29	0.30
		6	0.24	0.29	0.23	0.32	0.19	0.24	0.25

Firstly, one can see that the problems of types T10 and T12 (with increasing preference weights) are relatively easy to solve by the basic model MW2$_2$, and the influence of the valid inequalities is negligible (at least for such small problems), except maybe a few cases for the smallest size. Furthermore, when analyzing the impact of individual valid inequalities, constraint c$_3$, which limits maximum increase in the value of function $L(\mathbf{y}, \mathbf{p}, \alpha)$, stands out in positive way. It is especially apparent for problems of types T5–T8. Adding this constraint allows us to shorten several times the average solution times with respect to basic formulation. The results are similar for different numbers of facilities. The situation is somewhat different for problems T4. Here, constraint c$_3$ improves the average solution time only for cases with the greatest number of facilities. For problems T4 constraints c$_1$ and c$_2$, arising from the transitivity relation, achieve slightly better results. Constraint c$_4$ has negligible effect on the solution times for all types of problems. When considering the formulations with two valid inequalities, we can see that only for a few problems they obtain shorter solution times with respect to the formulation with single valid inequality. The formulation with both constraint c$_1$ and c$_2$ can be given as an example, which improves the solution times for problems of types T4 and T6. However, it is worth to remember that the number of constraints c$_1$ and c$_2$ is of order m^3 (exactly it is $\binom{m}{3}$), and the number of constraints c$_3$ is only of order m. Hence, as the problem size would increase, constraint c$_3$ would hinder the problem to a lesser degree.

The results for MILP model of WOWA show that, unlike in the case of LP model of WOWA, constraint c$_3$ allows to improve the solution times for some problem types. Basic MILP model of WOWA contains both the lower and upper bound of function $L(\mathbf{y}, \mathbf{p}, \alpha)$. In this case, however, the lower bound introduces more difficulties as it requires the binary variables. Since constraint c$_3$ limits the lower bound from above, it may give tighter description of relevant area of the feasible set, which might lead to better computational performance of the model.

6 Conclusions

This paper has investigated the Weighted Ordered Median Problem (WOMP), which extends the Order Median Problem by taking into account the demand requirements according to WOWA aggregation. This approach allows to obtain the optimal solution in terms of the distribution of outcomes given by the demand weights. In our research we have focused mainly on computational efficiency of such models. In case of non-increasing preference weights, thus consistent with equitable relation, WOWA aggregation can be formulated as LP optimization. This formulation is based on function $L(\mathbf{y}, \mathbf{p}, \alpha)$, which expresses the weighted average of the largest costs within the fixed demand portion of α. In general, when the preference weights do not satisfy the monotonicity condition, we have proposed the extended formulation, which can be applied for any non-negative preference weights. However, this flexibility requires the binary variables and related constraints, which substantially increase the computational complexity, and thus significantly limit the maximum size of problems that can be solved. We have also checked the possibility for improving performance by introducing the valid inequalities and have carried out the computational tests to examine the computational performance of particular formulations and the influence of the valid inequalities.

LP model of WOWA have performed very well with small problems, up to 30 locations, which have been solved in a few hundreds to a few seconds. Considering large problems, about 100 locations, the linear model have obtained reasonable solution times for problems with all non-zero (monotone) weights (T9, T11) and very good solution times for n-median problems (T1). Much worse results have been achieved for problems of types T2 and T3 — 3 of 5 problems of type T3 have been solved and none of type T2. The valid inequalities, in general, have not improved the computational performance of the linear model. This observation is similar to the results obtained for problems with OWA aggregation [30], where valid inequalities did not improve the performance of linear formulation of OWA either.

The comparison of two general MILP model of WOWA, for any non-negative preference weights, show better performance of model MW2, with reduced number of the binary variables. However, even this better model has exceeded the time limit of 600 s for some problems with only 10 locations. Some of the proposed valid inequalities have allowed for several times reduction of the solution times, and thus all problems with 10 locations have been solved. Nevertheless, we have not managed to solve all the problems for a slightly larger size. Thus it seems that in the case of non-monotonic preference weights MILP models of WOWA allow to solve problems with only a dozen or so locations. It suggests the need for the use of approximate method for problems of larger size. At present we are working on the adaptation of metaheuristic called Variable Neighborhood Search (VNS), which was previously applied for Order Median Problem.

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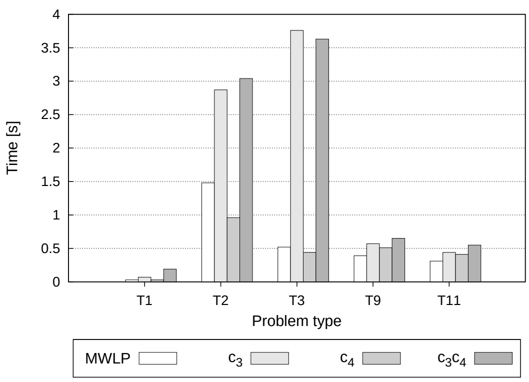

Figure 3: Average solution times [s] (within 60 problems with $m = 30$) for linear model MWLP

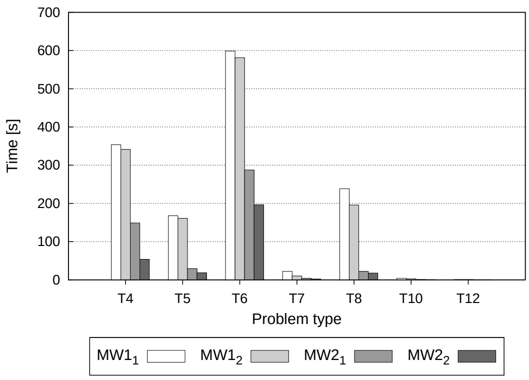

Figure 4: Average solution times [s] for problems with $m = 10$ by models MW1 and MW2

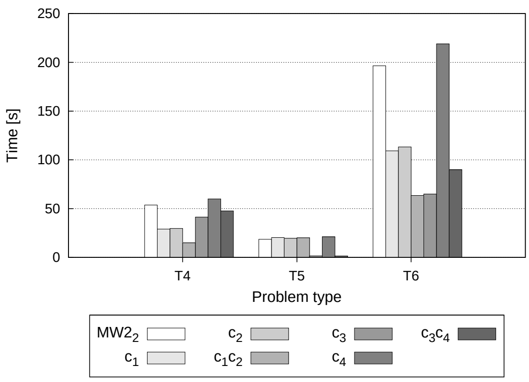

Figure 5: Average solution times [s] for T4–T6 problems with $m = 10$ by model $MW2_2$ with valid inequalities

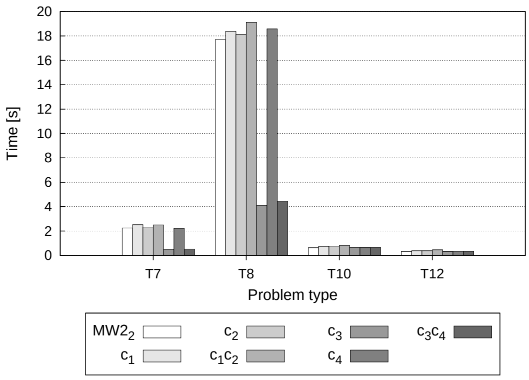

Figure 6: Average solution times [s] for T7, T8, T10, T12 problems with $m = 10$ by model $\mathrm{MW2}_2$ with valid inequalities